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ON $(n-1,n)$-Φ_m-PRIME AND $(n-1,n)$-WEAKLY PRIME SUBMODULES

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ABSTRACT. Let $m, n \geq 2$ be two positive integers, R a commutative ring with identity and M a unitary R-module. A proper submodule P of M is an $(n-1,n)$-Φ_m-prime ($(n-1,n)$-weakly prime) submodule if $a_1, \ldots, a_{n-1} \in R$, $x \in M$ with $a_1 \ldots a_{n-1}x \in P \backslash (P : M)^{m-1}P$ ($0 \neq a_1 \ldots a_{n-1}x \in P$) implies $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x \in P$, for some $i \in \{1, \ldots, n-1\}$ or $a_1 \ldots a_{n-1} \in (P : M)$. In this paper we study these type of submodules, giving some useful results and examples concerning them.

1. Introduction

Throughout this article, all rings are assumed to be commutative with identity and all modules are unital. We remind that a proper ideal P of R is called prime if, for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.

Let P be a submodule of M. Then $(P : M) = \{r \in R | rM \subseteq P\}$ is an ideal of R. An R-module M is faithful if $Ann_R M = (0 : M) = 0$. Also, M is a multiplication R-module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It is easy to show that $N = (N : M)M$.

A proper submodule P of M is called prime if, for $r \in R$, $x \in M$ and $rx \in P$ we have $x \in N$ or $r \in (P : M)$. It is easy to show that if P is a prime submodule of M, then $(P : M)$ is a prime ideal of R.

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In [10, 12], Ebrahimpour and Nekooei show that a proper submodule P of M is $(n-1,n)$-prime if $a_1...a_{n-1}x \in P$, implies $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$, for some $i \in \{1,\dots,n-1\}$ or $a_1...a_{n-1} \in (P:M)$, where $a_1,...,a_{n-1} \in R$ and $x \in M$. So, a $(1,2)$-prime submodule is just prime.

In [1], authors defined a weakly prime ideal as a proper ideal P of R with the property that for $a,b \in R$, if $0 \neq ab \in P$ then $a \in P$ or $b \in P$. The notion of a weakly prime element (i.e., an element $p \in R$ such that (p) is a weakly prime ideal) was introduced by Galovich [14] in his study of unique factorization rings with zero divisors. It is hoped that weakly prime elements and weakly prime ideals will prove useful in the study of commutative rings with zero divisors and in particular, factorization in such rings.

In [22], Nekooei extends this concept to the case of submodules. He defines a weakly prime submodule as a proper submodule P of M with the property that for $r \in R$ and $x \in M$, if $0 \neq rx \in P$ then $x \in P$ or $r \in (P:M)$.

In [10], Ebrahimpour and Nekooei defined a $(n-1,n)$-weakly prime ideal as a proper ideal P of R with the property that for $a_1,\dots,a_n \in R$, if $0 \neq a_1\dots a_n \in P$ then $a_1\dots a_{i-1}a_{i+1}\dots a_n \in P$, for some $i \in \{1,\dots,n\}$. So, a $(1,2)$-weakly prime ideal is just weakly prime and every proper ideal of a quasi-local ring (R,M) with $M^n=0$ is $(n-1,n)$-weakly prime.

Also, in [11], Ebrahimpour and Nekooei defined a proper submodule P of M as an $(n-1,n)$-weakly prime submodule such that for $a_1,\dots,a_{n-1} \in R$ and $x \in M$, $0 \neq a_1\dots a_{n-1}x \in P$ implies $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}x \in P$, for some $i \in \{1,\dots,n-1\}$ or $a_1...a_{n-1} \in (P:M)$. So a $(1,2)$-weakly prime submodule is just weakly prime.

In studying unique factorization domains, Bhatwadekar and Sharma [8] defined an almost prime ideal as a proper ideal P of R with the property that for $a,b \in R$, if $ab \in P\backslash P^2$ then $a \in P$ or $b \in P$. Thus, a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime.

In [14], Anderson and Bataineh defined a Φ-prime ideal as follows: Let R be a commutative ring and $S(R)$ be the set of all ideals of R. Let $\Phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Then a proper ideal P of R is called Φ-prime if for $a,b \in R$, $ab \in P\backslash\Phi(P)$ implies $a \in P$ or $b \in P$. They defined $\Phi_m : S(R) \to S(R) \cup \{\emptyset\}$ with $\Phi_m(J) = J^m$, for all $J \in S(R)$ and $m \geq 2$.

