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A GENERALIZATION OF CORETRACTABLE MODULES

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Abstract. Let R be a ring and M a right R-module. We call M, coretractable relative to $\overline{Z}(M)$ (for short, $\overline{Z}(M)$-coretractable) provided that, for every proper submodule N of M containing $\overline{Z}(M)$, there exists a nonzero homomorphism $f : \frac{M}{N} \to M$. We investigate some conditions under which two concepts of coretractable and $\overline{Z}(M)$-coretractable, coincide. For a commutative semiperfect ring R, we show that R is $\overline{Z}(R)$-coretractable if and only if R is a Kasch ring. Some examples are provided to illustrate different concepts.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let M be an R-module and N a submodule of M. We use $End_R(M)$, $ann_r(M)$, $ann_l(M)$ to denote the ring of endomorphisms of M, the right annihilator in R of M and the left annihilator in R of M, respectively. Let M be a module and K a submodule of M. Then K is essential in M denoted by $K \leq_e M$, if $L \cap K \neq 0$ for every nonzero submodule L of M. Dually, K is small in M ($K \ll M$), in case $M = K + L$ implies that $L = M$. We also recall that a module M is a small module in case there is a module L containing M such that $M \ll L$. It is well-known that a module M is small if and only if M is a small submodule

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of its injective hull. Of course, the concept of small submodules has a key role throughout the paper.

A submodule N of a module M is called *supplement* if there is a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. A module M is called *supplemented* if every submodule of M has a supplement. A module M is called *amply supplemented*, in case $M = A + B$ implies A contains a supplement A' of B in M. The reader can find more details about classes of all versions of supplemented modules in [7] and [13].

Let R be a ring and M a right R-module. Recall that M is singular provided that $Z(M) = M$ where $Z(M) = \{x \in M \mid xI = 0, I \leq_e R_R\}$. Suppose that $\mathcal{S}$ denotes the class of all small right R-modules. In [10] the authors defined $\overline{Z}(M)$ as the reject of $\mathcal{S}$ in M, i.e. $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U, U \in \mathcal{S}\}$. In this way, M is called *(non)cosingular*, in case $(\overline{Z}(M) = M)$ $\overline{Z}(M) = 0$. They investigated some general properties of $\overline{Z}(M)$. For a ring R, the submodule $\overline{Z}(R_R)$ $(\overline{Z}({}_RR))$ is a two-sided ideal of R by [3, Corollary 8.23]. Throughout the paper, for every R-module M, we suppose that $\overline{Z}(M) \neq M$ unless otherwise stated.

Khuri in [4] introduced the concept of a retractable module. A module M is retractable in case for every nonzero submodule N of M, there is a nonzero homomorphism $f : M \to N$, i.e $Hom_R(M, N) \neq 0$. Toloee and Vedadi in [11] studied retractable rings and their relations with other known rings. In the literature, there are some works about retractable modules (see [5, 14, 16]). Amini, Ershad and Sharif in [2] defined dual notation namely coretractable modules. A module M is *coretractable* provided that, $Hom_R(\frac{M}{N}, M) \neq 0$ for every proper submodule N of M. There are also some papers whose main subject is coretarctablity of modules. We refer readers to [1, 8, 15] for more information about coretractable modules.

This work is devoted to coretractable modules relative to just an important submodule namely $\overline{Z}(M)$. If in the definition of a coretractable module M, we fix the submodule $\overline{Z}(M)$ and focus just on nonzero homomorphisms from $\frac{M}{K}$ to M where K contains $\overline{Z}(M)$, we have a generalization of coretractable modules. We present some conditions to prove that when two concepts coretractable and $\overline{Z}(M)$-coretractable are equivalent. Among them, we show that if $\overline{Z}(M)$ is δ-small in M or it is a coretractable module, then M is coretractable if and only if M is $\overline{Z}(M)$-coretractable. We show that R_R is $\overline{Z}(R_R)$-coretractable if and only if every simple right R-module that annihilated by $\overline{Z}(R_R)$, can be embedded in R_R. As a consequence, we prove for a commutative semiperfect ring R that, R is a coretractable R-module if and only if R is a Kasch ring.

2. $\overline{Z}(M)$-coretractable modules

In this section we introduce a new generalization of coretractable modules namely, $\overline{Z}(M)$-coretractable modules.

Recall that a module M is *coretractable*, in case for every proper submodule N of M, there exists a nonzero homomorphism $f\colon \dfrac{M}{N} \to M$.

