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A NOTE ON $C_c(X)$ VIA A TOPOLOGICAL RING

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Abstract. Let $C_c(X)$ (resp., $C_c^*(X)$) denote the functionally countable subalgebra of $C(X)$ (resp., $C^*(X)$), consisting of all functions (resp., bounded functions) with countable image. $C_c(X)$ (resp., $C_c^*(X)$) as a topological ring via m_c-topology (resp., m_c^*-topology) and u_c-topology (resp., u_c^*-topology) is investigated and the equality of the latter two topologies is characterized. Topological spaces which are called N-spaces are introduced and studied. It is shown that the m_c-topology on $C_c(X)$ and its relative topology as a subspace of $C(X)$ (with m-topology) coincide if and only if X is an N-space. We also show that X is pseudocompact if and only if it is both a countably pseudocompact, and an N-space.

1. Introduction

Throughout this paper all given topological spaces X, are Tychonoff. $C(X)$ (resp., $C^*(X)$) denotes the ring of all real-valued (resp., bounded real-valued) continuous functions on X. The subring of continuous (resp., bounded continuous) functions with countable range on the space X, is denoted by $C_c(X)$ (resp., $C_c^*(X)$). In recent years $C_c(X)$ and $C_c^*(X)$ have been studied widely, by Karamzadeh and his colleagues, see [6, 7, 2]. We aim to investigate the similarities and differences between the ring $C(X)$ (resp., $C^*(X)$) and its functionally countable subring via m_c-topology (resp., m_c^*-topology) and u_c-topology (resp., u_c^*-topology), see also [1, 12]. It is shown that m_c^*-topology on $C_c^*(X)$ coincides with its relative topology as a subspace

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of $C_{c_m}(X)$. We introduce and study N-spaces, and show that a pseudocompact space X is an N-space. We will also observe that the m_c-topology on $C_c(X)$ and its relative topology as a subspace of $C(X)$ with the m-topology coincide if and only if X is an N-space. Pseudocompact spaces versus N-spaces are also investigated. We notice that $C_u(X)$ (resp., $C^*_u(X)$) is completely metrizable and $C_{c_u}(X)$ is metrizable but may not be complete. The reader is referred to [5, 8] for notations and fundamental terminologies concerning rings of real-valued continuous functions.

2. Functionally countable subring via m_c-topology

Let $\mathcal{U}^+(X)$ be the set of all positive units of $C(X)$. For each $f \in C(X)$ and $u \in \mathcal{U}^+(X)$, the subset $\mathcal{B}(f,u)$ is defined as follows:

$$\mathcal{B}(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x),\ \forall x \in X\}.$$

The family $\{\mathcal{B}(f,u) : u \in \mathcal{U}^+(X),\ f \in C(X)\}$ will be a base for a topology on $C(X)$ which is called m-topology. $C(X)$ endowed with the m-topology is denoted by $C_m(X)$ which is a Hausdorff topological ring, see [3, 4, 9, 10, 11]. We introduce the counterpart of the latter topology on the $C_c(X)$ (resp., $C^*_c(X)$), see also [1, 12].

Definition 2.1. Let $\mathcal{U}^+_c(X) = \{u \in \mathcal{U}^+(X) : u \in C_c(X)\}$. The m_c-topology on $C_c(X)$ is defined by taking the subset of the form

$$\mathcal{B}_c(f,u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x),\ \ \forall x \in X\}$$

as a base for a neighborhood system at f, for each $f \in C_c(X)$ and $u \in \mathcal{U}^+_c(X)$. The set $C_c(X)$ endowed with the m_c-topology is denoted by $C_{c_m}(X)$.

