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SEQUENTIALLY COMPACT S-ACTS

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Abstract. The investigation of equational compactness was initiated by Banaschewski and Nelson. They proved that pure injectivity is equivalent to equational compactness. Here, we define the so-called sequentially compact acts over semigroups and study some of their categorical and homological properties. Some Baer conditions for injectivity are also presented.

1. Introduction and Preliminaries

An algebra A is called *equationally compact* if every system of polynomial equations (with constants from A) has a solution in A provided that every finite subsystem of it has a solution in A.

The notion of equational compactness of universal algebras has been studied by Banaschewski and Nelson [3]. They show that equational compactness is equivalent to pure injectivity, injectivity with respect to all pure monomorphisms. Moreover, Banaschewski [2] deals with this notion in the special case of G-sets for a group G. There are some weaker notions of equational compactness, such as, 1-equational compactness or f-equational compactness, where systems of equations contain, respectively, one variable or finite number of variables. Normak [13] shows that for a monoid S, all S-acts are f-equationally compact if and only if all S-acts are 1-equationally compact.

Giuli [10] has introduced the category of *Projection Algebras* A over a monoid S, by a system of equations $xs = a_s (s \in S, a_s \in A)$ as an algebraic version of ultrametric spaces. Computer scientists use this

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notion as a convenient means for algebraic specification of process algebras. In this category, Ebrahimi and Mahmoudi [12] introduced the concept of sequential compactness and showed that sequential compactness coincides with injectivity and equational compactness.

Throughout this paper, S stands for a given semigroup and $\mathbf{Act}-\mathbf{S}$ will denote the category of all (right) S-acts, including $\emptyset$, and homomorphisms between them.

Based on the above mentioned points, we are persuaded to develop the notion of sequential compactness for a given semigroups S in the category $\mathbf{Act}-\mathbf{S}$.

In Section 3, we generalized the notion of sequential compactness for S-acts and study some basic and homological properties of it. Section 4 deals with the categorical properties of sequential compactness of S-acts, and Section 5 is devoted to the study of some Baer type Criteria for this notion.

Here we briefly recall the definition and the categorical and algebraic ingredients of the category **Act-S** of (right) S-acts over a semigroup S. For more information and the notions not mentioned here about this category, one may consult [11].

Recall that, for a semigroup S, a set A is a right S-*act* (or an S-*set*) if there is an S-*action* $\mu : A \times S \to A$, denoting $\mu(a,s) := as$, such that $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$.

A morphism $f : A \to B$ between S-acts A, B is called a *homomorphism* if, for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

A subset A of an S-act B is called a subact of B and in this case, B is said to be an extension of A, if for each $a \in A$ and $s \in S$, $as \in A$. Also, an element $a \in A$ ($t \in S$) is said to be a *fixed element* (*left zero element*) if $as = a$ ($ts = t$) for all $s \in S$. The set of all fixed elements of an S-act A and the set of all left zero elements of a semigroup S are denoted, respectively, by $Fix(A)$ and $Z(S)$. The semigroup S is called a left zero semigroup if all of its elements are left zeros. The S-act $A \cup \{0\}$ with a fixed element 0 adjoined to A is denoted by A^0. Notice that each semigroup S can be considered as an S-act with the action given by its multiplication, and adjoining an external left identity 1 to a semigroup S gives us an S-act $S^1 := S \cup \{1\}$. Also, here the empty set is allowed to be an S-act.

A subset I of S is a right ideal if for each $r \in I$, $s \in S$, $rs \in I$. A right ideal I of S is finitely generated if I has a finite subset $\{t_1, t_2, ..., t_n\}$ such that $I = \bigcup_{i=1}^{n} t_i S^1$.

For a family $\{A_i\}$ of S-acts, their cartesian product $\prod_{i \in I} A_i$ with the S-action defined by $(a_i)s = (a_i s)$ is the product of the family $\{A_i\}$ in

Act-S. The coproduct of a family $\{A_i\}$ in **Act-S** is their disjoint union $\coprod_{i\in I} A_i = \cup_{i\in I}(A_i \times \{i\})$ with the action of S defined by $(a,i)s = (as,i)$ for $s \in S,\ a \in A_i$. Recall that for a family $\{A_i : i \in I\}$ of S-acts each with a unique fixed element 0, the *direct sum* $\bigoplus_{i\in I} A_i$ is defined to be the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ except for a finite number of indices.

2. SEQUENTIAL COMPACTNESS

In this section, by means of the notion of the *sequential system of equations* on an S-act A we introduce the concepet of so called *sequentially compact* or briefly *s-compact* S-acts. Then, we study some basic properties of s-compactness.

Definition 2.1. [6] (1) Any set $\Sigma = \{xs = a_s : s \in S, a_s \in A\}$ is called a *sequential system of equations* with constants from an S-act A, or over A.

