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THE LATTICE OF CONGRUENCES ON A TERNARY SEMIGROUP

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Abstract. In this paper, we investigate some properties of congruences on ternary semigroups. We also define the notion of congruence on a ternary semigroup generated by a relation and we determine the method of obtaining a congruence on a ternary semigroup T from a relation R on T. Furthermore, we study the lattice of congruences on a ternary semigroup and we show that this lattice is not generally modular, it is not even semimodular. Then, we indicate some conditions under which this lattice is modular.

1. Introduction

The theory of ternary algebraic systems was introduced by D. H. Lehmer [5] in 1932, but before that such structures were studied by E. Kanser [3], who gave the idea of n-ary algebras. Lehmer [5] studied certain ternary algebraic systems called triplexes, commutative ternary groups in fact. Ternary structures and their generalization, the so called n-ary structures, are outstanding for their application in physics. The notion of ternary semigroup was known for the first time by S. Banach. By bringing an example, he showed that a ternary semigroup does not necessarily reduce to an ordinary semigroup ($T = \{-i, i\}$ is a ternary semigroup under the multiplication over complex numbers while T is not an ordinary semigroup under complex number multiplication). J. Los [6], studied some properties of ternary semigroups and

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he proved that every ternary semigroup can be embedded in an ordinary semigroup. The notion of congruence was first introduced by Karl Fredrich Gauss in the beginning of the nineteenth century. Congruences are a special type of equivalence relations which play a vital role in the study of quotient structures of different algebraic structures. In 1997, V. N. Dixit and S. Dewan [1], represented the concept of congruences on a ternary semigroups and they studied some interesting properties of them. In 2007, S. Kar and B. K. Maity [4], introduced some concepts such as cancelative congruences, group congruences and Rees congruences. They also investigated these congruences in ternary semigroups. In this paper, we define the notion of congruence on a ternary semigroup generated by a relation and we determine the method of obtaining a congruence on a ternary semigroup T from a relation R on T, when ternary semigroup T admits an element 1 as an identity. Making of congruences is important because we can gain new ternary semigroups (in fact quotiont semigroups) from them. We also study some properties of congruences on ternary semigroups. Moreover, we study the lattice of congruences on a ternary semigroup. By giving an example, we show that this lattice is not generally modular, it is not even semimodular. However, if every element of a ternary semigroup T is invertible, then the lattice of congruences on T is a modular lattice. We start with some elemantary notions that we need them in the next sections.

Definition 1.1. A non-empty set T is called a ternary semigroup if there exists a ternary operation $T \times T \times T \to T$, written as $(a, b, c) \to abc$ satisfying the following statement:

$$(abc)de = a(bcd)e = ab(cde) \text{ for all } a, b, c, d, e \in T.$$

Remark 1.2. Let T be a ternary semigroup, $m, n \in \mathbb{N}$ $(m \leq n)$ and $x_1, x_2, \ldots, x_{2n+1} \in T$. Then we can write

$$(x_1 x_2 ... x_{2n+1}) = (x_1 ... ((x_m x_{m+1} x_{m+2}) x_{m+3} x_{m+4}) ... x_{2n+1})$$

Example 1.3. Let $\mathbb{Z}_0^-$ be the set of non-positive integers. Then together with usual ternary multiplication of integers, $\mathbb{Z}_0^-$ forms a ternary semigroup.

Remark 1.4. Let S be an ordinary semigroup under the binary operation $(a, b) \mapsto a{*}b$. Then S with the ternary operation $(a, b, c) \mapsto (a{*}b){*}c$ is a ternary semigroup, while a ternary semigroup does not necessarily reduce to an ordinary semigroup.

Definition 1.5. An element e of a ternary semigroup T is called;
(i) a left identity (left unital element) if $eex = x$ for all $x \in T$;
(ii) a right identity (right unital element) if $xee = x$ for all $x \in T$;
(iii) a lateral identity (lateral unital element) if $exe = x$ for all $x \in T$;
(iv) a two-sided identity (bi-unital element) if $eex = xee = x$ for all $x \in T$;
(v) an identity (unital element) if $eex = exe = xee = x$ for all $x \in T$.

