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INTERSECTION OF ESSENTIAL IDEALS IN THE RING OF REAL-VALUED CONTINUOUS FUNCTIONS ON A FRAME

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Abstract. A frame L is called *coz-dense* if $\Sigma_{\text{coz}(\alpha)} = \emptyset$ implies $\alpha = \mathbf{0}$. Let $\mathcal{R}L$ be the ring of real-valued continuous functions on a coz-dense and completely regular frame L. We present a description of the socle of the ring $\mathcal{R}L$, based on minimal ideals of $\mathcal{R}L$ and zero sets in pointfree topology. We show that socle of $\mathcal{R}L$ is an essential ideal in $\mathcal{R}L$ if and only if the set of isolated points of ΣL is dense in ΣL if and only if the intersection of any family of essential ideals is essential in $\mathcal{R}L$. Besides, the counterpart of some results in the ring $C(X)$ is studied for the ring $\mathcal{R}L$. For example, an ideal E of $\mathcal{R}L$ is an essential ideal if and only if $\bigcap Z[E]$ is a nowhere dense subset of ΣL.

1. Introduction

A nonzero ideal in a commutative ring R is called essential if it intersects every nonzero ideal nontrivially. This concept was first introduced in [13] and plays an important role in the structure theory of noncommutative Noetherian rings, see [11] or [21].

The intersection of all essential ideals in any commutative ring R, or the sum of all minimal ideals of R is the socle of R denoted by $\text{Soc}(R)$ (see [10]). Let $C(X)$ be the ring of real-valued continuous functions on a completely regular Hausdorff space X. The socle of $C(X)$ denoted by $C_F(X)$ is a z-ideal (see [18]).

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The following propositions, which topologically characterize essential ideals and the socle of $C(X)$, are proved in [18] and [1], respectively.

Proposition 1.1. *The socle of $C(X)$ consists of all functions vanishing everywhere except on a finite number of points of X.*

Proposition 1.2. *If E is a nonzero ideal in $C(X)$, then the following statements are equivalent.*

(1) *E is an essential ideal in $C(X)$.*
(2) *$\bigcap Z[E]$ is a nowhere dense subset of X.*

The following theorem can be easily deduced from the two foregoing propositions.

Theorem 1.3. [2] *The intersection of any family of essential ideals is essential in the ring $C(X)$ if and only if the set of isolated points of X is dense in X.*

For the characterization of $C_F(X)$ cited in Proposition 1.1 above, the authors begin by characterizing minimal ideals of $C(X)$. This characterization is extended in [5, 6] to $\mathcal{R}L$, as follows:

$$\mathrm{Soc}(\mathcal{R}L) = \{\alpha \in \mathcal{R}L : \mathrm{coz}(\alpha) \text{ is a join of finitely many atoms}\}.$$

The approaches in [5] and [6] are completely different. In the former case, the author T. Dube starts by characterizing minimal ideals of $\mathcal{R}L$. In [6], minimal ideals of $\mathcal{R}L$ are not considered, and the description of the socle is obtained via the fact that, for a ring R and $a \in R$, we have:

$$a \in \mathrm{Soc}\, R \Leftrightarrow \mathrm{Ann}(a) \text{ is an intersection of finitely many maximal ideals.}$$

In this note, we are trying to describe the socle of the ring $\mathcal{R}L$ by using minimal ideals of $\mathcal{R}L$ and zero sets in the pointfree topology, as follows:

$$\mathrm{Soc}(\mathcal{R}L) = \{\alpha \in \mathcal{R}L : \Sigma_{\mathrm{coz}(\alpha)} \text{ is a finite subset of } \Sigma L\},$$

which is discussed in Proposition 4.3. Also, we prove that $\mathrm{Soc}(\mathcal{R}L)$ is a strongly z-ideal of $\mathcal{R}L$ and is direct sum of minimal ideals generated by idempotent elements of $\mathcal{R}L$ (Corollary 4.4).

The counterpart of Proposition 1.2 and Theorem 1.3 for the ring of real-valued continuous functions $\mathcal{R}L$ is given in Proposition 3.6 and Theorem 4.6, respectively. Also, for a coz-dense and completely regular frame L, it is shown that a point p in ΣL is isolated if and only if the ideal M_p of $\mathcal{R}L$ is nonessential (Proposition 3.8).