(2) We say that Σ is *solvable* in an extension B of A if there is some $b \in B$ such that for all $s \in S$, $bs = a_s$. The system Σ is said to be *consistent* if it has a solution in some extension B of A.

Note that there is a one to one correspondence between the set of all systems of equations Σ_A of the above form on an S-act A and the set of all maps $k : S \to A$. For any S-act B and $b \in B$, let us denote the left translation mapping $\lambda_b : S \to B$, defined by $\lambda_b(s) := bs$, by λ_b. In these notations, we have

Lemma 2.2. [6] *Let A be an S-act. A map $k : S \to A$ is a homomorphism if and only if there exists b in an extension B of A such that $k = \lambda_b$.*

Definition 2.3. An S-act A is said to be *sequentially compact* or simply, *s-compact* if every system of sequential equations $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution in A whenever this is the case for each finite subsystem of Σ.

In the following, by an easy observation, we obtain an equivalent condition to s-compactness which will be used in the proof of some results.

Lemma 2.4. *An S-act A is s-compact if and only if every map $f : S \to A$ is of the form λ_a for some $a \in A$ whenever for every finite subset T of S there is an element $a_T \in A$ such that $f\mid_T = \lambda_{a_T}$.*

In what follows, we are going to find some characterization for s-compactness based on purity, injectivity and retractness. To proceed, we lised some preliminaries.

Lemma 2.5. (i) *Let A be an S-act. If $f : S \to A$ is a map whose restriction to each finite subset T of S is of the form λ_{a_T} for some $a_T \in A$, then f is a homomorphism.*

(ii) *Let Σ be a system of sequential equations on A such that each of its finite subsets has a solution in A. Then Σ has a solution in some extension B of A.*

Proof. (i) Let $s, t \in S$. Considering $T = \{t, ts\}$, there is an element $a_T \in A$ such that $f \mid_T = \lambda_{a_T}$. Thus $f(t)s = (a_T t)s = a_T(ts) = f(ts)$ and hence f is a homomorphism.

(ii) Let $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ satisfying the property given in (ii). Consider the map $f : S \to A$ by $f(s) = a_s$. It is easy to check that the restriction of f to every finite subset T of S is of the form λ_{a_T} for some $a_T \in A$. So by part (i), f is a homomorphism. Let $E(A)$ be the injective hull of A, which exists by [8]. Thus $f : S \to A \hookrightarrow E(A)$ is extended to $\bar{f} : S^1 \to E(A)$, whence $\bar{f}(1)$ is a solution of Σ. □

Definition 2.6. (i) An S-act A is said to be *f-pure* in an S-act B if every $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution in A whenever this is the case for B and each finite subset of Σ. An S-act A is called *absolutely f-pure* if it is f-pure in each of its extensions.

(ii)[9] An S-act A is called an *s-dense* subact of B or an S-act B is called an s-dense extension of A, if for every $b \in B$, $bS \subseteq A$.

(iii)[5] An S-act A is called a *strongly-s-dense* subact of B if for every $b \in B$, $bS \subseteq A$ and for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt$ for all $t \in T$. In this case, an S-act B is said to be a strongly-s-dense extension of A.

Lemma 2.7. [5] *In* **Act-S**, *pushouts transfer strongly-s-dense monomorphisms. That is, for the following pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

we have h is strongly-s-dense if so is f.

An S-act A is said to be *strongly-s-dense injective* if for any strongly-s-dense monomorphism $g : B \to C$, any homomorphism $f : B \to A$ can be lifted to a homomorphism $\bar{f} : C \to A$. That is, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f\downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

An S-act A is said to be a *strongly-s-dense retract* of an its strongly-s-dense extension B, if there exists a morphism $f : B \to A$ such that $f\mid_A = id_A$. In this case, f is called a *strongly-s-dense retraction*. An S-act A is called an *absolute strongly-s-dense retract* if it is an strongly-s-dense retract of each of its strongly-s-dense extensions.

The following theorem shows that the s-compactness is equivalent to a kind of injectivity.

Theorem 2.8. *For an S-act A, the following assertions are equivalent*:

(i) *A is an s-compact S-act.*

(ii) *A is absolute strongly-s-dense retract.*

(iii) *A is strongly-s-dense retract of $A \cup \{b\}$, for every strongly-s-dense extension B of A and $b \in B$.*

(iv) *A is strongly-s-dense injective.*

(v) *A is absolutely f-pure.*

Proof. (i)⇒(ii) Let A be a strongly-s-dense subact of B and $b \in B$. Then $bS \subseteq A$ and for every finite subset T of S there is an element $a_{_T} \in A$ such that $a_{_T}t = bt$ for all $t \in T$. Since A is s-compact, by Lemma 2.4, there exists $a_b \in A$ such that $\lambda_b = \lambda_{a_b}$. Now, for every $b \in B \setminus A$ choose and fix such an $a_b \in A$. Define $\pi : B \to A$ by

$$\pi(b) = \begin{cases} b, & \text{if } b \in A \\ a_b, & \text{if } b \notin A \end{cases}$$

Which shows that π is a retraction.