Lemma 2.2. *$\mathcal{B}_c(f,u)$ is a contraction of $\mathcal{B}(f,u)$, i.e., for each $f \in C_c(X)$ and $u \in \mathcal{U}^+_c(X)$ we have $\mathcal{B}_c(f,u) = \mathcal{B}(f,u) \cap C_c(X)$.*

Remark 2.3. For each $f \in C^*(X)$ and $u \in \mathcal{U}^{*+}(X)$, where

$$\mathcal{U}^{*+}(X) = \{u \in \mathcal{U}^+(X) : u \in C^*(X)\},$$

consider the subset

$$\mathcal{B}^*(f,u) = \{g \in C^*(X)\ :\ |f(x) - g(x)| < u(x),\ \ \forall x \in X\}.$$

Then it can be shown that $\{\mathcal{B}^*(f,u) : u \in \mathcal{U}^{*+}(X), f \in C^*(X)\}$ is an open base for some topology on $C^*(X)$, which is called m^*-topology on $C^*(X)$ and the notion $C^*_m(X)$ is used for $C^*(X)$ endowed with m^*-topology. Similarly, if $f \in C^*_c(X)$ and $u \in \mathcal{U}^{*+}_c(X)$, where

$$\mathcal{U}^{*+}_c(X) = \{u \in \mathcal{U}^+(X)\ :\ u \in C^*_c(X)\},$$

then m_c^*-topology on $C_c^*(X)$ is defined by taking the subset

$$\mathcal{B}_c^*(f,u) = \{g \in C_c^*(X) \ : \ |f(x) - g(x)| < u(x), \ \forall x \in X\}$$

as a local base at f, for each $f \in C_c^*(X)$ and $u \in \mathcal{U}_c^{*+}(X)$. The set $C_c^*(X)$ endowed with the m_c^*-topology is denoted by $C_{c_m}^*(X)$.

Lemma 2.4.

(1) *If $f \in C^*(X)$ and $u \in \mathcal{U}^{*+}(X)$, then $\mathcal{B}(f,u) \cap C^*(X) = \mathcal{B}^*(f,u)$.*
(2) *If $f \in C_c^*(X)$ and $u \in \mathcal{U}_c^{*+}(X)$, then*

$$\mathcal{B}_c(f,u) \cap C_c^*(X) = \mathcal{B}_c^*(f,u).$$

The proofs of the two following propositions are similar and hence only one of them is proved.

Proposition 2.5. *The m^*-topology on $C^*(X)$ coincides with the relative topology on $C^*(X)$ as a subspace of $C_m(X)$.*

Proposition 2.6. *The m_c^*-topology on $C_c^*(X)$ coincides with the relative topology on $C_c^*(X)$ as a subspace of $C_{c_m}(X)$.*

Proof. Let τ_1 be the m_c^*-topology on $C_c^*(X)$ and τ_2 be the relative topology on $C_c^*(X)$ as a subspace of $C_{c_m}(X)$. If $G \in \tau_1$ and $f \in G$, then there exists $u_1 \in \mathcal{U}_c^{*+}$ such that $\mathcal{B}_c^*(f,u_1) \subseteq G$. So G contains $\mathcal{B}_c(f,u_1) \cap C_c^*(X)$ as an open subset of τ_2. Hence $f \in int_{\tau_2} G$ implies that $G \subseteq int_{\tau_2} G$, therefore $G \in \tau_2$. Now, let G be an open subset of $C_c(X)$ with m_c-topology. We must prove that $G \cap C_c^*(X)$ belongs to τ_1. If $G \cap C_c^*(X) \neq \emptyset$ and $f \in G \cap C_c^*(X)$, then there exists $u_1 \in \mathcal{U}_c^+$ such that $\mathcal{B}_c(f,u_1) \subseteq G$. If we take $v = \dfrac{u_1}{1+u_1}$, then $v \in \mathcal{U}_c^{*+}$. Hence $\mathcal{B}_c(f,v) \subseteq \mathcal{B}(f,v) \subseteq G$ which implies that $\mathcal{B}_c^*(f,v) \subseteq G$. □