Definition 2.1. Let M be a module. We say M is $\overline{Z}(M)$-coretractable in case for every proper submodule N of M containing $\overline{Z}(M)$, there is a nonzero homomorphism from $\dfrac{M}{N}$ to M.

Example 2.2. (1) Every coretractable module is coretractable relative to its $\overline{Z}$. In particular every semisimple module M is $\overline{Z}(M)$-coretractable.

(2) Let M be a noncosingular module. Then it is obvious that M is $\overline{Z}(M)$-coretractable. In other words, there is a noncosingular module which is not coretractable. Since $Hom_{\mathbb{Z}}(\dfrac{\mathbb{Q}}{\mathbb{Z}}, \mathbb{Q}) = 0$, then as an $\mathbb{Z}$-module $\mathbb{Q}$ is not coretractable. Note that $\mathbb{Q}$ is noncosingular.

Recall from [9] that a ring R is right GV (*generalized V-ring*), in case every simple singular right R-module is injective.

Proposition 2.3. *Let R be a right GV-ring. If M is an indecomposable module with $0 \neq \dfrac{M}{\overline{Z}(M)}$ having a maximal submodule, then M is $\overline{Z}(M)$-coretractable if and only if M is simple projective.*

Proof. Let M be $\overline{Z}(M)$-coretractable. By assumption there is a maximal submodule K of M containing $\overline{Z}(M)$. Now there is a monomorphism $g : \dfrac{M}{K} \to M$, since M is a $\overline{Z}(M)$-coretractable module. It follows that Img is a simple submodule of M. Then Img is either singular or projective. If Img is projective, then K is a direct summand of M and hence $K = 0$ or $K = M$. So that $K = 0$. If Img is singular, it will be injective as R is right GV. Therefore, Img is a summand of M and since $g \neq 0$ we conclude that $Img = M$, a contradiction. The converse is obvious. □

Note that for a cosingular module M, concepts coretractable and $\overline{Z}(M)$-coretractable coincide.

Let M be a module and N a submodule of M. Following [17], N is δ-small in M (denoted by $N \ll_\delta M$), in case $M = N + K$ with $\frac{M}{K}$ singular implies that $M = K$. Note that by definitions, every small submodule of M is δ-small in M. The sum of all δ-small submodules of M is denoted by $\delta(M)$. Also $\delta(M)$ is the reject of the class of all simple singular modules in M.

Lemma 2.4. *Let M be a module. In each of the following cases M is $\overline{Z}(M)$-coretractable if and only if M is coretractable.*
(1) $\overline{Z}(M) \ll_\delta M$ $(\overline{Z}(M) \ll M)$.
(2) $\overline{Z}(M)$ *is a coretractable module.*

Proof. (1) We shall prove the δ case. The other follows immediately. Let M be $\overline{Z}(M)$-coretractable and K a proper submodule of M. Suppose that $M \neq \overline{Z}(M) + K$. Since M is $\overline{Z}(M)$-coretractable, there is a homomorphism $f : \frac{M}{(\overline{Z}(M)+K)} \to M$. So that $fo\pi : \frac{M}{K} \to M$ is the required homomorphism where $\pi : \frac{M}{K} \to \frac{M}{(\overline{Z}(M)+K)}$ is natural epimorphism. Otherwise, $M = \overline{Z}(M)+K$. It follows from [17, Lemma 1.2], there is a decomposition $M = Y \oplus K$ where Y is a semisimple projective submodule of $\overline{Z}(M)$. Therefore, there is a monomorphism from $\frac{M}{K}$ to M since K is a direct summand of M. Therefore, M is coretractable. The converse is clear.

(2) Let K be a proper submodule of M. Then $K + \overline{Z}(M) \neq M$ or $K + \overline{Z}(M) = M$. If first one happens, then similar to (1), we will have required nonzero homomorphism. Now suppose that $K + \overline{Z}(M) = M$. Then $h : \frac{M}{K} \to \frac{\overline{Z}(M)}{(\overline{Z}(M) \cap K)}$ is an isomorphism induced from $M = \overline{Z}(M) + K$. Since $\overline{Z}(M)$ is coretractable, there is a nonzero homomorphism $g : \frac{\overline{Z}(M)}{(\overline{Z}(M) \cap K)} \to \overline{Z}(M)$. Therefore, $jogoh : \frac{M}{K} \to M$ is a nonzero homomorphism where $j : \overline{Z}(M) \to M$ is the inclusion. □