2. Preliminaries

We recall some basic notions and facts about frames and spaces. For further information see [3, 14, 22] on frames and [9] on spaces.

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of open subsets of a topological spase X is denoted by $\mathfrak{O}X$.

A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

The *pseudocomplement* of an element a of a frame L is the element

$$a^\star = \bigvee\{x \in L : x \wedge a = \bot\}.$$

An element x of a frame L is said to be:

(1) *dense* if $x^\star = \bot$,
(2) *prime* (or a *point*) if $x < \top$ and, for $a, b \in L$, $a \wedge b \leq x$ implies $a \leq x$ or $b \leq x$, and
(3) an atom if $\bot < x$ and, for any $a \in L, \bot \leq a \leq x$ implies $a = \bot$ or $a = x$.

An element a of a frame L is said to be *rather below* an element b, written $a \prec b$, provided that $a^\star \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\!\!\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a,\ c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$.

Recall that the contravariant functor Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* : M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. It is well known that h_* preserves primes and arbitrary meets.

Recall [3] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee\{(r, s)| : p < r < s < q\}$.

(R4) $\top = \bigvee\{(p,q) : p,q \in \mathbb{Q}\}$.

The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been studied as an f-ring in [3]. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p,q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p,q) = \bot$.

The *cozero map* is the map $\mathrm{coz} : \mathcal{R}L \to L$, defined by

$$\mathrm{coz}(\alpha) = \bigvee\{\alpha(p,0) \vee \alpha(0,q) : p,q \in \mathbb{Q}\} = \alpha((-,0) \vee (0,-))$$

where, for $r,s \in \mathbb{Q}$,

$$(r,-) = \bigvee\{(0,q)) : q \in \mathbb{Q}, q > r\}$$

and

$$(-,s) = \bigvee\{(p,0)) : p \in \mathbb{Q}, p < s\}.$$

For $A \subseteq \mathcal{R}L$, let $\mathrm{Coz}(A) = \{\mathrm{coz}(\alpha) : \alpha \in A\}$ with the cozero part of a frame L, $\mathrm{Coz}(\mathcal{R}L)$, called $\mathrm{Coz}\, L$ by previous authors. It is known that L is completely regular if and only if $\mathrm{Coz}\, L$ generates L. For more details about the *cozero map* and its properties used in this note see [3].

Let L be a frame, $a \in L$, and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-,r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s,-) \leq a\}$ are denoted by $L(a,\alpha)$ and $U(a,\alpha)$, respectively.

For $a \neq \top$, it is obvious that $r \leq s$, for each $r \in L(a,\alpha)$ and $s \in U(a,\alpha)$. In fact, we have:

Proposition 2.1. [19] Let L be a frame. If $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p,\alpha), U(p,\alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.

Proposition 2.2. [19] If p is a prime element of a frame L, then there exists a unique map $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ such that for each $\alpha \in \mathcal{R}L$, $r \in L(p,\alpha)$, and $s \in U(p,\alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.

By the following proposition, $\tilde{p}$ is an f-ring homomorphism.

Proposition 2.3. [19] If p is a prime element of a frame L, then $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$.

Recall [8] that for $\alpha \in \mathcal{R}L$, $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$, where $\alpha[p] = \tilde{p}(\alpha)$. $Z(\alpha)$ is called the *zero set* of α. For $A \subseteq \mathcal{R}L$, we write $Z[A]$ to designate the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. On the other hand, the family $Z[\mathcal{R}L]$ of all zero-sets in L will also be denoted, for simplicity, by $Z[L]$. Also for a subfamily $\mathcal{F}$ of $Z(L)$, we write

$$Z^{\leftarrow}[\mathcal{F}] = \{\alpha : Z(\alpha) \in \mathcal{F}\}.$$

The following lemma and proposition proved in [8] play important roles in the description of zero sets.