Theorem 2.7. *$C_{c_m}(X)$ is a regular Hausdorff space.*

Proof. Let $f, g \in C_c(X)$ and $f \neq g$, so there exists $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. We consider the constant function

$$u(x) = \frac{1}{3}|f(x_0) - g(x_0)|,$$

it is clear that $u \in \mathcal{U}_c^+$. We show that $\mathcal{B}_c(f,u) \cap \mathcal{B}_c(g,u) = \emptyset$. If $h \in \mathcal{B}_c(f,u) \cap \mathcal{B}_c(g,u)$, then for each $x \in X$, $|f(x) - h(x)| < u(x)$ and $|g(x) - h(x)| < u(x)$. Hence

$$\begin{aligned} |f(x_0) - g(x_0)| &\leq |f(x_0) - h(x_0)| + |g(x_0) - h(x_0)| \\ &< \frac{1}{3}|f(x_0) - g(x_0)| + \frac{1}{3}|f(x_0) - g(x_0)| \\ &= \frac{2}{3}|f(x_0) - g(x_0)| \end{aligned}$$

which is a contradiction. Now, we show that $C_{c_m}(X)$ is regular. For this main, let F be a closed subset of $C_{c_m}(X)$ and $g \in C_c(X) \setminus F$. So $g \notin cl_m(F)$, i.e., there exists $u \in \mathcal{U}_c^+$ such that $\mathcal{B}_c(g, u) \cap F = \emptyset$. Put $G = \mathcal{B}_c(g, \frac{u}{2})$ and $V = \bigcup_{f \in F} \mathcal{B}_c(f, \frac{u}{2})$, then $g \in G$ and $F \subseteq V$. It is sufficient to show that $G \cap V = \emptyset$. If $h \in G \cap V$, then $|h - g| < \frac{u}{2}$ and there exists $f_1 \in F$ such that $h \in \mathcal{B}_c(f_1, \frac{u}{2})$, i.e., $|f_1 - h| < \frac{u}{2}$. Therefore

$$|g - f_1| < |g - h| + |f_1 - h| < \frac{u}{2} + \frac{u}{2} = u,$$

so $f_1 \in \mathcal{B}_c(g, u)$ which is a contradiction. □

Definition 2.8. A topological space X is called an N-space, whenever for any $f \in \mathcal{U}^+$ there exists $g \in \mathcal{U}_c^+$ such that $g(x) \leq f(x)$ for each $x \in X$.

Lemma 2.9. *If X is an N-space, then for each $f \in \mathcal{U}^+$ there exists $g \in \mathcal{U}_c^+$ such that $f(x) \leq g(x)$ for all $x \in X$.*

Proof. If $f \in \mathcal{U}^+$, then $\frac{1}{f} \in \mathcal{U}^+$. Since X is an N-space, we infer that there exists $h \in \mathcal{U}_c^+$ such that $h \leq \frac{1}{f}$, therefore $\frac{1}{h} \geq f$. Now, if $g = \frac{1}{h} \in \mathcal{U}_c^+$, then $f(x) \leq g(x)$ for each $x \in X$. □

Example 2.10. The space $X = \bigcup_{k \in \mathbb{Z}} [2k, 2k+1]$ is an N-space. To see this, let $f \in \mathcal{U}^+(X)$. Then for each $x \in [2k, 2k+1]$, we define $g(x) = min\{f(x) : 2k \leq x \leq 2k+1\}$. It is evident that $g \in \mathcal{U}_c^+$ and $g(x) \leq f(x)$ for each $x \in X$. Therefore X is an N-space.

We remind the reader that whenever X is functionally countable, then X is an N-space, but the converse is not true as Example 2.10 shows.

Example 2.11. The space $\mathbb{R}$ is not an N-space. Consider

$$f(x) = e^x \in C(\mathbb{R}),$$

it is evident that $f \in \mathcal{U}^+$. Let $g \in \mathcal{U}_c^+$ such that for each $x \in \mathbb{R}$, $g(x) \leq f(x)$. Since $C_c(\mathbb{R}) = \mathbb{R}$, we infer that there exists $r \in \mathbb{R}$ such that $g(x) = r$. Hence, for each $x \in \mathbb{R}$, $r < e^x$ which implies that for each $x \in \mathbb{R}$, $\ln r < x$ and it is a contradiction.