In [10], Ebrahimpour and Nekooei defined an $(n-1,n)$-Φ_m-prime ideal as a proper ideal P of R with the property that for $a_1,\dots,a_n \in$

R, if $a_1 \ldots a_n \in P \backslash P^m$ then $a_1 \ldots a_{i-1} a_{i+1} \ldots a_n \in P$, for some $i \in \{1, \ldots, n\}$; $(m, n \geq 2)$.

In [24], Zamani defined ϕ-prime submodule as follows: Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. A proper submodule P of M is called ϕ-prime submodule if for $r \in R$ and $x \in M$, $rx \in P \backslash \phi(P)$ implies $r \in (P : M)$ or $x \in P$. He defined $\Phi_m : S(M) \rightarrow S(M) \cup \{\emptyset\}$ with $\Phi_m(N) = (N : M)^{m-1} N$, for all $N \in S(M)$; $(m \geq 2)$.

In [11], Ebrahimpour and Nekooei show that P is an $(n-1, n)$-ϕ_m-prime submodule of M if $a_1 \ldots a_{n-1} x \in P \backslash (P : M)^{m-1} P$ implies $a_1 \ldots a_{i-1} a_{i+1} \ldots a_{n-1} x \in P$, for some $i \in \{1, \ldots, n-1\}$ or $a_1 \ldots a_{n-1} \in (P : M)$, where $a_1, \ldots, a_{n-1} \in R$ and $x \in M$. So a $(1, 2)$-Φ_2-prime submodule is just almost prime. We shall call $(1, 2)$-Φ_m-prime submodules "Φ_m-prime". In this paper we study $(n-1, n)$-weakly prime and $(n-1, n)$-Φ_m-prime submodules, which are generalizations of weakly prime and almost prime submodules, respectively; $(n, m \geq 2)$.

Some of our results use the $R(+)M$ construction. Let R be a ring and M be an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$.

Let R be an integral domain with quotient field K and M a torsionfree R-module. Then the following conditions are equivalent, by [20].

1) For all $y \in K$ and all $x \in M, yx \in M$ or $y^{-1} M \subseteq M$.

2) For all $y \in K$, $yM \subseteq M$ or $y^{-1} M \subseteq M$. M is called *valuation R-module* if one of these conditions holds.

Let R be a ring and M an R-module. We show the set of all prime ideals of R by $Spec(R)$ and the set of all prime submodules of M by $Spec(M)$.

In this paper, we study $(n-1, n)$-weakly prime and $(n-1, n) - \phi_m$-prime submodules. Also, we prove some interesting results and give some examples about these types of submodules.

2. Main Results

It is clear that every $(n-1, n)$-prime submodule is $(n-1, n)$-weakly prime; $(n \geq 2)$. In Example 2.1 and Example 2.2, we show that the converse is not true in general.

Example 2.1. Let Q be a maximal ideal of a ring R. Then, $\overline{R} = \frac{R}{Q^2}$ is a quasi-local ring with unique maximal ideal $\overline{Q}$ and $\overline{Q}^2 = 0$. Let J be a proper ideal of $\overline{R}$. It is easy to show that J is $\overline{Q}$-primary. Suppose $S = \overline{R}[x]$ and $M = S$ as an S-module. We show that $J[x]$ is a weakly prime submodule of M. We know that $J[x]$ is $\overline{Q}[x]$-primary. Suppose $0 \neq fg \in J[x]$. Both f and g can not be in $\overline{Q}[x]$. Let $g \notin \overline{Q}[x]$. Since $J[x]$ is $\overline{Q}[x]$-primary, we have $f \in J[x]$. So, $J[x]$ is a weakly prime submodule and hence, it is an $(n-1, n)$-weakly prime submodule. Thus, $J[x]$ is an $(n-1, n)$-ϕ_m-prime submodule of M. Moreover, if $J \neq \overline{Q}$, then $J[x]$ is a weakly prime submodule that is not prime; $(n, m \geq 2)$.

Example 2.2. Let $R = \frac{K[x,y]}{(x^2y^2)}$, where K is a field, $M = \frac{K[x,y]}{(x,y)} \simeq K$ as an R-module, $D = R(+)M$ and $M' = D$ as an D-module. The submodule $N = 0(+)M$ of M' is a weakly prime submodule that is not prime.