Proposition 2.5. *Let M be a module such that $\frac{M}{\overline{Z}(M)}$ is coretractable. If $\frac{M}{\overline{Z}(M)}$ can be embedded in M (for example, $\frac{M}{\overline{Z}(M)}$ is semisimple and $\overline{Z}(M)$ is a direct summand of M), then M is $\overline{Z}(M)$-coretractable.*

Proof. Let K be a proper submodule of M containing $\overline{Z}(M)$. Then $\dfrac{K}{\overline{Z}(M)}$ is a proper submodule of $\dfrac{M}{\overline{Z}(M)}$. Since $\dfrac{M}{\overline{Z}(M)}$ is coretractable, there is a nonzero homomorphism $g : \dfrac{M}{K} \to \dfrac{M}{\overline{Z}(M)}$. Because, $\dfrac{M}{\overline{Z}(M)}$ can be embedded in M, we conclude that there will be a nonzero homomorphism from $\dfrac{M}{K}$ to M. □

Let M be a module and $K \leq M$. We say M is $\overline{Z}(K)$-*coretractable* if for every proper submodule T of M containing $\overline{Z}(K)$, there is a nonzero homomorphism $g : \dfrac{M}{T} \to M$.

Proposition 2.6. *Let $M = M_1 \oplus \ldots \oplus M_n$. If each M_i is $\overline{Z}(M_i)$-coretractable, then M is $\overline{Z}(M)$-coretractable.*

Proof. The proof is exactly similar to proof of [2, Proposition 2.6]. Note that $\overline{Z}(M_1 \oplus \ldots \oplus M_n) = \overline{Z}(M_1) \oplus \ldots \oplus \overline{Z}(M_n)$. □

Lemma 2.7. (1) *Let $M = \bigoplus_{i=1}^{n} M_i$ be a $\overline{Z}(M_i)$-coretractable module for at least one $i \in \{1, \ldots, n\}$. Then M is $\overline{Z}(M)$-coretractable.*

(2) *Let M be $\overline{Z}(M)$-coretractable. If $\overline{Z}(M)$ contains no nonzero image of any endomorphism of M, then $\dfrac{M}{\overline{Z}(M)}$ is coretractable.*

(3) *Let M be $\overline{Z}(M)$-coretractable. If $\dfrac{M}{\overline{Z}(M)}$ has a maximal submodule, then $Soc(M) \neq 0$. In particular, if M is a finitely generated $\overline{Z}(M)$-coretractable module, then $Soc(M) \neq 0$.*

Proof. (1) This is straightforward.

(2) Let $\dfrac{T}{\overline{Z}(M)}$ be a proper submodule of $\dfrac{M}{\overline{Z}(M)}$. Then $\overline{Z}(M) \subseteq T \subset M$. Since M is $\overline{Z}(M)$-coretractable, there exists a nonzero homomorphism $g : \dfrac{M}{T} \to M$. Now define $h : \dfrac{\frac{M}{\overline{Z}(M)}}{\frac{T}{\overline{Z}(M)}} \to \dfrac{M}{\overline{Z}(M)}$ by $h(x + \overline{Z}(M) + \dfrac{T}{\overline{Z}(M)}) = g(x + T)$ for every $x \in M$. If $Imh = \overline{Z}(M)$, then $Img \subseteq \overline{Z}(M)$, a contradiction. So that, $\dfrac{M}{\overline{Z}(M)}$ is coretractable.

(3) Let $\frac{K}{\overline{Z}(M)}$ be a maximal submodule of $\frac{M}{\overline{Z}(M)}$. Then K is a maximal submodule of M also containing $\overline{Z}(M)$. So there is a $h : \frac{M}{K} \to M$. It follows that Imh is a simple submodule of M.

□

Let M be a module and $N \leq M$. Then N is called *fully invariant*, if for every $f \in End_R(M)$, $f(N) \subseteq N$. There are some well-known fully invariant submodules of a module M such as $Rad(M), Soc(M), \overline{Z}(M)$.