(ii)⇒(iii) This is trivial.

(iii)⇒(i) Let every finite subset of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A. Then, by Lemma 2.5, there is an extension B of A and $b \in B$ which is a solution for Σ. Therefore, A is strongly-s-dense in $A \cup \{b\}$ which implies that there is a retraction $g : A \cup \{b\} \to A$. So for every $s \in S$, $g(b)s = g(bs) = g(a_s) = a_s$, which means that Σ has a solution $g(b)$ in A.

The equivalence of (ii) and (iv) flows from [7, Lemma 3.5(i)] and Lemma 2.7.

(i)⇒(v) Let B be an extension of A and $b \in B$ be a solution for a system $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ and also let Σ have a solution $a_{_T}$ for each finite subset T of S. Since A is s-compact, Σ has a solution in A. So A is f-pure in B.

(v)⇒(i) Let every finite subset Σ' of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A. By Lemma 2.5, there is an extension B of A such that Σ has a solution $b \in B$. So, for every finite subset T of S there is an element $a_{_T} \in A$ such that $a_{_T}t = bt(t \in T)$ and $bS \subseteq A$. Thus Σ has a solution in A and hence it is an s-compact S-act. □

Theorem 2.9. *A subact A of an s-compact S-act B is s-compact if and only if A is f-pure in B.*

Proof. Let B be an s-compact f-pure extension of A and each finite subsystem of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A. Since B is s-compact, Σ has a solution $b \in B$ and since B is an f-pure extension of A, Σ has a solution in A.

Conversely, by Theorem 2.8, an s-compact S-act A is f-pure in B. $\square$

Corollary 2.10. *Every retract of an s-compact S-act is s-compact.*

Proof. Using Theorem 2.9, suffices to show that every retract of an S-act B is an f-pure subact of it B. Let A be a retract of B and $g : B \to A$, a homomorphism such that $g\mid_A = id_A$. Suppose that $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution b in B and each finite subset of Σ has a solution in A. So for every $s \in S$, $bs = g(bs) = g(b)s$. Then Σ has a solution $g(b)$ in A. $\square$

In the following, we give some classification results for semigroups based on s-compactness.

Lemma 2.11. *If S is finitely generated as an S-act, then every S-act is s-compact.*

Proof. Using Lemmas 2.4 and 2.5, the proof is clear. $\square$

The following definition motivated from the equivalency of (i) and (iv) in Theorem 2.8.

Definition 2.12. An S-act A is called *quasi s-compact* if every homomorphism $f : C \to A$ from a strongly-s-dense subact C of A can be extended to A.

Theorem 2.13. *Let S be a countably generated semigroup. Then the following assertions are equivalent*:

(i) *All S-acts are s-compact.*

(ii) *All direct sums of s-compact S-acts are s-compact.*

(iii) *All countable direct sums of injective S-acts are s-compact.*

(iv) *S is finitely generated as an S-act.*

(v) *Each countable direct sum of each family of quasi s-compact S-acts is s-compact.*

Proof. The implications (i)⇒(ii), (ii)⇒(iii) and (i)⇒(v) are obvious. The assertion (iv)⇒(i) is obtained from Lemma 2.11. Let us show (iii)⇒(iv). Suppose that $S = \bigcup_{n\in\mathbb{N}} t_n S^1$. Take $I_n = t_1S^1 \cup \cdots \cup t_nS^1$, for $n \in \mathbb{N}$. Then we have an infinite chain of right ideals

$I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$, and $S = \bigcup_{n\in\mathbb{N}} I_n$. Consider the Rees factor acts S^1/I_n for $n \in \mathbb{N}$, and let $E_n(S^1/I_n)$ be the injective hull of S^1/I_n. Then $E = \bigoplus_{n\in\mathbb{N}} E_n(S^1/I_n)$ is s-compact by the hypothesis. Denote $a_n = \{t_n/I_i\}_{i\in\mathbb{N}} = (t_n/I_1, t_n/I_2, \cdots, t_n/I_{n-1}, \theta, \theta, \theta, \cdots) = (1/I_1, 1/I_2, \cdots, 1/I_{n-1}, \theta, \theta, \theta, \cdots)t_n = a'_n t_n$. The system $\Sigma = \{xt_i = a_i \mid i = 1, 2, \cdots\}$ is finitely solvable in E. For this, let $\Sigma' = \{xt_i = a_i | i = 1, 2, \cdots, k\}$ be a finite subset of Σ. For each $1 \leq i \leq k$, $a'_k t_i = (1/I_1, 1/I_2, \cdots, 1/I_{k-1}, \theta, \theta, \cdots)t_i = (t_i/I_1, t_i/I_2, \cdots, t_i/I_{i-1}, \theta, \theta, \theta, \cdots) = a_i$. So Σ has a solution $a = (x_1, x_2, \cdots, x_N, \theta, \theta, \cdots) \in E$ for some $n \in \mathbb{N}$. Therefore, as every element in a direct sum has only finitely many non-zero coordinates, we have for every $i > N+1, t_i \in I_{N+1}$. So, $S \subseteq I_{N+1}$, which gives $S = I_{N+1}$ is finitely generated as an S-act.