Example 2.3. Let (R, Q) be a quasi-local ring, M an R-module and P a proper submodule of M. If $P \cap Q^{n-1}M = 0$, then P is an $(n-1, n)$-weakly prime submodule of M. For, if $0 \neq a_1 \ldots a_{n-1}x \in P$, then $a_1 \ldots a_{n-1} \notin Q^{n-1}$. So, there exists an $i \in \{1, \ldots, n-1\}$ such that a_i is a unit of R. Thus $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x \in P$. Similarly, if $P \cap Q^{n-1}M \subseteq (P : M)^{m-1}P$, then P is an $(n-1, n)$-Φ_m-prime submodule of M, where $m, n \geq 2$. For example, let $R = \frac{k[|x,y|]}{(x)(x,y)}$, where k is a field, $Q = (\overline{x}, \overline{y})$ be the unique maximal ideal of R and $M = R$ as an R-module. Let $P = (\overline{x})$. Then, $P \cap Q^{n-1}M = 0$; $(n \geq 2)$. Therefore, P is $(n-1, n)$-weakly prime and hence is $(n-1, n)$-Φ_m-prime; $(m \geq 2)$.

Next, we show that for a principal submodule Rx with $ann(x) = 0$ $(Rx \cong R)$ the concepts $(n-1, n)$-Φ_m-prime and $(n-1, n)$-prime are the same; $(m, n \geq 2)$.

Lemma 2.4. *Let R be a ring, M an R-module and x a non-zero element of M such that $Rx \neq M$ and $ann(x) = 0$. Let $m, n \geq 2$ be two integers. If Rx is not an $(n-1, n)$-prime submodule, then there exist $a_1, \ldots, a_{n-1} \in R$ and $y \in M$ such that $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}y \notin Rx$, for all $i \in \{1, \ldots, n-1\}$ and $a_1 ... a_{n-1} \notin (Rx : M)$ and $a_1 \ldots a_{n-1}y \in Rx$. But, $a_1 \ldots a_{n-1}y \notin (Rx : M)^{m-1}Rx$.*

Proof. Since Rx is not $(n-1, n)$-prime, there exist $a_1, \ldots, a_{n-1} \in R$ and $y \in M$ such that $a_1 \ldots a_{n-1}y \in Rx$ but $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}y \notin Rx$, for all $i \in \{1, \ldots, n-1\}$ and $a_1 ... a_{n-1} \notin (Rx : M)$. If $a_1 \ldots a_{n-1}y \notin (Rx : M)^{m-1}Rx$, we are done. So, we assume that $a_1 \ldots a_{n-1}y \in (Rx : M)^{m-1}Rx$.

Let $y' = y+x$. Then, $a_1 \dots a_{n-1}y' \in Rx$ and $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}y' \notin Rx$, for all $i \in \{1, \dots, n-1\}$ and $a_1 \dots a_{n-1} \notin (Rx : M)$. If $a_1 \dots a_{n-1}y' \in (Rx : M)^{m-1}Rx$, then $a_1 \dots, a_{n-1}x \in (Rx : M)^{m-1}Rx$. So, there exists $r \in (Rx : M)^{m-1}$ such that $a_1 \dots a_{n-1}x = rx$ and hence $a_1 \dots a_{n-1} = r \in (Rx : M)^{m-1} \subseteq (Rx : M)$, which is a contradiction. Therefore, $a_1 \dots a_{n-1}y' \notin (Rx : M)^{m-1}Rx$, as wanted. □

Corollary 2.5. *Let R be a ring, M an R-module and x a non-zero element of M such that $Rx \neq M$ and $ann(x) = 0$. Then, Rx is an $(n-1, n)$-Φ_m-prime submodule of M if and only if Rx is $(n-1, n)$-prime; $(n, m \geq 2)$.*

Proof. ($\Leftarrow$) If Rx is an $(n-1, n)$-prime submodule of M, then it is clear that Rx is $(n-1, n)$-Φ_m-prime.