Proposition 2.8. (1) *Let M be a module, $K, L \leq M$ with $\overline{Z}(L) = L$ and K is a fully invariant supplement of L in M. If M is $\overline{Z}(L)$-coretractable, then K is coretractable.*

(2) *Let M be a module such that $\overline{Z}(M)$ has a fully invariant supplement K in M. If $\overline{Z}^2(M) = \overline{Z}(M)$ and M is $\overline{Z}(M)$-coretractable, then K is coretractable.*

Proof. (1) Let N be a proper submodule of K. Consider the submodule $N + \overline{Z}(L)$ of M. If $N + \overline{Z}(L) = M$, then by modularity $N + (K \cap \overline{Z}(L)) = K$ which implies that $N = K$, a contradiction (note that $K \cap \overline{Z}(L) \subseteq K \cap L \ll K$). It follows that $N + \overline{Z}(L)$ is a proper submodule of M. Being M, $\overline{Z}(L)$-coretractable, implies that there is nonzero homomorphism $g : \frac{M}{(N+\overline{Z}(L))} \to M$. Now $(go\pi)(K) \subseteq K$ as K is fully invariant where $\pi : M \to \frac{M}{N+\overline{Z}(L)}$ is natural epimorphism. Define the homomorphism $h : \frac{K}{N} \to K$ by $h(x+N) = g(x+N+\overline{Z}(L))$. Since g is nonzero, there is a $x \in M \setminus (N + \overline{Z}(L))$ such that $g(x + N + \overline{Z}(L)) \neq 0$. Set $x = k+l$ where $k \in K$ and $l \in L$. To contrary, suppose that $k \in N$. Now $x \notin N+L$ implies that $l \notin L$, which is a contradiction. Therefore, $h(k + N) = g(k + l + N + \overline{Z}(L)) = g(x + N + \overline{Z}(L)) \neq 0$. Hence K is coretractable.

(2) This case is a direct consequence of (1).

□

Let M be an R-module. A submodule K is said to be *dense* in M if, for any $y \in M$ and $0 \neq x \in M$, there exists $r \in R$ such that $xr \neq 0$ and $yr \in K$. Obviously, any dense submodule of M is essential. It follows from [6, Proposition 8.6] that, K is dense in M if and only if $Hom_R(\frac{P}{K}, M) = 0$ for every submodule $K \subseteq P \subseteq M$.

Remark 2.9. Let M be a module such that $\overline{Z}(M) \neq M$. If $\overline{Z}(M)$ is dense in M, then M is not $\overline{Z}(M)$-coretractable. In fact for a $\overline{Z}(M)$-coretractable module M with $\overline{Z}(M) \neq M$, we have $\overline{Z}(M)$ is not dense in M. This follows from the fact that if M is $\overline{Z}(M)$-coretractable such that $\overline{Z}(M) \neq M$, then there is a nonzero homomorphism from $\frac{M}{\overline{Z}(M)}$ to M.

Proposition 2.10. *Let M be a module such that $\overline{Z}(M) \neq M$. If M is quasi-injective or every proper submodule of M is contained in a maximal submodule, then M is $\overline{Z}(M)$-coretractable if and only if every proper submodule of M containing $\overline{Z}(M)$ is not dense in M.*

Proof. (1) Let M be a quasi-injective module such that every proper submodule of M containing $\overline{Z}(M)$ is not dense in M. Suppose that K is a proper submodule of M containing $\overline{Z}(M)$. Since K is not dense in M, there is a $f : \frac{P}{K} \to M$ where P is a submodule of M containing K. It follows that $fo\pi : P \to M$ is a nonzero homomorphism such that $\pi : P \to \frac{P}{K}$ is natural epimorphism. Consider inclusion homomorphism $j : P \to M$. Since M is quasi-injective, there exists $h : M \to M$ such that $hoj = fo\pi$. By defining $\overline{h} : \frac{M}{K} \to M$ with $\overline{h}(m + K) = h(m)$ we conclude that M is $\overline{Z}(M)$-coretractable. Note that $\overline{h}$ is nonzero. Conversely, if M is $\overline{Z}(M)$-coretractable and $\overline{Z}(M) \subseteq K < M$, then there is a homomorphism $g : \frac{M}{K} \to M$ which shows that K is not dense in M.