Lemma 2.4. *Let p be a prime element of frame L. For $\alpha \in \mathcal{R}L$, $\alpha[p] = 0$ if and only if $\operatorname{coz}(\alpha) \leq p$. Hence $Z(\alpha) = \Sigma L - \Sigma_{\operatorname{coz}(\alpha)}$.*

Proposition 2.5. *For every $\alpha, \beta \in \mathcal{R}L$, we have*

(1) *For every $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*
(2) *$Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$.*
(3) *$Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.*
(4) *If α is a unit of $\mathcal{R}L$, then $Z(\alpha) = \emptyset$.*
(5) *$Z(L)$ is closed under the countable intersection.*

In [12], using the technique of sublocales, the authors present zero sublocales. A sublocale S of a frame L is a zero sublocale if it is of the form

$$(f^*(0, -))^* \wedge (f^*(-, 0))^*$$

for some localic map $f : L \to \mathcal{L}(\mathbb{R})$ as the right Galois adjoint of a frame homomorphism $h = f^* : \mathcal{L}(\mathbb{R}) \to L$. The zero sets used in this paper are different from the zero sublocales.

3. On essential ideals

We shall now commence our discussion on coz-dense frames; so we start by formalizing the definition stated in the abstract.

Definition 3.1. A frame L is coz-dense if for any $c \in \operatorname{Coz} L$, $\Sigma_c = \emptyset$ implies $c = \bot$.

It is clear from the definition that if L has no primes, that is, if $\Sigma L = \emptyset$, then L cannot be coz-dense. At the other extreme, every spatial frame is coz-dense. The following result gives a clearer picture of when a completely regular frame with at least one prime is coz-dense.

Lemma 3.2. *The following are equivalent for a completely regular frame L that has primes.*

(1) *L is coz-dense.*
(2) *For $a \in L$, $\Sigma_a = \emptyset$ implies $a = \bot$.*
(3) *The frame homomorphism $L \to \mathfrak{O}\Sigma L$ given by $a \mapsto \Sigma_a$ is dense.*
(4) *The bottom element of L is the only element which is below every prime of L.*

Proof. (1) $\Rightarrow$ (2). For any $c \in \operatorname{Coz} L$ with $c \leq a$, $\Sigma_c = \emptyset$ which implies $c = \bot$ by coz-density. Therefore $a = \bot$ since, by complete regularity, a is the join of cozero elements below it. The other implications required to prove all the equivalences are immediate. $\square$

Remark 3.3. We have stated above that spatial frames are coz-dense. In spite of a great deal of effort expended, we have not been able to determine if (by way of some sort of converse) every coz-dense completely regular frame is spatial. We however have the following result regarding Boolean frames.

Proposition 3.4. *Every coz-dense Boolean frame with primes is spatial.*

Proof. Let L be a coz-dense frame with primes, and let $a \in L$ be complemented. If $a = \perp$ or $a = \top$, the result is immediate. So suppose $\perp\ < a < \top$. For brevity, put $T = \Sigma L - \Sigma_a$, so that T is the set of primes above a. Write $t = \bigwedge T$. We aim to show that $a = t$. Clearly, $a \leq t$. Since a is complemented, it is co-linear, which is to say

$$a \vee \bigwedge\nolimits_\alpha b_\alpha = \bigwedge\nolimits_\alpha (a \vee b_\alpha)$$

for all $\{b_\alpha\} \subseteq L$. Since $\Sigma_a \neq \emptyset$, for any $p \in \Sigma_a$, $a \nleq p$, which implies $a \vee p = \top$ since primes are maximal elements in regular frames. Consequently,

$$t = t \wedge (a \vee p) = (t \wedge a) \vee (t \wedge p) = a \vee (t \wedge p).$$

Taking meets over all $p \in \Sigma_a$ we have

$$\begin{aligned} t &= \bigwedge\{a \vee (t \wedge p) : p \in \Sigma L\} \\ &= a \vee \big(\bigwedge\{t \wedge p : p \in \Sigma L\}\big) \\ &= a \vee \big(t \wedge \bigwedge \Sigma_a\big) \\ &= a \vee \big(\bigwedge T \wedge \bigwedge \Sigma_a\big) \\ &= a \vee \bigwedge \Sigma L \\ &= a \vee \perp \\ &= a. \end{aligned}$$

Therefore a is a meet of primes above it. □

Proposition 3.5. *Let L be a coz-dense and completely regular frame. For every $a \in L$, the following statements are equivalent.*

(1) *Σ_a is dense in ΣL.*
(2) *a is dense in L.*

Proof. (1) $\Rightarrow$ (2). Let $b \in L$ and $a \wedge b = \perp$. Then $\Sigma_a \cap \Sigma_b = \Sigma_{a\wedge b} = \Sigma_\perp = \emptyset$, it follows that $\Sigma_b = \emptyset$. So, by Lemma 3.2, $b = \perp$. Therefore $a^\star = \perp$, and hence a is dense in L.