($\Rightarrow$) If Rx is not $(n-1, n)$-prime, then there exist $a_1 \dots a_{n-1} \in R$ and $y \in M$ such that $a_1 ... a_{n-1}y \in Rx \setminus (Rx : M)^{m-1}Rx$ such that $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}y \notin Rx$, for all $i \in \{1, \dots, n-1\}$ and $a_1 ... a_{n-1} \notin (Rx : M)$, by Lemma 2.4, which is a contradiction. □

Lemma 2.6. *Let R be an integral domain, M a multiplication R-module, J and K two submodules such that $(J : M)$ and $(K : M)$ are non-zero, finitely generated ideals of R, $(J : M) \nsubseteq (K : M)$ and $(K : M) \nsubseteq (J : M)$. If $((J \cap K) : M)$ is Φ_n-prime, then the radical of J equals the radical of K; $(n \geq 2)$.*

Proof. We show that $K \subseteq radJ$. If $(K : M) \subseteq (P : M)$, for every minimal prime submodule P over J, then $(K : M) \subseteq (radJ : M)$. Since M is a multiplication module, we have $K \subseteq radJ$.

So, we assume that there exists a prime submodule P of M minimal over J such that $(K : M) \nsubseteq (P : M)$. Set $(P : M) = q$. Choose an element $r \in (K : M) \setminus q$. Clearly, for $(J : M)^{(n)} = (J : M)^n R_q \cap R$, we have $(J : M)^n R_q = (J : M)^{(n)} R_q$. Since $(J : M)$ is finitely generated, $(J : M)R_q \neq (J : M)^n R_q$. Thus, $(J : M)R_q \neq (J : M)^{(n)} R_q$, and consequently $(J : M) \nsubseteq (J : M)^{(n)}$. Since $(J : M) \nsubseteq (K : M)$ and $(J : M) \nsubseteq (J : M)^{(n)}$, we have $(J : M) \nsubseteq (K : M) \cup (J : M)^{(n)}$.

Choose an element $s \in (J : M) \setminus ((K : M) \cup (J : M)^{(n)})$. Then, $r, s \notin (J : M) \cap (K : M)$, but $rs \in (J : M) \cap (K : M) = ((J \cap K) : M)$. We claim that $rs \notin ((J \cap K) : M)^n$. Otherwise, $rs \in (J : M)^n \subseteq (J : M)^{(n)}$. However, $r \in R \setminus q$ and $rs \in (J : M)^{(n)}$ implies that $s \in (J : M)^{(n)}$, which is a contradiction. So, we have $rs \in ((J \cap K) : M) \setminus ((J \cap K) : M)^n$ and $r, s \notin (J \cap K : M)$. This contradicts the fact that $((J \cap K) : M)$ is Φ_n-prime. Thus $K \subseteq radJ$, and then $radK \subseteq radJ$. Similarly, $radJ \subseteq radK$. Therefore we have $radK = radJ$, as desired. □

Theorem 2.7. *Let R be a Noetherian domain, M a finitely generated multiplication R-module and P be a submodule of M such that $(P:M)$ is a non-zero Φ_n-prime ideal of R. Then, P is a primary submodule of M; $(n \geq 2)$.*

Proof. If P is not primary, then every minimal primary decomposition of P must have at least two components. Take a minimal primary decomposition of P and let Q be a primary component of this decomposition. If K is the intersection of all other primary components in the decomposition, then $P = Q \cap K$, where $Q \nsubseteq K$ and $K \nsubseteq Q$ and $rad(Q:M) \neq rad(K:M)$. Since $Q \nsubseteq K$ and M is a multiplication module, we have $(Q:M) \nsubseteq (K:M)$. Similarly, we get $(K:M) \nsubseteq (Q:M)$. So, by Lemma 2.6, we have $radQ = radK$. Thus, $(radQ:M) = (radK:M)$. Since M is finitely generated, we have $rad(Q:M) = rad(K:M)$, by [19, Theorem 4.4], which is a contradiction. $\square$

Let R be a ring, M a multiplication R-module and N_1, N_2 be two submodules of M. There exist ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Ameri in [2] defined the product of N_1N_2 by I_1I_2M. We use this notion in proving the following theorem.

A ring R is said to be locally UFD, if R_p is a UFD, for every $p \in Spec(R)$. It is clear that if R is a UFD, then it is locally UFD. Example 2.8 shows that the converse is not true in general.

Example 2.8. Let $R = \mathbf{Z}[\sqrt{5}i]$, where $\mathbf{Z}$ is the ring of integers. We know that R is a Dedekind domain and the element 2 does not admit a factorization into prime elements, by [15, Example 13.8]. So, R is not a UFD. But, R_p is a PID, for every $p \in Spec(R)$.