(v)⇒(iii) follows from the fact that each injective S-act is quasi s-compact. □

Corollary 2.14. *For a semigroup S, if all S-acts are s-compact, then each countably generated right ideal of S as an S-act is finitely generated.*

By the following definition we have another homological classification of those S for which all S-acts are s-compact.

Definition 2.15. A semigroup S is said to has *local left identities* if for every finite subset T of S there is an element $y_T \in S$ such that for each $t \in T, y_T t = t$.

Example 2.16. Note that all monoids and all lattices as a semigroup (by the action given by $xy = x \wedge y$), in particular chains, have local left identities.

Also if each nonempty right ideal of S is generated by an idempotent element, then S has local left identities.

Theorem 2.17. *For a semigroup S, the following are equivalent*:

(i) *Every S-act A is s-compact and S has local left identities.*

(ii) *S is s-compact as an S-act and has local left identities.*

(iii) *S has a left identity.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii) Since S has local left identities, the restriction of the identity map $id : S \to S$ on any finite subset T is of the form λ_{a_T}. So, by Lemma 2.4, $id = \lambda_{s_0}$ for some $s_0 \in S$, which implies that s_0 is a left identity element of S.

(iii)⇒(i) Apply Lemma 2.11. □

Corollary 2.18. *If S has local left identities, then the following are equivalent*:

(i) *S is an s-compact S-act.*
(ii) *Every S-act A is s-compact.*
(iii) *S has a left identity.*

Proposition 2.19. *If S has local left identities, then every S-act A has an s-compact strongly-s-dense extension.*

Proof. Let B be an s-dense extension of A and T be a finite subset of S. Since S has local left identities, there is an element $y_T \in S$ such that $y_{_T}t = t(t \in T)$. So for each $b \in B, bt = (by_T)t$ and $by_T \in A$. Then each s-dense extension is strongly-s-dense extension. Also it is clear that $S^2 = S$. hence, by [9, Theorem 3.10], each S-act has an s-dense injective s-dense extension. Since every s-dense injective is s-compact, the assertion holds. □

3. Categorical properties

In this section we consider the behaviour of s-compactness of S-acts with respect to products, coproducts, and direct sums. Lemma 2.4 will play a fundamental role to prove all the results of this section.

Theorem 3.1. *The product $\prod_{i\in I} A_i$ of S-acts is s-compact if and only if each A_i is s-compact.*

Proof. Necessity. Let $A = \prod_{i\in I} A_i$ be an s-compact S-act and for some $k \in I$, $f : S \to A_k$ be a map whose restriction to every finite subset T of S is of the form $\lambda_a(a \in A_k)$. Consider the map $\bar{f} : S \to \prod_{i\in I} A_i$ defined by

$$\bar{f}(s)(i) = \begin{cases} f(s), & \text{if } i = k \\ a_i s, & \text{if } i \neq k \end{cases}$$

where for any $i \neq k$, $a_i \in A_i$ is chosen and fixed by the axiom of choice. Let T be a finite subset of S. For every $t \in T$,

$$\bar{f}(t)(i) = \begin{cases} f(t) = a_{_T}t, & \text{if } i = k \\ a_i t, & \text{if } i \neq k \end{cases}$$

which means that $\bar{f}(t) = \lambda_{\{a_i\}}(t)$ $(a_k = a_{_T})$. Since $\prod_{i\in I} A_i$ is s-compact, $\bar{f} = \lambda_{\{a_i\}}$ and thus $f = \lambda_{a_k}$.