(2) Suppose that every proper submodule of M contained in a maximal submodule of M. Let $\overline{Z}(M) \subseteq K \subset M$. Then there is a maximal submodule L of M such that $K \leq L$. Since L is not dense in M, there is a nonzero homomorphism $h : \frac{M}{L} \to M$. Since $f : \frac{M}{K} \to \frac{M}{L}$ with $f(x + K) = x + L$ is a nonzero homomorphism, then hof is nonzero. It follows that M is $\overline{Z}(M)$-coretractable. The converse is the same as the converse of (1). □

Theorem 2.11. *Let R be a ring. Then the following are equivalent:*

(1) *R_R is $\overline{Z}(R_R)$-coretractable;*

(2) *Every finitely generated free right R-module F is $\overline{Z}(F)$-coretractable;*

(3) *For every right ideal I containing $\overline{Z}(R_R)$, $ann_l(I) \neq 0$;*

(4) *Every simple right R-module annihilated by $\overline{Z}(R_R)$ can be embedded in R_R.*

Proof. (1) $\Leftrightarrow$ (2) Follows from Proposition 2.6.

(1) $\Rightarrow$ (3) Let I be a right ideal containing $\overline{Z}(R_R)$. Since R_R is $\overline{Z}(R_R)$-coretractable, there is a nonzero homomorphism $f : \frac{R}{I} \to R$. Consider the endomorphism $g = fo\pi : R \to R$ where π is the natural epimorphism from R to $\frac{R}{I}$. Then there is an element $a \in R$ such that $g(x) = ax$. Let $y \in I$. Then $g(y) = ay = 0$ as $I \subseteq Kerg$.

(3) $\Rightarrow$ (1) Let I be a right ideal containing $\overline{Z}(R_R)$. Since $ann_l(I) \neq 0$, there exists an element of R such as a which $aI = 0$ and $a \neq 0$. Define $f : \frac{R}{I} \to R$ by $f(x + I) = ax$. It is easy to check that f is an R-homomorphism and in particular $f \neq 0$.

(1) $\Rightarrow$ (4) Let $M \cong \frac{R}{K}$ be a simple right R-module such that $M\overline{Z}(R_R) = 0$. It follows that $\overline{Z}(R_R) \subseteq K$. Since R is $\overline{Z}(R_R)$-coretractable, there is a nonzero homomorphism $f : \frac{R}{K} \to R$.

(4) $\Rightarrow$ (1) Let T be a proper right ideal of R containing $\overline{Z}(R_R)$. Now there exists a right maximal ideal K of R such that $\overline{Z}(R_R) \subseteq T \subseteq K$. Consider the simple right R-module $M = \frac{R}{K}$. Since $M\overline{Z}(R_R) = 0$, there is a nonzero homomorphism $g : \frac{R}{K} \to R$ by assumption. Being T a submodule of K, there exists $f : \frac{R}{T} \to \frac{R}{K}$ defined by $f(x+T) = x+K$. Hence gof is the desired homomorphism. □

Remark 2.12. Let R be a ring with $ann_l(\overline{Z}(R_R)) = 0$. Then R_R is not $\overline{Z}(R_R)$-coretractable. By [12, Proposition 2.1], $J(R) \subseteq ann_l(\overline{Z}(R_R))$. So $J(R) = 0$.

Corollary 2.13. *Let R be a semiperfect ring with $\overline{Z}(R_R) \neq R$. Then the following statements are equivalent:*

(1) *R is $\overline{Z}(R_R)$-coretractable;*

(2) *Every simple cosingular right R-module can be embedded in R_R.*

Proof. (1) $\Rightarrow$ (2) It follows from (1) $\Rightarrow$ (4) of Theorem 2.11 and the fact that over a semiperfect ring, a simple module is annihilated by $\overline{Z}(R_R)$ if and only if it is cosingular ([10, Theorem 3.5]).

(2) $\Rightarrow$ (1) This is a consequence of (4) $\Rightarrow$ (1) of Theorem 2.11 and the fact the over a semiperfect ring, a simple module is annihilated by $\overline{Z}(R_R)$ if and only if it is cosingular. □

Recall from [6], a ring R is right (left) *Kasch* in case every simple right (left) R-module can be embedded in R_R ($_RR$). In [2, Theorem 2.14], the authors proved that R is right Kasch if and only if R_R is coretractable. The following maybe an analogue for commutative semiperfect rings. We should note that a ring R is *semilocal* in case $\frac{R}{J(R)}$ is a semisimple ring.

Corollary 2.14. *Let R be a commutative semiperfect ring with $\overline{Z}(R) \neq R$. Then the following statements are equivalent:*

(1) *R is $\overline{Z}(R)$-coretractable;*
(2) *Every simple cosingular R-module can be embedded in R;*
(3) *R is a Kasch ring.*

Proof. (1) $\Leftrightarrow$ (2) See Corollary 2.13.