Remark 1.6. There is no need any ternary semigroup to have unique identity. For example, the set of all integers $\mathbb{Z}$, with usual ternary multiplication of integers is a ternary semigroup and both of 1 and -1 are identity elements of $\mathbb{Z}$.

Definition 1.7. A ternary semigroup T is called a ternary monoid if it has an identity.

Definition 1.8. An element a of a ternary semigroup T is said to be invertible in T if there exists an element b in T such that $abx = bax = xab = xba = x$, for all $x \in T$.

Definition 1.9. Let X be a set. Then every subset ρ of the cartesian product $X \times X$ is called a (binary) relation on X. We denote the set of all binary relations on X by B_X and we define the binary operation o on B_X by the rule that, for all ρ, $\sigma \in B_X$,

$$\rho o \sigma = \{(x, y) \in X \times X \mid (x, z) \in \rho \text{ and } (z, y) \in \sigma \text{ for some } z \in X\}.$$

It is clear that (B_X, o) is a ternary semigroup.

Definition 1.10. For every $\rho \in B_X$, we denote $\rho o \rho$ by ρ^2, $\rho o \rho o \rho$ by ρ^3, etc. We also denote the set $\{(x, y) \in X \times X \mid (y, x) \in \rho\}$ by ρ^{-1}. Also, let R be a relation on X. Then, the smallest equivalence on X containing R (the intersection of all equivalence relations on X containing R) is called the equivalence relation on X generated by R and it is denoted by R^e. Moreover, if R is a reflexive relation on X then we denote $\cup_{n\geq 1} R^n$ by R^∞, and we call it the transitive closure of the relation R.

Proposition 1.11. *For every relation R on a set X, we have $R^e = (R \cup R^{-1} \cup 1_X)^\infty$.*

Proof. See [2, Proposition 1.4.9]. □

Corollary 1.12. *Let R be a relation on a set X. Then $(x, y) \in R^e$ if and only if either $x = y$ or for some $n \in \mathbb{N}$, there is a sequence $x = z_1, z_2, \ldots, z_{n-1}, z_n = y$ of elements of T such that, for each $i \in \{1, 2, \ldots, n-1\}$, either $(z_i, z_{i+1}) \in R$ or $(z_{i+1}, z_i) \in R$.*

We denote by $E(X)$, the set of all equivalence relations on set X. By [2, Section 5] we can see that $(E(X), \subseteq, \cap, \cup^e)$ is a complete lattice, for every set X.

2. Congruence on a ternary semigroup generated by a relation

In this section, we investigate some properties of congruences on ternary semigroups. Furthermore we try to obtain a congruence on a ternary semigroup T from a relation R on T, when semigroup T admits an identity.

Definition 2.1. A relation ρ on a ternary semigroup T is said to be,
(i) a left compatible relation if for every $a, b \in T$, $a\rho b$ implies $at_1t_2\rho bt_1t_2$ for all $t_1, t_2 \in T$;
(ii) a right compatible relation if for every $a, b \in T$, $a\rho b$ implies $t_1t_2a\rho t_1t_2b$ for all $t_1, t_2 \in T$;
(iii) a lateral compatible relation if for every $a, b \in T$, $a\rho b$ implies $t_1at_2 \, \rho \, t_1bt_2$ for all $t_1, t_2 \in T$;
(iv) a compatible relation if for all $a, b, c, a', b', c' \in T$, $a\rho a', b\rho b', c\rho c'$ imply $abc \, \rho \, a'b'c'$.

We note that a compatible relation on a ternary semigroup may not be a left (right, lateral) compatible relation.

Example 2.2. Let $T = \{a, b, c, d\}$. Define a ternary operation $[\ ]$ on T as $[abc] = a\cdot b\cdot c$, where $\cdot$ is the binary operation defined as following:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	a	a	a
d	a	a	c	d

Then $(T, [\ ])$ is a ternary semigroup. Consider the relation $\rho = \{(a, d)\}$ on T. Then ρ is compatible, since $a^n = a$ and $d^n = d$ for every n, but ρ is not a left (right, lateral) compatible relation, because $(bda, bdd) = (a, b) \notin \rho$ $((adc, ddc) = (a, c) \notin \rho$, $(bad, bdd) = (a, b) \notin \rho)$.