Sufficiency. Let $f : S \to \prod_{i\in I} A_i$ be a map such that for every finite subset T of S, $f|_{_T}$ is of the form $\lambda_{a_{_T}}(a_{_T} \in \prod_{i\in I} A_i)$. So for every projection map $\pi_i : \prod_{i\in I} A_i \to A_i$, $\pi_i \circ f|_{_T}$ is of the form $\lambda_{a_{_T}}(a_{_T} \in A_i)$. Then $\pi_i \circ f = \lambda_{a_i}$ for some $a_i \in A_i$ and hence $f = \lambda_{\{a_i\}}$. □

Theorem 3.2. *The coproduct $\coprod_{i\in I} A_i$ of S-acts is an s-compact S-act if and only if so is each A_i, $i \in I$.*

Proof. Necessity. Suppose that $f : S \to A_j$ is a map such that for every finite subset T of S there is an element $a_T \in A_j$ such that $f(t) = a_T t$ for each $t \in T$. For the j-th injection map $\tau_j : A_j \to \coprod_{i\in I} A_i$, $\tau_j f$ has the same property as f. So it is of the form λ_a for some $a \in \coprod_{i\in I} A_i$, by s-compactness of $\coprod_{i\in I} A_i$. Now we get $a \in A_i$, and $f = \lambda_a$.

Sufficiency. Let $\{A_i \mid i \in I\}$ be a family of s-compact S-acts and $f : S \to \coprod_{i\in I} A_i$ be a map such that its restriction to each finite subset T of S is of the form λ_{a_T}. Consider $s_1, s_2 \in S$ such that $f(s_1) \in A_m$ and $f(s_2) \in A_n (m \neq n)$. It is easily seen that, for a set $T = \{s_1, s_2\}$, $f|_T$ is not of the form λ_{a_T}. So there exists $k \in I$ such that $f(S) \subseteq A_k$ and since A_k is s-compact, $f = \lambda_a (a \in A_k)$. Therefore, $\coprod_{i\in I} A_i$ is an s-compact S-act. □

For an S-act A and $a \in Fix(A)$, the one element S-act $\{a\}$ is s-compact. So the following corollary is straightforward.

Corollary 3.3. *For each S-act A, $Fix(A)$ is s-compact.*

In Theorems 3.1 and 3.2, there are equivalent conditions for product and coproduct of S-acts. But this situation does not hold for direct sums. In the following, we have investigated the behavior of direct sum about the s-compactness of a family of S-acts.

Theorem 3.4. *For a family $\{A_i \mid i \in I\}$ of S-acts each with a unique fixed element, if $\bigoplus_{i\in I} A_i$ is s-compact, then each $A_j (j \in I)$ is s-compact.*

Proof. Let $f : S \to A_j$ be a map with the property as in Lemma 2.4. Then for the injection map $\sigma_j : A_j \to \bigoplus_{i\in I} A_i$, the map $\sigma_j f : S \to \bigoplus_{i\in I} A_i$ has the same property as f. So $\sigma_j f = \lambda_a$ for some $a \in \bigoplus_{i\in I} A_i$ and thus $f = \lambda_{a_j}$. □

Proposition 3.5. *The following assertions are equivalent:*

(i) *Each direct sum of s-compact S-acts is s-compact.*

(ii) *Each direct sum of s-compact S-acts is f-pure in the direct product of them.*

Proof. (i)⇒(ii) Apply Theorem 2.8.

(ii)⇒(i) Applying Lemma 2.9 and Theorem 3.1, we get the result. □

Proposition 3.6. *Let $\{A_i : i \in I\}$ be a family of s-compact S-acts each with at least one fixed element. The direct sum $\bigoplus_{i\in I} A_i$ is s-compact if and only if every map $S \to \bigoplus_{i\in I} A_i$ which its restrictions to all finite* *subsets are of the form λ_x, factors through a direct sum of finitely many A_i.*

Proof. Let $\bigoplus_{i\in I} A_i$ be s-compact. Take a homomorphism $f : S \to \bigoplus_{i\in I} A_i$ with such property. By Lemma 2.4, there exists $a = (a_i)_{i\in I} \in \bigoplus_{i\in I} A_i$ such that $f = \lambda_a$. Consider a finite subset $J \subseteq I$ such that $a_i = 0$ for all $i \notin J$. Then, since $f(s) = as$ for each $s \in S$, f factors through the homomorphism $f : S \to \bigoplus_{i\in J} A_i \to \bigoplus_{i\in I} A_i$. The converse also holds by applying Lemma 2.4 and Theorem 3.1. □

The following result generalizes Theorem 3.4, in the case of quasi s-compactness.

Theorem 3.7. *If a direct sum $\bigoplus_{i\in I} A_i$ of S-acts A_i is quasi s-compact, then so is each $A_i, i \in I$.*

Proof. Let $k \in I$, $h : B \hookrightarrow A_k$ be a strongly-s-dense monomorphism and $f : B \to A_k$ be a homomorphism. Then $h \bigoplus (\bigoplus_{i\neq k} id_{A_i}) : B \bigoplus (\bigoplus_{i\neq k} A_i) \to \bigoplus_{i\in I} A_i$ is a strongly-s-dense monomorphism, and $f \bigoplus (\bigoplus_{i\neq k} id_{A_i}) : B \bigoplus (\bigoplus_{i\neq k} A_i) \to \bigoplus_{i\in I} A_i$ is a homomorphism. So, by the assumption, there exists a homomorphism $g : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} A_i$ extending $f \bigoplus (\bigoplus_{i\neq k} id_{A_i})$. Now, the homomorphism $p_k g \tau_k : A_k \to A_k$ extends f which $\tau_k : A_k \to \bigoplus_{i\in I} A_i$, is the k-th injection map and $p_k : \bigoplus_{i\in I} A_i \to A_k$ is the k-th projection map. □

The converse of Theorem 3.4 does not hold in general, but it holds for finitely s-compact S-acts as follows.