Proposition 2.12. *A pseudocompact space X is an N-space.*

Proof. Let X be pseudocompact and $f \in \mathcal{U}^+$. Since f in $C^*(X)$ is invertible, we infer that it is far away from zero, i.e., there exists $r > 0$ such that $f(x) > r$ for each $x \in X$. So if $g(x) = r$, for each $x \in X$, then $g \in \mathcal{U}_c^+$ and it is clear that $r = g(x) \leq f(x)$ for each $x \in X$. □

By Example 2.10, we conclude that the converse of Proposition 2.12 does not hold.

Proposition 2.13. *Let X be a connected space. Then X is an N-space if and only if X is a pseudocompact space.*

Proof. Let X be a pseudocompact space, so by Proposition 2.12, X is an N-space. If X is a connected N-space, then $1 + |f| \in \mathcal{U}^+$ for each $f \in C(X)$. Hence there exists $g \in \mathcal{U}_c^+$ such that $1 + |f| \leq g$. Since $g \in C(X)$ and X is connected, we infer that there exists $r \in \mathbb{R}^+$ such that $g = r$. So for every $x \in X$, we have $1 + |f| \leq r$, i.e., $f \in C^*(X)$. Therefore $C(X) = C^*(X)$ and we are done. □

The following example shows that every subspace of an N-space may not be an N-space.

Example 2.14. The topological space $X = [0, 1] \cup [2, 3]$ is an N-space, but $Y = (0, 1)$ is a connected space which is not pesudocompact. Hence Y is a subspace of an N-space X that is not an N-space.

Theorem 2.15. *The m_c-topology on $C_c(X)$ and the relative topology on $C_c(X)$ as a subspace of $C_m(X)$ coincide if and only if X is an N-space.*

Proof. Let X be an N-space and τ_1, τ_2 be the m_c-topology on $C_c(X)$ and the relative topology on $C_c(X)$ respectively. If $G \in \tau_1$ and $f \in G \subseteq C_c(X)$, then there exists $u_1 \in U_c^+$ such that $\mathcal{B}_c(f, u_1) \subseteq G$, so $\mathcal{B}(f, u_1) \cap C_c(X) \subseteq G$. Hence $f \in \mathcal{B}(f, u_1) \cap C_c(X) \subseteq G$, but $\mathcal{B}(f, u_1) \cap C_c(X) \in \tau_2$ which implies that $G \in \tau_2$ and therefore $\tau_1 \subseteq \tau_2$. Now, let G be an open subset of $C_m(X)$ and hence $G \cap C_c(X) \in \tau_2$. We prove that $G \cap C_c(X) \in \tau_1$. If $G \cap C_c(X) = \emptyset$ we are done. Otherwise, let $f \in G \cap C_c(X)$, then there exists $u \in \mathcal{U}^+$ such that $\mathcal{B}(f, u) \subseteq G$. But $f \in C_c(X)$ implies that $\mathcal{B}(f, u) \cap C_c(X) \subseteq G$. Therefore

$$\{g \in C_c(X) : |f(x) - g(x)| < u(x) \ : \ \forall x \in X\} \subseteq G.$$

Since X is an N-space, we infer that there exists $u_1 \in \mathcal{U}_c^+$ such that $u_1(x) \leq u(x)$ for each $x \in X$. Therefore

$$\begin{aligned} f &\in \{g \in C_c(X) : |f(x) - g(x)| < u_1(x), \forall x \in X\} \\ &\subseteq \{g \in C_c(X) : |f(x) - g(x)| < u(x), \forall x \in X\} \\ &\subseteq G \end{aligned}$$

and so $G \in \tau_1$, hence $\tau_2 \subseteq \tau_1$. Conversely, let $\tau_1 = \tau_2$. We prove that X is an N-space. For this main, let $u \in \mathcal{U}^+$ be an arbitrary element, so for each $f \in C_c(X)$ we have $\mathcal{B}(f,u) \cap C_c(X) \in \tau_1 = \tau_2$. Since $f \in \mathcal{B}(f,u) \cap C_c(X) \in \tau_1$ we infer that there exists $u_1 \in \mathcal{U}_c^+$ such that $\mathcal{B}_c(f,u_1) \subseteq \mathcal{B}(f,u) \cap C_c(X)$. Hence, $\mathcal{B}_c(f,u_1) \subseteq \mathcal{B}_c(f,u)$. It is evident that $f + \frac{1}{2}u_1 \in \mathcal{B}_c(f,u_1)$, hence $\frac{1}{2}u_1 + f \in \mathcal{B}(f,u)$. Therefore