(2) $\Rightarrow$ (1). Let $b \in L$ and $\Sigma_a \cap \Sigma_b = \emptyset$. Then, by Lemma 3.2, $a \wedge b = \perp$. Since a is dense in L, we conclude that $b = \perp$, it follows that $\Sigma_b = \emptyset$. Therefore, Σ_a is dense in ΣL. □

An ideal I in $\mathcal{R}L$ is called a strongly z-ideal if $Z(\alpha) \in Z[I]$ implies $\alpha \in I$, that is $I = Z^{\leftarrow}[Z[I]]$. Also, an ideal I in $\mathcal{R}L$ is a z-ideal if for any $\alpha \in \mathcal{R}L$ and $\beta \in I, \operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ implies $\alpha \in I$. Note that in the ring $\mathcal{R}L$, every strongly z-ideal is a z-ideal. But the reverse is not true, see [8] for more details.

Let L be a completely regular frame. (a) An ideal E of $\mathcal{R}L$ is essential if and only if $\bigvee \operatorname{Coz}(E)$ is dense in L (see Lemma 4.3 in [5]). (b) Notice that every nonzero ideal I of $\mathcal{R}L$ contains a nonzero z-ideal. To see this, consider any $\alpha \in I$ with $\alpha \neq \mathbf{0}$. Then the set

$$J = \{\beta \in \mathcal{R}L : \operatorname{coz}(\beta) \prec\!\!\prec \operatorname{coz}(\alpha)\}$$

is a nonzero z-ideal contained in I. To see that $J \subseteq I$, recall from [4, Lemma 4.4] that if $\operatorname{coz}(\beta) \prec\!\!\prec \operatorname{coz}(\alpha)$, then β is a multiple of α. J is of course nonzero by complete regularity. Therefore, for every ideal E of $\mathcal{R}L$, we have that E is essential if and only if $I \cap E \neq 0$, for every nonzero z-ideal I of $\mathcal{R}L$.

In any semiprime ring, it is well known that

$$E \text{ is an essential ideal if and only if } \operatorname{Ann}(E) = 0.$$

Notice that $\mathcal{R}L$ is a semiprime ring for any frame L. By a semiprime ring we mean one with no nonzero nilpotent elements.

According to these descriptions, for every completely regular frame L and an ideal E in $\mathcal{R}L$, we have that E is an essential ideal if and only if $Ann(E) = 0$ if and only if $I \cap E \neq 0$, for every non-zero z-ideal I of $\mathcal{R}L$.

Drawing on "strongly z-ideals are z-ideals". For every completely regular frame L and an ideal E in $\mathcal{R}L$, we can immediately present that E is an essential ideal if and only if

$$\text{for every non-zero strongly } z\text{-ideal } I \text{ of } \mathcal{R}L,\ I \cap E \neq 0.$$

A subset S of a topological space X is said to be nowhere dense if

$$\operatorname{int}_X(\operatorname{cl}_X S) = \emptyset.$$

As we already noted, the following proposition is the counterpart of Proposition 1.2, which is uesd for the proof of Propositions 3.8, 3.10 and Theorem 4.6.

Proposition 3.6. *Let L be a coz-dense and completely regular frame and E be an ideal of $\mathcal{R}L$. Then the following statements are equivalent.*

(1) *E is an essential ideal.*
(2) *$\bigcap Z[E]$ is a nowhere dense subset in ΣL.*

Proof. It is obvious that

$$\mathrm{cl}_{\Sigma L}(\Sigma L - \bigcap Z[E]) = \mathrm{cl}_{\Sigma L} \bigcup_{\alpha \in E} (\Sigma L - Z(\alpha))$$

$$= \mathrm{cl}_{\Sigma L} \bigcup_{\alpha \in E} \Sigma_{\mathrm{coz}(\alpha)} = \mathrm{cl}_{\Sigma L} \Sigma_{\bigvee \mathrm{Coz}(E)}.$$