(1) $\Rightarrow$ (3) From [12, Corollary 2.7(3)], we have $Soc(R) = \overline{Z}(R)$ since R is a commutative semilocal ring. Now let K be a proper essential ideal of R. Then $Hom_R(\frac{R}{K}, R) \neq 0$ because $\overline{Z}(R) \subseteq K$. Therefore, R is a coretractable R-module. Hence R is a Kasch ring (see [2, Theorem 2.14]).

(3) $\Rightarrow$ (1) In this case R is a coretractable R-module and hence $\overline{Z}(R)$-coretractable. □

Example 2.15. (1) Let $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ where K is a field. Then $J(R) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$. It is easy to check that R is a semilocal ring as $\frac{R}{J(R)} \cong K \times K$ which is a semisimple ring. Now by [3, Exercise 10, Page 113] and [12, Corollary 2.7], $\overline{Z}(R_R) = Soc(_RR) = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$. However, $\overline{Z}(_RR) = Soc(R_R) = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$. Set $m_1 = \overline{Z}(R_R)$ and $m_2 = \overline{Z}(_RR)$. Then both m_1 and m_2 are left maximal and right maximal ideals of R. A quick calculation shows that $ann_l(m_1) = m_2$, $ann_l(m_2) = 0$, $ann_r(m_1) = 0$ and $ann_r(m_2) = m_1$. Now by Theorem 2.11, R_R is $\overline{Z}(R_R)$-coretractable while R_R is not $\overline{Z}(_RR)$-coretractable. Also left version of Theorem 2.11, implies that $_RR$ is $\overline{Z}(_RR)$-coretractable but it is not $\overline{Z}(R_R)$-coretractable. Since the simple right R-module $\frac{R}{m_2}$ can not be embedded in R_R and the simple left R-module $\frac{R}{m_1}$ can not be embedded in $_RR$, the ring R is neither right Kasch nor left Kasch.

(2) Let K be a division ring and

$$R = \left\{ A = \begin{bmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \mid a, b, c, d, e \in K \right\}$$

.

Then, $J(R) = \{A \in R \mid a = 0 = e\}$, $Soc(R_R) = ann_l(J(R)) = \{A \in R \mid a = 0\}$, $Soc({}_RR) = ann_r(J(R)) = J(R)$. Since $\frac{R}{J(R)} \cong K \times K$, R is a semilocal ring. Now from [12, Corollary 2.7], we have $\overline{Z}({}_RR) = Soc(R_R) = \{A \in R \mid a = 0\}$ and $\overline{Z}(R_R) = Soc({}_RR) = J(R)$. From [6, Example 8.29], $\overline{Z}({}_RR)$ is a left maximal and right maximal ideal of R. Since $ann_r(\overline{Z}({}_RR)) = \{A \in R \mid a = e = 0\} = J(R) \neq 0$, it follows from [6, Corollary 8.28], $\frac{R}{\overline{Z}({}_RR)}$ can be embedded in ${}_RR$ (see also Theorem 2.11). Therefore, ${}_RR$ is $\overline{Z}({}_RR)$-coretractable. Now an easy computation shows that $ann_l(\overline{Z}({}_RR)) = \{A \in R \mid a = c = d = e = 0\} \neq 0$. So $\frac{R}{\overline{Z}({}_RR)}$ can be embedded in R_R by [6, Corollary 8.28]. As $\overline{Z}({}_RR)$ is a maximal right ideal of R, then R_R is $\overline{Z}({}_RR)$-coretractable. Also from [6, Example 8.29], R is a right Kasch ring while it is not a left Kasch ring.

(3) Let K be a field and $R = K \times K \times K \times \ldots$. It is well-known that R is a Von Nuemann regular V-ring. By [10, Corollary 2.6], every R-module is noncosingular. So every R-module M is $\overline{Z}(M)$-coretractable. In particular R as a ring is $\overline{Z}(R)$-coretractable. Now consider the ideal $I = K \oplus K \oplus \ldots$ of R. Then $ann(I) = 0$ and of course $ann(m) = 0$ for every maximal ideal m of R containing I. Hence the simple R-module $\frac{R}{m}$ can not be embedded in R (see [6, Corollary 8.28]). Therefore, R is not a Kasch ring.