$$|\frac{1}{2}u_1(x) + f(x) - f(x)| \leq u(x)$$

for each $x \in X$, so $\frac{1}{2}u_1(x) \leq u(x)$ for each $x \in X$. Now, if $v = \frac{1}{2}u_1$, then $v \in \mathcal{U}_c^+$ and for each $x \in X$, $v(x) \leq u(x)$, hence we are done. □

Corollary 2.16. *If X is a pseudocompact space, then the m_c-topology on $C_c(X)$ and relative topology on $C_c(X)$ as a subspace of $C_m(X)$ coincide.*

Corollary 2.17. *If X is a N-space, then the m_c^*-topology on $C_c^*(X)$ and relative topology on $C_c^*(X)$ as a subspace of $C_{c_m}(X)$ coincide.*

3. u_c-topology on $C_c(X)$

Definition 3.1. For a function $f \in C(X)$ and each positive real number ε, the subset $u(f,\varepsilon)$ is defined as follows:

$$u(f,\varepsilon) = \{g \in C(X) : |f(x) - g(x)| < \varepsilon,\ \forall x \in X\}.$$

The family

$$\{u(f,\varepsilon) : \varepsilon \in \mathbb{R}^+,\ f \in C(X)\}$$

will be a base for a neighborhood system at f and this topology on $C(X)$ is called uniform topology which is denoted by u-topology. The notion $C_u(X)$ is used for $C(X)$ endowed with the u-topology. Similarly, the uniform topology on $C^*(X)$, $C_c(X)$ and $C_c^*(X)$ are defined and denoted by $C_u^*(X)$, $C_{c_u}(X)$, and $C_{c_u}^*(X)$, respectively.

It is evident that m_c-topology (resp., m_c^*-topology) is finer than u_c-topology (resp., u_c^*-topology). The equality of the latter two topologies is investigated in the following facts.

Proposition 3.2. *Let*

$$u_c(f,\varepsilon) = \{g \in C_c(X) : |f(x) - g(x)| < \varepsilon,\ \forall x \in X\}.$$

If $f \in C_c(X)$ and $\varepsilon > 0$, then $u_c(f,\varepsilon) = u(f,\varepsilon) \cap C_c(X)$.

Proof. Let $g \in u_c(f,\varepsilon)$, so $g \in C_c(X)$ and $|f(x) - g(x)| < \varepsilon$. Therefore $g \in u(f,\varepsilon) \cap C_c(X)$. Conversely, if $g \in u(f,\varepsilon) \cap C_c(X)$, then $g \in C_c(X)$ and for each $x \in X$, $|f(x) - g(x)| < \varepsilon$. Hence,

$$g \in \{h \in C_c(X) : |f(x) - h(x)| < \varepsilon \ , \ \forall x \in X\}$$

which implies that $g \in u_c(f, \varepsilon)$. □

Proposition 3.3. *The u_c-topology on $C_c(X)$ coincides with the relative topology on $C_c(X)$ as a subspace of $C_u(X)$.*

Proof. Let τ_1 be the u_c-topology on $C_c(X)$ and τ_2 be the relative topology on $C_c(X)$ as a subspace of $C_u(X)$. If $G \in \tau_1$ and $f \in G$, then there exists $\varepsilon > 0$ such that $u_c(f, \varepsilon) \subseteq G$. So $u(f, \varepsilon) \cap C_c(X) \subseteq G$ which implies that $f \in u(f, \varepsilon) \cap C_c(X) \subseteq G$ where $u(f, \varepsilon) \cap C_c(X) \in \tau_2$. Therefore $f \in int_{\tau_2} G$, so $G \in \tau_2$. If $H \in \tau_2$, then there exists $G \subseteq C(X)$ such that G is open and $H = G \cap C_c(X)$. Now, if $f \in H$, then $f \in G \cap C_c(X)$, so $f \in C_c(X)$ and there exists $\varepsilon > 0$ such that $\mathcal{U}(f, \varepsilon) \subseteq G$ which implies that $f \in \mathcal{U}_c(f, \varepsilon) \subseteq \mathcal{U}(f, \varepsilon) \subseteq G$. Hence, $f \in int_{\tau_1} G$ and we are done. □