Hence,

$$\mathrm{cl}_{\Sigma L}(\Sigma L - \bigcap Z[E]) = \Sigma L \quad \Leftrightarrow \quad \mathrm{cl}_{\Sigma L} \Sigma_{\bigvee \mathrm{Coz}(E)} = \Sigma L.$$

Therefore, by Proposition 3.5, $\bigvee \mathrm{Coz}(E)$ is a dense element in L if and only if $\bigcap Z[E]$ is a nowhere dense subset in ΣL. Now, by Lemma 4.3 in [5], the proof is complete. □

Let I be any ideal in $\mathcal{R}L$. If $\bigcap Z[I]$ is nonempty, we call I a *strongly fixed ideal*; if $\bigcap Z[I] = \emptyset$, then I is a *strongly free ideal*, see [7] for more details. By the foregoing proposition, strongly free ideals are essential ideals. Also for every nonisolated point $p \in \Sigma L$, the ideal $M_p = \ker \widetilde{p}$ is essential.

Let $f : \Sigma L \to \mathbb{R}$ be a continuous function. For every $p, q \in \mathbb{Q}$, define $\widehat{f}(p,q) = \bigvee\{a \in L : f(\Sigma_a) \subseteq \langle p, q\rangle\}$, where $\langle p, q\rangle = \{x \in \mathbb{R} : p < x < q\}$. By Lemma 4.5 in [20], $\widehat{f} : \Re \to L$ is a frame map.

Remark 3.7. Let $p \in \Sigma L$ be an isolated point. We define

$$f_p(x) = \begin{cases} 0 & \text{if } x = p \\ 1 & \text{if } x \in \Sigma L - \{p\}. \end{cases}$$

It is clear that $f_p : \Sigma L \to \mathbb{R}$ is a continuous map, in fact $f_p = \mathbf{1} - \chi_p$. Therefore, if L is a coz-dense frame, then

$$\widehat{f_p}\mathcal{R}L = M_p = \{f \in \mathcal{R}L : f[p] = 0\} = \ker \widetilde{p}$$

is the maximal ideal in $\mathcal{R}L$ and $\widehat{f_p}^2 = \widehat{f_p}$ (see Lemma 4.14 in [20]).

Clearly every maximal ideal in any commutative ring with unity is either essential or else is generated by an idempotent in this case we call it isolated (see [16, 17] for more details).

Proposition 3.8. *Let L be a coz-dense and completely regular frame. Then, a point $p \in \Sigma L$ is isolated if and only if the ideal M_p in $\mathcal{R}L$ is nonessential.*

Proof. $\Rightarrow$) Let $p \in \Sigma L$ be an isolated point. Then, by Remark 3.7, we have $\widehat{f_p}\mathcal{R}L = M_p$. So $\bigcap Z[M_p] = Z(\widehat{f_p}) = Z(f_p) = \{p\}$ (see Proposition 4.12 in [20]). Hence, $\mathrm{int}_{\Sigma L}(\mathrm{cl}_{\Sigma L} \bigcap Z[M_p]) = \{p\}$. Therefore, by Proposition 3.6, the ideal M_p in $\mathcal{R}L$ is nonessential.

$\Leftarrow$) Let the ideal M_p in $\mathcal{R}L$ be non-essential. Then there exists $f \in \mathcal{R}L$ such that $M_p = f\mathcal{R}L$ and $f^2 = f$. So, we can conclude that

$$Z(f) = \{p\} \text{ and } Z(f - \mathbf{1}) = \Sigma L - \{p\}.$$

Therefore, the proof is complete. $\square$

A frame L is called *coz-disjoint* if $\operatorname{Coz} L \cap \downarrow p \neq \operatorname{Coz} L \cap \downarrow q$, for every two distinct prime elements $p, q \in L$. Notice that every completely regular frame is a coz-disjoint frame (for more details see [20]).

Proposition 3.9. [20] Let L be a coz-disjoint frame. If P is a prime ideal of $\mathcal{R}L$, then $|\bigcap Z[P]| \leq 1$.

Proposition 3.10. *Let L be a coz-dense and completely regular frame. If P is a prime and nonessential ideal in $\mathcal{R}L$, then there exists an isolated point p in ΣL and an idempotent element $f \in \mathcal{R}L$ such that*

$$P = M_p \text{ and } Z(f) = \{p\}.$$

Also, P is a minimal prime ideal in $\mathcal{R}L$.