Definition 3.8. An S-act A is said to be *finitely s-compact* if it is strongly-s-dense injective with respect to finitely generated strongly-s-dense subacts.

Theorem 3.9. *For a family $\{A_i \mid i \in I\}$ of S-acts each with a unique fixed element, $\bigoplus_{i\in I} A_i$ is finitely s-compact if and only if each $A_j (j \in I)$ is finitely s-compact.*

Proof. *Sufficiency*. Using the same argument as in the proof of Theorem 3.4, the assertion holds.

Necessity. It is not difficult to show that each homomorphism $F \to \bigoplus_{i\in I} A_i$ from a finitely generated S-act F, factors through a direct sum of finitely many A_i. Now since the direct sum of a finitely many S-acts is the direct product of them, by using [1, Proposition 10.40], the proof is complete. □

Proposition 3.10. *For a semigroup S, all S-acts are finitely s-compact if and only if all finitely generated S-acts are s-compact.*

Proof. ($\Rightarrow$) Let A be a finitely generated S-act. By hypothesis, A is finitely s-compact. Since A is a finitely generated S-act, it is easy to check that A is an absolute strongly-s-dense retract. Now, by Theorem 2.8, A is an s-compact S-act.

($\Leftarrow$) Let A be an S-act and $h : F \hookrightarrow B$ be a strongly-s-dense monomorphism for a finitely generated S-act F and $f : F \to A$ be a homomorphism. By hypothesis, F is s-compact and so a strongly-s-dense retract, by Theorem 2.8. The reminder is straightforward. $\square$

As a consequence of Theorems 3.1 and 3.4, we conclude the following theorem. First we recall the following definition.

Definition 3.11. An s-compact S-act A is called *countably* $\sum$-*s-compact* if every countable direct sum of A is s-compact.

Theorem 3.12. *For countably generated semigroup S, the following assertions are equivalent*:

(i) *All S-acts are s-compact.*

(ii) *Each s-compact S-act is countably $\sum$-s-compact.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i) Suppose that $\{E_n \mid n \in \mathbb{N}\}$ is a family of injective S-acts. By Theorem 2.8(iv), every injective S-act is s-compact and by Theorem 3.1, $A = \prod_{n\in\mathbb{N}} E_n$ is an s-compact S-act, and so by (ii), $\bigoplus_{n\in\mathbb{N}} A$ is also an s-compact S-act. Moreover, $A = \prod_{n\in\mathbb{N}} E_n = E_m \bigoplus \prod_{n\neq m} E_n$, for each $m \in \mathbb{N}$ and $\bigoplus_{m\in\mathbb{N}} A = \bigoplus_{m\in\mathbb{N}}(E_m \bigoplus \prod_{n\neq m} E_n) = (\bigoplus_{m\in\mathbb{N}} E_m) \bigoplus (\bigoplus_{m\in\mathbb{N}} \prod_{n\neq m} E_n)$. This facts together with Theorem 3.4 imply that $\bigoplus_{m\in\mathbb{N}} E_m$ is s-compact. Now Theorem 2.13(iii) completes the proof. $\square$

4. Some Baer type criterias for injectivity of S-acts

Injectivity is one of the central notions in many branches of mathematics. One usually takes a subclass $\mathcal{M}$ of monomorphisms in a category $\mathcal{A}$, members of which may be called $\mathcal{M}$-morphisms, and give the following definition.

An object A of $\mathcal{A}$ is said to be $\mathcal{M}$-*injective* if for any $\mathcal{M}$-morphism $g : B \to C$, any morphism $f : B \to A$ can be lifted to a morphism $\bar{f} : C \to A$ of $\mathcal{A}$. That is, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f\downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

One line of study in this regard is to investigate the relation between injectivity with respect to a subclass $\mathcal{M}_1$ of monomorphisms, which is called $\mathcal{M}_1$-injectivity, and injectivity with respect to another subclass $\mathcal{M}_2$ of monomorphisms, the results of which may be called the **Baer type criteria**. Note that if $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{M}_2$-injectivity implies $\mathcal{M}_1$-injectivity. The Baer type problem is about the converse of this fact.