In the following theorems the m-topology (resp., m^*-topology) and u-topology (resp., u^*-topology) on $C(X)$ (resp., $C^*(X)$) are compared. It is shown that the coincidence of these topologies on $C(X)$ (resp., $C^*(X)$) is equivalent to the pseudocompactness of X, see [11] and [8, 2N.2].

Theorem 3.4. *A space X is pseudocompact if and only if*

$$C_m(X) = C_u(X).$$

Theorem 3.5. *space X is pseudocompact if and only if*

$$C^*_m(X) = C^*_u(X).$$

Corollary 3.6. *For a space X, $C_m(X) = C_u(X)$ if and only if*

$$C^*_m(X) = C^*_u(X).$$

Remark 3.7. According to [6], a space X is called countably pseudocompact whenever $C_c(X) = C^*_c(X)$. Every pseudocompact space is countably pseudocompact, but the converse may not be hold. For instance, let $X = (0, 1) \cup \{2\}$ and $f \in C_c(X)$, then there exists $r \in \mathbb{R}$ such that $f((0, 1)) = r$. Now, if $k = max\{r, f(2)\}$, then it is clear that $|f| \leq k$, i.e., $f \in C^*_c(X)$ and X is countably pseudocompact. Since $f(x) = tan\dfrac{\pi x}{2}$ for each $x \in (0, 1)$ and $f(2) = -1$ is a continuous function which is not bounded, we infer that X is not pseudocompact.

We remind the reader that the next theorem is in fact, the same as [12, Proposition 2.2].

Theorem 3.8. *A space X is countably pseudocompact if and only if $C_{c_m}(X) = C_{c_u}(X)$.*

Theorem 3.9. *A space X is countably pseudocompact if and only if $C^*_{c_m}(X) = C^*_{c_u}(X)$.*

Proof. Let X be countably pseudocompact and τ_1, τ_2 be the m-topology and u-topology on $C^*_c(X)$ respectively. It is sufficient to show that $\tau_1 \subseteq \tau_2$. For this main, let $G \in \tau_1$ and $f \in G$, then there exists $u_1 \in \mathcal{U}^{*+}_c$ such that $\mathcal{B}^*_c(f, u_1) \subseteq G$. Now, if $\varepsilon = inf\{u_1(x) : x \in X\}$, then $\varepsilon > 0$, for $u_1 \in \mathcal{U}^{*+}_c$ is far away from zero. Therefore

$$\mathcal{U}^*_c(f, \varepsilon) \subseteq \mathcal{B}^*_c(f, u_1) \subseteq G$$

and we are done. Conversely, let $\tau_1 = \tau_2$. If X is not countably pseudocompact and $f \in C_c(X) \setminus C^*_c(X)$, then put $u = \frac{1}{|f| \vee 1}$. It is evident that $u \in \mathcal{U}^+_c$, so $\mathcal{B}_c(0, u) \cap C^*_c(X) \in \tau_1$. We show that $\mathcal{B}_c(0, u) \cap C^*_c(X) \notin \tau_2$ and we are done. If $\mathcal{B}_c(0, u) \cap C^*_c(X) \in \tau_2$, then there exists $\varepsilon > 0$ such that $u^*_c(0, \varepsilon) \subseteq \mathcal{B}_c(0, u) \cap C^*_c(X)$. Since $\frac{\varepsilon}{2} \in u^*_c(0, \varepsilon)$, then $\frac{\varepsilon}{2} \in \mathcal{B}^*_c(0, u)$ which implies that $|\frac{\varepsilon}{2} - 0| < u$. So $\frac{\varepsilon}{2} < \frac{1}{|f| \vee 1}$, therefore $|f| \vee 1 < \frac{2}{\varepsilon}$ which is a contradiction, for f is unbounded. □