Theorem 2.9. *Let R be a domain and M a finitely generated faithful multiplication R-module. If every proper submodule of M is a product of Φ_n-prime submodules, then R is locally UFD; $(n \geq 2)$.*

Proof. Let $p \in Spec(R)$, I_p be a proper principal ideal of R_p and $N = IM$. If $IM = N = M = RM$, then by [13, Thoorem 3.1], we have $I = R$, which is a contradiction. So, N is a proper submodule of M. Suppose that $N = N_1 ... N_k$, where N_i is a ϕ_n-prime submodule of M, for all $i \in \{1, \ldots, k\}$. Then $N = I_1 ... I_k M$, where $N_i = I_iM$, for $i \in \{1, ..., k\}$. Thus, $I = I_1 ... I_k$, by [13, Theorem 3.1]. Since N_i is ϕ_n-prime, then I_i is ϕ_n-prime, by [11, Lemma 4.3(i)].

In domain R_p, every nonzero principal ideal is invertible. We know that a factor of an invertible ideal is also invertible. Since R_p is quasi-local, an invertible ideal is principal. So, I_p is a product of principal

Φ_n-prime ideals and hence I_p is a product of principal prime ideals, by Corollary 2.5. Therefore, R_p is a UFD and R is locally UFD. $\square$

In Example 2.10, we show that a finitely generated faithful multiplication R-module M is not cyclic in general.

Example 2.10. Let R be a Dedekind domain. We know that every ideal of R is a multiplication R-module, by [16, Page 223]. Also, every ideal of a Dedekind domain is generated by at most two elements, by [23, Corollary 2, Page 125]. So, every ideal of R is a finitely generated faithful multiplication R-module.

Let R be an integral domain with quotient field K and M a torsion-free R-module. In [20], Moghaderi and Nekooei proved that for $y = \frac{r}{s} \in K$ and $x \in M$, $yx \in M$ if there exists $m \in M$ such that $rx = sm$. They also proved that M is a valuation R-module if M satisfies one of the following equivalent conditions:

(1) For every $y \in K$ and every $x \in M$, $yx \in M$ or $y^{-1}M \subseteq M$.

(2) For every $y \in K$, $yM \subseteq M$ or $y^{-1}M \subseteq M$.

Next, we show that in a multiplicative valuation module, the concepts $(n-1,n)$-Φ_n-prime and $(n-1,n)$-prime are the same, for some submodules; $(n \geq 2)$.

Lemma 2.11. *Let R be a ring, M a multiplicative valuation R-module and N_1, N_2 be two submodules of M. Then, $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.*

Proof. Let N_1, N_2 be two submodules of M. Since M is a multiplication R-module, then there exist the ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. We know that R is a valuation ring, by [20, Lemma 2.11]. So, $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$, and therefore $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$, as desired. $\square$

Theorem 2.12. *Let R be a ring, M a multiplicative valuation R-module and P a submodule of M with $ann(x) = 0$, for some $0 \neq x \in P$. Then, P is $(n-1,n)$-Φ_n-prime if and only if it is $(n-1,n)$-prime; $(n \geq 2)$.*

Proof. $(\Rightarrow)$ Let $a_1, \ldots, a_{n-1} \in R$ and $x \in M$ with $a_1 \ldots a_{n-1}x \in P$. Assume $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x \notin P$, for all $i \in \{1, \ldots, n-1\}$ and $a_1...a_{n-1} \notin (P : M)$. So $(a_i) \nsubseteq (P : M)$, for all $i \in \{1, \ldots, n-1\}$. We know that R is a valuation ring, by [20, Lemma 2.11]. So we have $(P : M) \subseteq (a_i)$, for all $i \in \{1, \ldots, n-1\}$. Hence $(P : M)^{n-1}P \subseteq (a_1 \ldots a_{n-1}x)$. If $(P : M)^{n-1}P \neq (a_1...a_{n-1}x)$, then $a_1...a_{n-1}x \in P\backslash(P : M)^{n-1}P$. Since P is $(n-1,n)$-Φ_n-prime, this implies that $a_1 \ldots a_{i-1}a_{i+1} \ldots a_{n-1}x \in P$, for some $i \in \{1, \ldots, n-1\}$ or $a_1...a_{n-1} \in$

$(P : M)$, which are contradictions. So we have $(a_1 \ldots a_{n-1}x) = (P : M)^{n-1}P$. Then P being a factor of a principal submodule is principal, by Lemma 2.11. So there exists $y \in M$ such that $P = Ry$. Hence $x = ry$, for some $r \in R$. So we have $ann(y) = 0$. Thus, by Corollary 2.5, P is $(n-1, n)$-prime.