Proposition 2.16. *Let R be a ring such that every free right R-module F is $\overline{Z}(F)$-coretractable. Then for every nonzero cosingular right R-module M, $Hom_R(M, R) \neq 0$.*

Proof. Let M be a cosingular right R-module. Then there is a free right R-module F and a submodule K of F such that $M \cong \frac{F}{K}$. Since M is cosingular, $\overline{Z}(F) \subseteq K$. Now there is a nonzero homomorphism $f : \frac{F}{K} \to F$ (note that F is $\overline{Z}(F)$-coretractable). The homomorphism

$\pi o f : M \to R$ is the required one where $\pi : F \to R$ is natural epimorphism. □

Proposition 2.17. *Let R be a ring having a radical right R-module M with $\overline{Z}(M) \neq M$. If for every right ideal I of R, $Rad(I) \neq I$, then there is a free right R-module F which is not $\overline{Z}(F)$-coretractable.*

Proof. Let $Rad(M) = M$ and $\overline{Z}(M) \neq M$. There exists a free right R-module F and a submodule K of F such that $\frac{M}{\overline{Z}(M)} \cong \frac{F}{K}$. Being M radical implies that $\frac{M}{\overline{Z}(M)}$ is radical. So, $Hom_R(\frac{M}{\overline{Z}(M)}, R) = 0$. It follows that $Hom_R(\frac{F}{K}, F) = 0$. Now being $\frac{F}{K}$ cosingular implies that $\overline{Z}(F) \subseteq K$ (note that $\frac{M}{\overline{Z}(M)}$ is cosingular). Therefore, F is not $\overline{Z}(F)$-coretractable. □

Corollary 2.18. *Let R be a semiperfect ring which is not right perfect. If R has a radical module, then there is a free right R-module F which is not $\overline{Z}(F)$-coretractable.*

Proposition 2.19. *Let M be an amply supplemented module such that every proper submodule of $0 \neq \frac{M}{\overline{Z}(M)}$ is contained in a maximal submodule. If for every $x \in M$, the module xR is $\overline{Z}(xR)$-coretractable, then M is $\overline{Z}(M)$-coretractable.*

Proof. Let M be amply supplemented. Suppose that K is a submodule of M containing $\overline{Z}(M)$. By assumption, K is contained in a maximal submodule L of M. For every $x \in M \setminus L$, we know $\frac{M}{L} \cong \frac{xR}{xR \cap L}$ as $xR + L = M$. Note that $\frac{M}{L}$ is cosingular. Otherwise, $\frac{M}{L} = \overline{Z}(\frac{M}{L}) = \overline{Z}^2(\frac{M}{L}) = \frac{\overline{Z}^2(M) + L}{L} = 0$, which is a contradiction (see [10, Theorem 3.5]). Now $\overline{Z}(xR) \subseteq xR \cap L$. Because xR is $\overline{Z}(xR)$-coretractable, $Hom_R(\frac{xR}{xR \cap L}, xR) \neq 0$. Hence there is a nonzero homomorphism $f : \frac{M}{L} \to M$. Therefore, $Hom_R(\frac{M}{K}, M) \neq 0$ as $K \subseteq L$. □

The following result follows from Proposition 2.19 and the fact that over a (semiperfect) right perfect ring, every (finitely generated) right R-module is amply supplemented.

Corollary 2.20. *Let R be a (semiperfect) right perfect ring such that every cyclic R-module xR is $\overline{Z}(xR)$-coretractable. Then every (finitely generated) right R-module M is $\overline{Z}(M)$-coretractable.*

Corollary 2.21. *Let R be a commutative (semiperfect) perfect ring such that every cyclic R-module xR is $\overline{Z}(xR)$-coretractable. Then every (finitely generated) projective R-module is coretractable. In particular, R is a Kasch ring.*

Proof. From Corollary 2.20, every (finitely generated) projective R-module M is $\overline{Z}(M)$-coretractable. It follows from [12, Corollary 2.7(3)], $Soc(M) = \overline{Z}(M)$ for every (finitely generated) projective R-module. It is clear that for every proper essential submodule N of M and hence for every proper submodule N of M, there is a nonzero homomorphism $f : \dfrac{M}{N} \to M$ (note that if $N \leq_e M$, then $Soc(M) \subseteq N$). This completes the proof. □

Definition 2.22. Let $\mathcal{SC}$ be the class of all simple cosingular (small) right R-modules. Then we set $\overline{wZ}(R_R) = Rej_R(\mathcal{SC})$. By [3, Corollary 8.23], $\overline{wZ}(R_R)$ is a two-sided ideal of R.