The object A is said to be an $\mathcal{M}$-*retract* of its $\mathcal{M}$-extension B if there exists a morphism $f : B \to A$ such that $f\mid_A = id_A$, in this case f is said to be an $\mathcal{M}$-*retraction*. An object A is called an *absolute* $\mathcal{M}$-*retract* if it is an $\mathcal{M}$-retract of each of its $\mathcal{M}$-extensions. When $\mathcal{M}$ is the class of all monomorphisms, we do not mention $\mathcal{M}$ and get the ordinary notions of injectivity, retract, and absolute retract.

Here we consider some classes of monomorphisms such as s-dense or strongly-s-dense monomorphisms instead of $\mathcal{M}$.

Consider $\mathcal{M}_1$, the class of all strongly-s-dense monomorphisms and $\mathcal{M}_2$, the class of all s-dense monomorphisms and $\mathcal{M}$, the class of all monomorphism. By Theorem 2.8, every $\mathcal{M}_1$-injective S-act is s-compact. It is clear that $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}$, thus every injective S-act is s-dense injective and every s-dense injective S-act is s-compact.

Here we give an example which is an s-compact S-act but it is not s-dense injective. By Corollary 3.3, the left zero semigroup S, as an S-act, is s-compact. We know that if S has more than two elements, then it has no left identity. Let S be an s-dense injective S-act. So the identity map $id_S : S \to S$ can be extended to $f : S^1 \to S$. Thus $f(1)$ is a left identity of S, which is a contradiction.

From now on, we use injectivity with respect to these classes of monomorphisms to give some Baer type results about injectivity. Note that each injective S-act A has a *fixed* element and injective hulls of S-acts always exist (see [11]).

The following results are the Baer type ones for $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

Theorem 4.1. *If S has local left identities, then every s-compact S-act is s-dense injective.*

Proof. Let A be an s-compact S-act and $f : S \to A$ be a homomorphism. By Lemma 2.2, the set $\Sigma = \{xs = f(s) \mid s \in S\}$ has a solution b in an extension B of A. Since S has local left identities, for every finite subset T of S there is an element $y_T \in S$ such that $y_T t = t(t \in T)$. So for every $t \in T, (by_T)t = f(t)$ and $by_T = f(y_T) \in A$. Thus every finite subset of Σ has a solution in A. Since A is an s-compact S-act, Σ has a solution $a \in A$. Therefore, $f = \lambda_a$ for some $a \in A$ and by [9, Theorem 3.3], A is s-dense injective. $\square$

Proposition 4.2. *If every nonempty proper right ideal of S is generated by a central idempotent element, then every s-compact S-act with at least one fixed element is injective.*

Proof. By Example 2.16, S has local left identities. Now Theorems 4.1 and [4, Theorem 11] give the result. $\square$

An important semigroup satisfying proposition 4.2 is $S = (\mathbb{N}, min)$. The category of all S-acts over this semigroup, so called *projection algebras*, has been studied by Giuli [10] and Ebrahimi and Mahmoud [12]. These types of S-acts are mostly used in computer science.

A semigroup S is called a *Clifford semigroup* (see, [11]) if each $\alpha \in S$ has a pseudoinverse element (there exists $\beta \in S$ such that $\alpha\beta\alpha = \alpha$) and the set of all idempotents is a subset of the set of all central elements.

It is easy to check that if β is a pseudoinverse of $\alpha \in S$, $\alpha\beta$ is an idempotent element and $\alpha S^1 = \alpha\beta S^1$. So the following corollary is straighforward.

Corollary 4.3. *Let S be a Clifford semigroup. If each proper nonempty right ideals of S is principal, then every s-compact S-act with at least one fixed element is injective.*

The following two classes of semigroups are special cases of Clifford semigroups.

(i) Commutative chains with the order relation $(x \leq y \Leftrightarrow xy = x)$ or $(x \leq y \Leftrightarrow xy = y)$.

(ii) Commutative bands (all elements of S are idempotents).

From now on, for a subact A of an S-act B and $b \in B$, consider $\bar{A} = \{b \in B \mid bS \subseteq A\}$ and $I_b = \{s \in S \mid bs \in A\}$. The subset $\bar{A}$ is a subact of B and I_b is a right ideal of S. It is clear that $A \subseteq \bar{A}$ and if $A_1 \subseteq A_2$, then $\bar{A}_1 \subseteq \bar{A}_2$. Also for each homomorphism $f : B \to C$ and $A \subseteq B$, $f(\bar{A}) \subseteq \overline{f(A)}$. These properties show that the map $C : Sub(B) \to Sub(B)$ defined by $C(A) = \bar{A}$ is a closure operator.

Definition 4.4. An S-act B is an *essential extension* of A, if each homomorphism $f : B \to C$ is a monomorphism, whenever $f\mid_A$ is a monomorphism.