Example 2.3. Let T be a set such that $|T| > 3$ and let 0 be a fixed element of T. Then, T with the ternary operation defined by

$$xyz = \begin{cases} x & \text{if } x = y = z \\ 0 & \text{otherwise} \end{cases},$$

is a ternary semigroup. Suppose that $0 \neq x \in T$ and consider relation $\rho = \{(x, x)\}$. Then, ρ is a compatible relation on T, but ρ is not a left (right, lateral) compatible relation.

In the following example, we will see that a left, right and lateral compatible relation on a ternary semigroup may not be compatible.

Example 2.4. Let $A = \{1, 2\}$ and $T = A \times A$. Then, under ternary operation $(x_1, x_2)(y_1, y_2)(z_1, z_2) = (x_1, z_2)$, T is a ternary semigroup. Consider the relation $R = 1_T \cup \{(x, y), (x, z), (y, t), (z, t)\}$ where $x = (1, 1)$, $y = (1, 2)$, $z = (2, 1)$ and $t = (2, 2)$. It is easy to see that R is left, right and lateral compatible, while it is not a compatible relation, because $x\, R\, z$ and $x\, R\, y$ but $(x, t) = (xxx, zzy) \notin R$

Due to the previous examples, a compatible relation on a ternary semigroup T is not generally left (right, lateral) compatible and a left, right and lateral compatible relation on a ternary semigroup may not be a compatible. However, we have the following proposition.

Proposition 2.5. *Let S be a ternary semigroup and let R, C^l, C^m, C^r, C and T denote respectively, the set of reflexive, left compatible, lateral compatible, right compatible, compatible and transitive relation on S. Then, we have:*
(1) $C \cap R \subseteq (C^l \cap C^m \cap C^r) \cap R$.
(2) $(C^l \cap C^m \cap C^r) \cap T \subseteq C \cap T$.

Proof. (1) Let $\rho \in C \cap R$. Then, ρ is a reflexive and compatible relation on S. Suppose that $a\rho b$ and $s, t \in S$ are arbitrary elements. Then, by reflexivity, $s\rho s$ and $t\rho t$ and hence $sta\rho stb$, $ast\rho bst$ and $sat\rho sbt$. Thus, ρ is a left, right and lateral compatible relation on S. Consequently, $\rho \in (C^l \cap C^m \cap C^r) \cap R$.

(2) Let $\rho \in (C^l \cap C^m \cap C^r) \cap T$. Then, ρ is a left, right and lateral compatible relation on S and ρ is transitive. Suppose that $a\rho a', b\rho b'$ and $c\rho c'$. Then, $abc\, \rho\, a'bc, a'bc\, \rho\, a'b'c$ and $a'b'c\, \rho\, a'b'c'$. This implies $abc\, \rho\, a'b'c'$. Thus, ρ is compatible. Consequently, $\rho \in C \cap T$. □

Corollary 2.6. *Let S be a ternary semigroup and let R, C^l, C^m, C^r, C and T denote respectively, the set of reflexive, left compatible, lateral compatible, right compatible, compatible and transitive relation on S. Then, $C \cap (R \cap T) = (C^l \cap C^m \cap C^r) \cap (R \cap T)$ i.e., a reflexive and transitive relation on S is compatible if and only if it is left, right and lateral compatible.*

Proposition 2.7. *Let T be a ternary semigroup with an identity element e. Then every left and right compatible relation on T is a lateral compatible relation on T.*

Proof. Let ρ be a left and right compatible relation on T and suppose that $(a, b) \in \rho$ and $s, t \in T$ are arbitrary elements. Then, $(esa, esb) \in \rho$. Hence, $((esa)te, (esb)te) \in \rho$. Therefore, $(sat, sbt) = (e(sat)e, e(sbt)e) \in \rho$. Consequently, ρ is lateral compatible. □

Proposition 2.8. *Let R be a left (right, lateral) compatible relation on a ternary semigroup T. Then, R^n is a left (right, lateral) compatible relation on T for every $n \geq 1$.*