Example 2.23. (1) Since every simple cosingular right $\mathbb{Z}$-module has the form $\dfrac{\mathbb{Z}}{p\mathbb{Z}}$ where p is a prime number, then $\overline{wZ}(\mathbb{Z}) = 0$.

(2) Let R be a local ring which is not a V-ring. Then the only simple cosingular right R-module is $\dfrac{R}{J(R)}$. So $\overline{wZ}(R_R) = J(R)$.

(3) Let R be a local ring with at least three proper ideals. Then by [12, Corollary 2.7(1)], $\overline{Z}(R_R) = Soc(_RR)$. By (2), we have $\overline{wZ}(R_R) = J(R)$. Note that $\overline{Z}(R_R) \subseteq \overline{wZ}(R_R)$. For instance $\overline{Z}(\mathbb{Z}_8) = \{0, 4\}$ while $\overline{wZ}(\mathbb{Z}_8) = \{0, 2, 4, 6\}$.

Some basic properties of $\overline{wZ}(R_R)$ are listed below. The proof is straightforward and omitted.

Lemma 2.24. *Let R be a ring. Then;*

(1) *$\overline{Z}(R_R) \subseteq \overline{wZ}(R_R)$ and $J(R) \subseteq \overline{wZ}(R_R)$.*

(2) *$\dfrac{R}{\overline{wZ}(R_R)}$ is a cosingular right R-module.*

(3) *$\overline{wZ}(R_R) = R$ if and only if R is a right V-ring.*

(4) *$\overline{wZ}(R_R)$ is the largest right ideal of R that annihilates all simple cosingular right R-modules.*

(5) *If R is semilocal, then $\dfrac{R}{\overline{wZ}(R_R)}$ is semisimple cosingular.*

Proposition 2.25. *Let R be a ring with $J(R) = 0$. If R_R is $\overline{Z}(R_R)$-coretractable, then $Soc(R_R) + \overline{wZ}(R_R) = R$. In particular, if $\overline{wZ}(R_R)$ is semisimple, then R is semisimple.*

Proof. In contrary, suppose that $I = Soc(R_R) + \overline{wZ}(R_R) \neq R$. Since I contains $\overline{wZ}(R_R)$ and R_R is $\overline{Z}(R_R)$-coretractable, we have $K = ann_l(I) \neq 0$. It follows that $(IK)(IK) = 0$. Now $J(R) = 0$, implies that $IK = 0$. Since R_R is $\overline{Z}(R_R)$-coretractable, every simple cosingular right R-module can be embedded in R_R. It follows that $MK = 0$ for every simple cosingular right R-module. Hence $K \subseteq \overline{wZ}(R_R)$. Since $\overline{wZ}(R_R)K = 0$, we conclude that $K^2 = 0$. Therefore $K \subseteq J(R) = 0$, which is a contradiction. For the last part, suppose that $\overline{wZ}(R_R)$ is semisimple. So, $I = Soc(R_R) = R$. This completes the proof. □

Corollary 2.26. *Let R be a ring with $J(R) = 0$ and $Soc(R_R) \subseteq \overline{wZ}(R_R)$. If R_R is $\overline{Z}(R_R)$-coretractable, then R is a right V-ring.*

Proof. From the proof of last proposition, we get $I = \overline{wZ}(R_R) = R$. Then, every simple right R-module is injective. It then follows that R is a right V-ring. □

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A GENERALIIZATION OF CORETRACTABLE MODULES

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یک تعمیم از مدول‌های مسطح

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فرض کنید R یک حلقه و M یک R-مدول راست باشد. مدول M را مسطح نسبت به $\overline{Z}(M)$ ($\overline{Z}(M)$-مسطح) می‌گوییم هرگاه برای هر زیرمدول محض مانند N از M که شامل $\overline{Z}(M)$ است، یک همریختی غیرصفر مانند $f: \frac{M}{N} \to M$ موجود باشد. ما در این مقاله شرایطی را بررسی می کنیم که دو مفهوم مسطح و $\overline{Z}(M)$-مسطح با هم معادل باشند. برای یک حلقهٔ جابجایی نیمه‌کامل مانند R نشان می‌دهیم که R نسبت به $\overline{Z}(R)$ مسطح است اگر وتنها اگر R یک حلقهٔ کَش باشد. در نهایت چند مثال نیز برای توضیح و بیان تفاوت مفاهیم متفاوت ارائه می‌دهیم.

کلمات کلیدی: مدول مسطح، مدول $\overline{Z}(M)$-مسطح، حلقهٔ کَش.