($\Leftarrow$) This holds for any module. $\square$

Let R be a ring, M an R-module and N a submodule of M. We know that $radN = \cap_{N \subseteq P \in Spec(M)} P$. If there exists no prime submodule over N, then $radN = M$. Moreover, N is said to be a radical submodule if $radN = N$.

Prüfer modules has been defined by Naoum and Al-Alwan in [21, page 407]. The next example shows that in a Prüfer Module M, the above result is not necessarily true.

Example 2.13. Let R be the ring of all algebraic integers and $M = R$ as an R-module. Then every radical submodule of M is idempotent. So, let $N_1 \neq N_2$ be two maximal submodules of M. Hence, $N_1N_2 = N_1 \cap N_2$ and $(N_1N_2)^2 = N_1N_2$. So, N_1N_2 is $(n-1, n)$-Φ_n-prime, but not a prime submodule; $(n \geq 2)$.

Let R be a ring, M an R-module and $D = R(+)M$. We observe that for an ideal I of R and a positive integer $n \geq 1$, we have $(I(+)M)^n = I^n(+)I^{n-1}M$.

Theorem 2.14. *Let R be a ring, M a finitely generated faithful multiplication R-module and P a ϕ_n-prime submodule of M. If for $a, b \in R$ and $(x, y) \in M$ with $(a, x)(b, y) \in (P : M)^n(+)M$ and $a, b \notin (P : M)$; $ay + bx \in (P : M)^{n-1}M$, then $(P : M)(+)M$ is a Φ_n-prime ideal of $D = R(+)M$; $(n \geq 2)$.*

Proof. Set $q = (P : M)$. Suppose that $(a, x)(b, y) \in q(+)M \backslash (q(+)M)^n$. So $(a, x)(b, y) \in q(+)M \setminus (q^n(+)q^{n-1}M)$ and hence $ab \in q$. If $ab \notin q^n$, then $ab \in q \setminus q^n$. Since $P = qM$ is ϕ_n-prime, then q is ϕ_n-prime, by [11, Lemma 4.3]. So, $a \in q$ or $b \in q$. Hence, $(a, x) \in q(+)M$ or $(b, y) \in q(+)M$. Now assume that $ab \in q^n$. If $a, b \notin q$, then $ay + bx \in q^{n-1}M$, by hypothesis. So, $(a, x)(b, y) \in q^n(+)q^{n-1}M = (q(+)M)^n$, a contradiction. $\square$

Now, we study the converse of Theorem 2.14, for $n = 2$.

Theorem 2.15. *Let R be a ring, M a finitely generated faithful multiplication R-module and P a submodule of M. If $(P : M)(+)M$ is an almost prime ideal of $D = R(+)M$ and there exists a $Q \in Max(R)$ such that $(P : M) \subseteq Q$, $(P : M) \cap Q^2 = 0$ and $\bigcap_{n \geq 1} Q^n = 0$, then P is an almost prime submodule of M.*

Proof. Set $q = (P : M)$. Suppose that $q(+)M$ is almost prime. Let $ab \in q \setminus q^2$, where $a, b \in R$. Then $(a,0)(b,0) \in (q(+)M) \setminus (q^2(+)qM)$, and hence $(a,0)(b,0) \in (q(+)M) \setminus (q(+)M)^2$. So, $a \in q$ or $b \in q$. Thus q is almost prime and $P = qM$ is an almost prime submodule of M, by [11, Lemma 4.3]. □

Theorem 2.16. *Let R be a ring and M a finitely generated faithful multiplication R-module. Then every prime ideal of $D = R(+)M$ is of the form $(P : M)(+)M$, for some prime submodule P of M.*

Proof. We know from [17, Theorem 25.1 (3)] that every prime ideal of $D = R(+)M$ is of the form $q(+)M$, for some prime ideal q of R. Let P be a prime submodule of M. So, $(P : M)$ is a prime ideal of R, by [18, Proposition 1]. Thus, $(P : M)(+)M$ is a prime ideal of D.