Corollary 3.10. *For a space X, $C_{c_m}(X) = C_{c_u}(X)$ if and only if $C^*_{c_m}(X) = C^*_{c_u}(X)$.*

Theorem 3.11. *A space X is pseudocompact if and only if X is countably pseudocompact and N-space.*

Proof. If X is pseudocompact, then it is countably pseudocompact and by Proposition 2.12, it is an N-space. Conversely, let X be a countably pseudocompact N-space. For each $f \in C(X)$, we have $1+|f| \in \mathcal{U}^+$. So there exists $g \in \mathcal{U}^+_c$ such that $1+|f| < g$. Since $g \in \mathcal{U}^+_c = \mathcal{U}^{*+}_c$, we infer that there exists $k > 0$ such that $|g| \leq k$ and hence $f \in C^*(X)$. □

Corollary 3.12. *A space X is countably pseudocompact if and only if $C_{c_m}(X) = C_{c_u}(X)$.*

Corollary 3.13. *A space X is an N-space and $C^*_{c_m}(X) = C^*_{c_u}(X)$ if and only if X is pseudocompact.*

Proof. It is immediate by Theorem 3.9 and Theorem 3.11. □

We recall that a metric space M is called complete (or a Cauchy space) if every Cauchy sequence of points in M has a limit that is also in M. A topological space (X, τ) is called completely metrizable space whenever there exists at least one metric d on X such that (X, d) is a complete metric space and d induces the topology τ. One can easily show that $C_u(X)$ ($C_u^*(X)$) is metrizable.

Theorem 3.14. *$C_{c_u}(X)$ is metrizable for any space X.*

Proof. We define the function $\rho : C_c(X) \times C_c(X) \to \mathbb{R}$ by

$$\rho(f,g) = \underset{x\in X}{Sup}\, \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}.$$

It is evident that $\rho(f,g) = \rho(g,f)$, $\rho(f,g) \geq 0$, and $\rho(f,g) = 0$ if and only if $f = g$. Since the function $f(x) = \dfrac{x}{1+x}$ for each $x \geq 0$ is extremally increasing and $|\alpha-\beta| \leq |\alpha-\gamma| + |\gamma-\beta|$, we infer that

$$\begin{aligned} \frac{|\alpha-\beta|}{1+|\alpha-\beta|} &\leq \frac{|\alpha-\gamma|+|\gamma-\beta|}{1+|\alpha-\gamma|+|\gamma-\beta|} \\ &= \frac{|\alpha-\gamma|}{1+|\alpha-\gamma|+|\gamma-\beta|} + \frac{|\gamma-\beta|}{1+|\alpha-\gamma|+|\gamma-\beta|} \\ &\leq \frac{|\alpha-\gamma|}{1+|\alpha-\gamma|} + \frac{|\gamma-\beta|}{1+|\gamma-\beta|}. \end{aligned}$$

Therefore $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$. We show that the metric space $C_c(X)$ with metric ρ is the same space as $C_{c_u}(X)$. Let $u_c(f,\varepsilon)$ be a neighborhood of f in $C_{c_u}(X)$, we show that $N_\rho(f, \dfrac{\varepsilon}{1+\varepsilon}) \subseteq u(f,\varepsilon)$. If $g \in N_\rho(f, \dfrac{\varepsilon}{1+\varepsilon})$, then

$$\frac{|f-g|}{1+|f-g|} < \frac{\varepsilon}{1+\varepsilon}$$

which implies that $|f-g| < \varepsilon$, i.e., $g \in u_c(f,\varepsilon)$. Conversely, let $N_\rho(f,\varepsilon)$ be an open ball in $C_c(X)$. For $r = min\{\dfrac{1}{2}, \varepsilon\} > 0$, we have $g \in u_c(f, \dfrac{r}{1-r})$, i.e., $|f-g| < \dfrac{r}{1-r}$. Hence