Theorem 4.5. *If S has local left identities and for every nontrivial right ideal I of S, $\bar{I} \neq I$, then every s-compact S-act is injective.*

Proof. Let A be an s-compact S-act, B be an essential extension of A and $b \in \bar{A} \setminus A$. It is clear that $A \cup \{b\}$ is an essential extension of

A. Since S has local left identities, for every finite subset T of S there exists $y_{_T} \in S$ such that $y_{_T}t = t$. So $bt = (by_{_T})t$ which $by_{_T} \in A$ and hence $A \cup \{b\}$ is a strongly-s-dense extension of A. It follows frome Theorem 2.8 that there is a homomorphism $g : A \cup \{b\} \to A$ with $g\mid_{_A} = id_{_A}$, which implies that g is an isomorphism. Thus $b \in A$ and hence $\bar{A} = A$.

Now let $b \in B \backslash A$. If $I_b = S$, then $b \in \bar{A} = A$ which is a contradiction. Thus $I_b \neq S$ and by hypothesis $\bar{I_b} \neq I_b$. For each $s \in \bar{I_b} \setminus I_b$, it is not difficult to show that $bs \in \bar{A} = A$ and hence $s \in I_b$ which is a contradiction. So $B = A$ and [11, Proposition III.1.20] deduced that A is an injective S-act. □

Recall that an S-act A is called *quasi injective* if every homomorphism $f : B \to A$ from a subact B of A can be extended to A.

Theorem 4.6. *Every s-compact S-act is injective if and only if every s-compact S-act is quasi injective and has a fixed element.*

Proof. We show only the unclear direction. Let A be an s-compact S-act and $E(A)$ be the injective hull of A. Choose and fix elements $0 \in Fix(A)$ and $\theta \in Fix(E(A))$. It is clear that two S-acts A and $\{0\} \times A$ are isomorphic. Consider $\{0\}\times A \overset{\tau_1}{\hookrightarrow} \{0\}\times E(A) \overset{\tau_2}{\hookrightarrow} A\times E(A)$, where τ_1 and τ_2 are inclusions and define the map $\tau_A : \{0\} \times A \to A \times E(A)$, by $\tau_A(0, a) = (a, \theta)$. Since every injective S-act is s-compact, by Theorem 3.1, the S-act $A \times E(A)$ is s-compact and hence it is quasi injective. So there exists a homomorphism $g : A \times E(A) \to A \times E(A)$ such that $g\tau_2\tau_1 = \tau_A$. For the homomorphism $p_{_A} : A \times E(A) \to \{0\} \times A$, defined by $p_{_A}(a, b) = (0, a)$, we have $p_{_A}g\tau_2\tau_1(0, a) = p_{_A}\tau_{_A}(0, a) = (0, a)$ and hence $\{0\} \times A$ is a retract of the injective S-act $\{0\} \times E(A)$. Thus A is injective. □

Theorem 4.7. *An S-act A is injective if and only if it is s-compact as well as f-pure injective.*

Proof. Let A be an s-compact and f-pure injective S-act and $E(A)$ be the injective hull of A. Since A is s-compact, by Theorem 2.8, it is f-pure subact of $E(A)$ and hence it is a retract of $E(A)$, which implies that A is injective. □

Theorem 4.8. *If every essential extension of any S-act A is strongly s-dense extension, then every s-compact S-act is injective.*

Proof. Let A be an s-compact S-act and $\tau : A \to E(A)$ be the injective hull of A. Since τ is essential, it is strongly s-dense extension and since A is s-compact, by Theorem 2.8, it is strongly s-dense injective. So there exists a homomorphism $g : E(A) \to A$ such that $g\tau = id_A$, which is a monomorphism by essentiality of τ. furthermore, $g\tau g = id_A g = g = g id_{E(A)}$ which implies that $\tau g = id_{E(A)}$. Thus τ is an isomorphism and hence A is injective. □

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SEQUENTIALLY COMPACT S-ACT

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فشردگی معادله‌ای در رسته S -سیستم‌ها

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مطالعات در مورد فشردگی معادله‌ای یک جبر را در ابتدا بناشفسکی و نلسون آغاز کردند. آنها نشان دادند که در جبرهای جامع مفاهیم خلوص انژکتیوی و فشردگی معادله ای هم ارز هستند. در اینجا ما مفهوم فشردگی دنباله‌ای روی یک نیمگروه S را معرفی کرده و برخی از خواص رسته‌ای و همولوژیکی آن را در رسته S سیستم‌ها مورد مطالعه قرار می‌دهیم. در نهایت تعدادی محک بئر برای انژکتیوی نیز بیان شده است.

کلمات کلیدی: فشردگی معادله‌ای، f-خالص انژکتیوی، سیستم انژکتیو.