Now, assume that $q(+)M$ is a prime ideal of D. Set $P = qM$. If $P = qM = M = RM$, then $q = R$, by [13, Theorem 3.1], a contradiction. So, $P \in Spec(M)$, by [13, Lemma 2.10], and therefore $(P : M) = q$, by [13, Theorem 3.1]. □

Unlike the case of prime ideals, an $(n-1,n)$-weakly prime or $(n-1,n)$-Φ_m-prime ideal of $D = R(+)M$ need not have the form $I(+)M$. For example, $0(+)0$ is $(n-1,n)$-weakly prime and as a result $(n-1,n)$-ϕ_m-prime; $(n, m \geq 2)$.

Let R be a ring and M an R-module. Two elements $x, y \in M$ are associates, denoted $x \sim y$, if $Rx = Ry$. A non-zero element $x \in M$ with $Rx \neq M$ is n-irreducible if $x = a_1 \dots a_{n-1}y$, where $a_1, ..., a_{n-1} \in R$ and $y \in M$, implies $x \sim a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}y$, for some $i \in \{1, \dots, n-1\}$ or $a_1 ... a_{n-1} \in (Rx : M)$; $(n \geq 2)$.

Theorem 2.17. *Let R be a ring, M an R-module and P a proper submodule of M. Suppose that every non-zero element of P is n-irreducible. Then P is $(n-1,n)$-weakly prime and hence $(n-1,n)$-Φ_m-prime; $(n, m \geq 2)$.*

Proof. Let $a_1 \dots a_{n-1}x \in P \setminus \{0\}$, where $a_1, \dots, a_{n-1} \in R$ and $x \in M$. So, $a_1 \dots a_{n-1}x$ is n-irreducible and hence

$$(a_1 \dots a_{n-1}x) = (a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x),$$

for some $i \in \{1, \dots, n-1\}$. Therefore $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P$ or $a_1 ... a_{n-1} \in ((a_1 ... a_{n-1}x) : M)$. But $((a_1 ... a_{n-1}x) : M) \subseteq (P : M)$. So, P is $(n-1,n)$-weakly prime. □

Corollary 2.18. *Let (R, Q) be a quasi-local ring, M an R-module and $x \in M$. If x is n-irreducible and $Qx = 0$, then Rx is $(n-1,n)$-weakly prime; $(n \geq 2)$.*

Proof. Since x is n-irreducible and $Qx = 0$, every non-zero element of Rx is an associate of x and hence n-irreducible. So Rx is $(n-1, n)$-weakly prime, by Theorem 2.17. □

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ON $(n-1,n)$-Φ_m-PRIME AND $(n-1,n)$-WEAKLY PRIME SUBMODULES

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زیرمدول‌های $(n-1,n)$-Φ_m- اول و $(n-1,n)$- اول ضعیف

مهدیه ابراهیم پور [1] و فاطمه میرزائی [2]
دانشگاه ولیعصر رفسنجان [1] و دانشگاه شهید باهنر کرمان [2]

فرض کنید $m,n \geq 2$ دو عدد صحیح و مثبت، و R حلقه ای جابجایی و یکدار و M یک R-مدول یکانی باشد. زیرمدول سره P از M یک زیرمدول $(n-1,n)$-Φ_m- اول ($(n-1,n)$-اول ضعیف) است اگر از $a_1,\ldots,a_{n-1} \in R$ و $x \in M$ که $a_1\ldots a_{n-1}x \in P\backslash(P:M)^{m-1}P$ ($0 \neq a_1\ldots a_{n-1}x \in P$) بتوان نتیجه گرفت $a_1\ldots a_{i-1}a_{i+1}\ldots a_{n-1}x \in P$، که در آن $i \in \{1,\ldots,n-1\}$، یا $a_1 \ldots a_{n-1} \in (P:M)$. در این مقاله به مطالعه این زیرمدول‌ها می‌پردازیم. برخی نتایج مفید و مثال‌هایی از این نوع زیرمدول‌ها ارائه می‌شود.

کلمات کلیدی: زیر مدول‌های اول ضعیف، زیر مدول‌های $(n-1,n)$-اول ضعیف، زیرمدول‌های $(n-1,n)$-Φ_m- اول.