$$\frac{|f-g|}{1+|f-g|} < \frac{r}{(1-r)+r},$$

so $\rho(f,g) < r$ which implies that $g \in N_\rho(f,r)$. Therefore

$$u_c(f, \frac{r}{1-r}) \subseteq N_\rho(f,r) \subseteq N_\rho(f,\varepsilon).$$

□

Remark 3.15. For a space X, $C_{c_u}(X)$ may not be complete. For instance, let $X = Q^c \cap [0,1]$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ where $f_n(x) = \frac{[nx]}{n}$ is a Cauchy sequence in $C_c(X)$, but it is not convergent in $C_c(X)$. For this main, we must prove that for arbitrary $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for each $m, n \geq M$, $\rho(f_m, f_n) < \varepsilon$. We note that $[mx] = mx - p_1$ and $[nx] = nx - p_2$ such that $0 \leq p_1, p_2 < 1$. For each $\varepsilon > 0$, if $M_1 \in \mathbb{N}$ and $\frac{1}{M_1} < \frac{\varepsilon}{2}$, then for each $m, n \geq M_1$,

$$\begin{aligned}
|f_m(x) - f_n(x)| &= \left|\frac{[mx]}{m} - \frac{[nx]}{n}\right| \\
&= \left|\frac{n[mx] - m[nx]}{mn}\right| \\
&= \left|\frac{mnx - np_1 - mnx + mp_2}{mn}\right| \\
&= \left|\frac{mp_1 - np_2}{mn}\right| \\
&= \left|\frac{p_2}{n} - \frac{p_1}{n}\right| \\
&\leq \frac{p_2}{n} + \frac{p_1}{m} \\
&< \frac{1}{n} + \frac{1}{m} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which implies that for each $m, n \geq M_1$, $|f_m(x) - f_n(x)| < \varepsilon$. So

$$\frac{|f_m(x) - f_n(x)|}{1 + |f_n(x) - f_n(x)|} < \frac{\varepsilon}{1+\varepsilon},$$

i.e., $\rho(f_m, f_n) < \frac{\varepsilon}{1+\varepsilon}$. For each $n \in \mathbb{N}$, f_n has countable image. Also if $x_0 \in X$ and $[nx_0] = k$, then for

$$r = min\left[\frac{nx_0 - k}{2n}, \frac{k+1-nx_0}{2n}\right]$$

we have $x \in \mathbb{N}_r(x_0)$, i.e., $|f_n(x) - f_n(x_0)| = 0$. Therefore $\{f_n\}$ with metric ρ is a Cauchy sequence in $C_c(X)$. Since for each $x \in X$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{[nx]}{n} = x$, we infer that for each $x \in X$, $f_n \longrightarrow f$ where $f(x) = x$. But $f \notin C_c(X)$, so $C_c(X)$ with metric ρ is not complete.

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A NOTE ON $C_c(X)$ VIA A TOPOLOGICAL RING

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$C_c(X)$ به عنوان یک حلقه توپولوژی

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فرض می‌کنیم $C_c(X)$ ($C_c^*(X)$) نمایش زیرجبر شماراتابعی $C(X)$ ($C^*(X)$)، متشکل از توابع (توابع کراندار) با برد شمارا باشد. $C_c(X)$ ($C_c^*(X)$) به عنوان حلقه توپولوژی به‌وسیله m_c-توپولوژی (m_c^*-توپولوژی) و u_c-توپولوژی بررسی شده و تساوی دو توپولوژی اخیر شناسایی شده است. فضاهای توپولوژی که N-فضا نامیده می‌شوند، معرفی و مطالعه شده‌اند. نشان داده شده است که m_c-توپولوژی روی $C_c(X)$ و توپولوژی نسبی به‌عنوان زیرفضای $C(X)$ (با m-توپولوژی) منطبق هستند اگر و تنها اگر X یک N-فضا باشد. همچنین نشان می‌دهیم که X شبه‌فشرده است اگر و تنها اگر X شمارا شبه‌فشرده و N-فضا باشد.

کلمات کلیدی: زیرجبر شمارا تابعی، m_c-توپولوژی، u_c-توپولوژی، N-فضا.