Proof. Let P be a nonessential ideal of $\mathcal{R}L$. By Propositions 3.9 and 3.6, there exists an isolated point p in ΣL such that $\bigcap Z[P] = \{p\}$. By Remark 3.7, $P \subseteq M_p = \widehat{f_p}\mathcal{R}L$ is the maximal ideal of $\mathcal{R}L$ and $\widehat{f_p}^2 = \widehat{f_p}$. On the other hand $\widehat{f_p}(\mathbf{1} - \widehat{f_p}) = \mathbf{0} \in P$ implies that $\widehat{f_p} \in P$ or $\mathbf{1} - \widehat{f_p} \in P$. If $\mathbf{1} - \widehat{f_p} \in P$, then $\mathbf{1} \in M_p$, which is a contradiction. Therefore $P = M_p$. Now, we show that M_p is a minimal prime ideal of $\mathcal{R}L$. Let Q be a prime ideal such that $Q \subseteq P$, then $\widehat{f_p}(\mathbf{1} - \widehat{f_p}) = \mathbf{0} \in Q$ implies that $\widehat{f_p} \in Q$, i.e., $Q = P$. $\square$

4. Socle of $\mathcal{R}L$

In this section, the socle of $\mathcal{R}L$ is characterized by the ideal consisting of all functions vanishing everywhere except on a finite number of points of L. For the proof of the next lemma, we will use the fact that in a regular frame prime elements are maximal.

Lemma 4.1. *Let L be a regular frame.*

(1) *ΣL is a Hausdorff space.*
(2) *If $\alpha \in \mathcal{R}L$ and $\Sigma_{\operatorname{coz}(\alpha)} = \{p_1, p_2, \ldots, p_n\}$, then each p_k, $k = 1, 2, ..., n$, is an isolated point of ΣL.*

Proof. (1). Suppose that $p, q \in \Sigma L$ and $p \neq q$. By regularity, $p = \bigvee_{x \prec p} x$. Since $p \not\leq q$, there exists $x \in L$ such that $x \prec p$ and $x \not\leq q$. So $x^\star \not\leq p$ and $x \not\leq q$. Therefore $p \in \Sigma_{x^\star}$ and $q \in \Sigma_x$. On the other hand, we have $\Sigma_{x^\star} \cap \Sigma_x = \Sigma_0 = \emptyset$. Hence, the proof is complete.

(2). By (1), it is obvious. $\square$

That the spectrum of a regular frame is Hausdorff was proved by Isbell in [15, 2.3]

Proposition 4.2. [20] *Suppose that L is a coz-disjoint and coz-dense frame. Let I be a non-zero ideal of $\mathcal{R}L$. The following statements are equivalent.*

(1) *I is a minimal ideal.*
(2) $|Z[I]| = 2$.
(3) *There exists $p \in \Sigma L$ such that $Z[I] = \{\Sigma L, \Sigma L - \{p\}\}$.*

Proposition 4.3. *Let L be a coz-dense and completely regular frame. Then*

$$\mathrm{Soc}(\mathcal{R}L) = \{\alpha \in \mathcal{R}L : \Sigma_{\mathrm{coz}(\alpha)} \text{ is a finite subset of } \Sigma L\}.$$

Proof. We note that if $\mathrm{Soc}(\mathcal{R}L) = (\mathbf{0})$ then by considering the empty set as a finite set, the result is trivially true. Hence, suppose $\mathrm{Soc}(\mathcal{R}L) = \sum \oplus I_k$, where I_k runs over the set of minimal ideals of $\mathcal{R}L$. Now, for each $0 \neq \alpha \in \mathrm{Soc}(\mathcal{R}L)$, we have $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where each $0 \neq \alpha_i$ belongs to some minimal ideal in $\mathcal{R}L$. Then, by Proposition 4.2, each α_i is zero everywhere except at an isolated point p_i of ΣL. Thus $Z(\alpha) = \Sigma L - \{p_1, p_2, \ldots, p_n\}$.

Conversely, let $\Sigma_{\mathrm{coz}(\beta)}$ be a finite set, then we have to show that $\beta \in \mathrm{Soc}(\mathcal{R}L)$. If $\Sigma_{\mathrm{coz}(\beta)} = \{p_1, p_2, \ldots, p_n\}$, then, by Lemma 4.1, each p_k, $k = 1, 2, ..., n$, is an isolated point of ΣL. Now for each p_k, there exists a minimal ideal I_k such that $Z(\alpha) = \Sigma L - \{p_k\}$, for all nonzero $\alpha \in I_k$. But each I_k is of the form $I_k = e_k\mathcal{R}L$, where e_k is an idempotent in $\mathcal{R}L$. Then, clearly,

$$\beta = e_1\beta + e_2\beta + \cdots + e_n\beta \in I_1 + I_2 + \cdots + I_n \subseteq \mathrm{Soc}(\mathcal{R}L).$$

□

Let $\alpha \in \mathcal{R}L$. Define $\overline{\alpha} : \Sigma L \to \mathbb{R}$ given by $\overline{\alpha}(p) = \widetilde{p}(\alpha)$. By Remark 4.15 in [20], $\overline{\alpha}$ is a continuous function.

Corollary 4.4. *Let L be a coz-dense and completely regular frame and*

$$A = \Sigma L \setminus \bigcap_{\alpha \in \mathrm{Soc}(\mathcal{R}L)} Z(\alpha).$$

Then the following statements hold.

(1) *$\mathrm{Soc}(\mathcal{R}L)$ is a strongly z-ideal of $\mathcal{R}L$.*
(2) *For every $\alpha \in \mathrm{Soc}(\mathcal{R}L)$, $\Sigma_{\mathrm{coz}(\alpha)}$ is a closed subset in ΣL.*
(3) *If $\alpha \in \mathrm{Soc}(\mathcal{R}L)$, then p is an isolated point in ΣL, for every $p \in \Sigma_{\mathrm{coz}(\alpha)}$.*
(4) *If $p \in A$, $I_p = \widehat{\chi_p}\mathcal{R}L$ is a minimal ideal of $\mathcal{R}L$.*

(5) $\mathrm{Soc}(\mathcal{R}L) = \sum_{p\in A} \oplus I_p$.

Proof. By using Proposition 4.3, (1), (2) and (3) are obvious.

(4). Let $J \subseteq I_p$, and $J \neq 0$. if $\beta \in J$ and $\beta \neq \mathbf{0}$, then we have $\beta \in I_p$. Hence $Z(\beta) \supseteq Z(\widehat{\chi_p}) = \Sigma L - \{p\}$, and so $Z(\beta) = \Sigma L - \{p\}$. Thus by Lemma 4.14 in [20] $\beta = \widehat{\chi_p}\beta$. Let $k = \overline{\beta}(p) \neq 0$. Then, by Proposition 4.16 in [20], $\overline{\widehat{\chi_p}} = \chi_p = \frac{\mathbf{1}}{\mathbf{k}}(\overline{\beta}) = \overline{\frac{\mathbf{1}}{\mathbf{k}}\beta}$, and hence, by Proposition 4.16 in [20], $\widehat{\chi_p} = \frac{\mathbf{1}}{\mathbf{k}}\beta \in J$. Therefore $J = I_p$, and I_p is minimal.

(5). By using (4) and Proposition 4.3, it is obvious. $\square$

In the ring $C(X)$, we can state:

$C_F(X)$ is an essential ideal if and only if the set of isolated points of X is dense in space X.

Now about the socle of the ring $\mathcal{R}L$, we have:

Proposition 4.5. *Let L be a coz-dense and completely regular frame. Then the following statements are equivalent:*

(1) $\mathrm{Soc}(\mathcal{R}L)$ *is an essential ideal in* $\mathcal{R}L$.
(2) *The set of isolated points of* ΣL *is dense in* ΣL.

Proof. We show the set of isolated points of ΣL by H. First we prove that $\bigcap Z[\mathrm{Soc}(\mathcal{R}L)] = \Sigma L - H$. For every $\alpha \in \mathrm{Soc}(\mathcal{R}L)$, by Proposition 4.3, $\Sigma_{\mathrm{coz}(\alpha)} \subseteq H$, that is to say, $\bigcup_{\alpha\in\mathrm{Soc}(\mathcal{R}L)} \Sigma_{\mathrm{coz}(\alpha)} \subseteq H$, and hence

$$\Sigma L - H \subseteq \bigcap Z[\mathrm{Soc}(\mathcal{R}L]. \qquad (1)$$

Now, suppose that $p \in \bigcap Z[\mathrm{Soc}(\mathcal{R}L]$ is an isolated point of ΣL. If

$$\beta(x) = \begin{cases} 1 & \text{if } x = p \\ 0 & \text{if } x \neq p, \end{cases}$$

then $\beta \in C(\Sigma L)$. So, by Lemma 4.5 in [20], $\widehat{\beta} \in \mathcal{R}L$. On the other hand, by Propositions 4.12 in [20] and 4.3, we have $\widehat{\beta} \in \mathrm{Soc}(\mathcal{R}L)$. Consequently, $p \in Z(\widehat{\beta})$ which is a contradiction. So p is a nonisolated point, i.e., $p \in \Sigma L - H$, in other words,

$$\Sigma L - H \supseteq \bigcap Z[\mathrm{Soc}(\mathcal{R}L]. \qquad (2)$$

According to the equations 1 and 2, we have $\Sigma L - H = \bigcap Z[\mathrm{Soc}(\mathcal{R}L]$, and hence

$$\begin{aligned} \mathrm{int}_{\Sigma L} \bigcap Z[\mathrm{Soc}(\mathcal{R}L] = \emptyset \ &\Leftrightarrow \mathrm{int}_{\Sigma L}(\Sigma L - H) = \emptyset \\ &\Leftrightarrow \Sigma L - \mathrm{cl}_{\Sigma L} H = \emptyset \\ &\Leftrightarrow \mathrm{cl}_{\Sigma L} H = \Sigma L. \end{aligned}$$

Therefore, by Proposition 3.6, statements (1) and (2) are equivalent. □

It is obvious that the intersection of any finite family of essential ideals is essential in the commutative rings. But this is not true in general for countable intersection even (for example see [2]).

The next theorem is a result similar to Theorem 1.3 for the ring $\mathcal{R}L$, which can easily be deduced by Propositions 3.6 and 4.5.

Theorem 4.6. *Let L be a coz-dense and completely regular frame. Then the intersection of any family of essential ideals is essential in the ring $\mathcal{R}L$ if and only if the set of isolated points of ΣL is dense in ΣL.*

Question 4.7. We end with a question whether every coz-dense and completely regular frame is spatial. In spite of a great deal of effort expended, we have not been able to answer this question.

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INTERSECTION OF ESSENTIAL IDEALS IN THE RING OF REAL-VALUED CONTINUOUS FUNCTIONS ON A FRAME

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اشتراک ایده‌آل‌های اساسی در حلقه توابع پیوسته حقیقی-مقدار روی یک قاب

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یک قاب L را همصفر چگال می‌نامیم اگر $\Sigma_{coz(\alpha)} = \emptyset$، آنگاه نتیجه بگیریم که $\alpha = \mathbf{0}$. فرض می‌کنیم $\mathcal{R}L$ حلقه توابع پیوسته حقیقی-مقدار روی یک قاب همصفر چگال و کاملاً منظم است. توصیفی از ساکل حلقه $\mathcal{R}L$ بر اساس ایده‌آل‌های مینیمال از $\mathcal{R}L$ و مجموعه‌های صفر در توپولوژی بدون نقطه ارائه می‌کنیم. نشان می‌دهیم که ساکل حلقه $\mathcal{R}L$ یک ایده‌آل اساسی در $\mathcal{R}L$ است اگر و تنها اگر مجموعه نقاط منزوی ΣL در ΣL چگال است اگر و تنها اگر اشتراک هر خانواده از ایده‌آل اساسی در $\mathcal{R}L$، اساسی است. علاوه بر این، همتای برخی نتایج در حلقه $C(X)$ برای حلقه $\mathcal{R}L$ مطالعه شده است. برای مثال، یک ایده‌آل E از $\mathcal{R}L$ ایده‌آلی اساسی است اگر و تنها اگر $\bigcap Z[E]$ یک زیرمجموعه هیچ جا چگال از ΣL است.

کلمات کلیدی: قاب، ایده‌آل اساسی، ساکل، مجموعه‌های صفر در توپولوژی بدون نقطه، حلقه توابع پیوسته حقیقی-مقدار روی یک قاب.