Proof. Let R be left compatible and let $(a, b) \in R^n$. Then, there exist $c_1, c_2, \ldots, c_{n-1} \in T$ such that $(a, c_1), (c_1, c_2), \ldots, (c_{n-1}, b) \in R$. Since R is left compatible, it follows that

$$(sta, stc_1), (stc_1, stc_2), \ldots, (stc_{n-1}, stb) \in R,$$

for every $s, t \in T$. Hence, $(sta, stb) \in R^n$ for every $s, t \in T$. Consequently, R^n is left compatible. We have a similar proof when R is right or lateral compatible. □

Definition 2.9. An equivalence relation ρ on a ternary semigroup T is said to be a right (left, lateral) congruence if it is a right (left, lateral) compatible relation. Furthermore, a compatible equivalence relation ρ on a ternary semigroup T is called a congruence on T.

Proposition 2.10. *An equivalence relation ρ on a ternary semigroup T is congruence if and only if it is left, right and lateral congruence.*

Proof. By attention to Corollary 2.6, the result follows. □

Proposition 2.11. *Let ρ_1 and ρ_2 be two left (right, lateral) congruences on a ternary semigroup T. Then, $\rho_1 o \rho_2$ is a left (right, lateral) congruence on T.*

Proof. See [4, Proposition 3.8]. □

Corollary 2.12. *Let ρ_1 and ρ_2 be two congruences on a ternary semigroup T. Then $\rho_1 o \rho_2$ is a congruence on T.*

Proposition 2.13. *The non-empty intersection of any family of congruences on a ternary semigroup T is a congruence on T.*

Proof. See [4, Proposition 3.11]. □

Notice that if T is an ordinary semigroup and ρ is congruence on T, then ρ is congruence on the ternary semigroup T with ternary operation, defined by the binary operation of semigroup T.

Proposition 2.14. *Let T be a ternary semigroup whose elements are invertible. Then, $\rho o \sigma = \sigma o \rho$ for any two congruences ρ and σ on T.*

Proof. Suppose that $(a, b) \in \rho o \sigma$. Then, there exists $c \in T$ such that $(a, c) \in \rho$ and $(c, b) \in \sigma$. Since c is invertible, then there exists $c' \in T$ such that $cc'x = c'cx = xcc' = xc'c = x$ for all $x \in T$. On the other hand, ρ and σ are congruences on T. Thus, $(ac'b, cc'b) \in \rho$ and $(ac'c, ac'b) \in \sigma$. Hence, $(ac'b, b) \in \rho$ and $(a, ac'b) \in \sigma$. Consequently, $(a, b) \in \sigma o \rho$ and therefore $\rho o \sigma \subseteq \sigma o \rho$. Similarly, we can show that $\sigma o \rho \subseteq \rho o \sigma$, by interchanging the roles of ρ and σ. □

Definition 2.15. A ternary monoid T is called a ternary group if for every $a, b, c \in T$, the equation $abx = c$, $axb = c$ and $xab = c$ have solutions in T.

Proposition 2.16. *A ternary monoid T is a ternary group if and only if every element of T is invertible.*

Proof. Let T be a ternary group and e is an identity of T and let $a \in T$ be an arbitrary element. Then, there exists $b \in T$ such that $abe = e$. Now, suppose that $x \in T$. Then, we have $abx = a(bee)x = (abe)ex = eex = x$ and $xab = xa(bee) = x(abe)e = xee = x$. Moreover, there exist $c, d \in T$ such that $bec = x$ and $dea = x$. Now, we have $bax = ba(bec) = b(abe)c = bec = x$ and $xba = (dea)ba = d(eab)a = dea = x$. Therefore, a is invertible.
Conversely, suppose that every element of T is invertible and let $a, b, c \in T$. Then, since a and b are invertible, so there exist $a', b' \in T$ such that for every $x \in T$, $aa'x = a'ax = xaa' = xa'a = x$ and $bb'x = b'bx = xbb' = xb'b = x$. Hence, $ab(b'a'c) = (abb')a'c = aa'c = c$, $(cb'a')ab = cb'(a'ab) = cb'b = c$ and $a(a'cb')b = (aa'c)b'b = cb'b = c$. Consequently, T is a ternary group. □

Corollary 2.17. *Let T be a ternary group. Then, $\rho o \sigma = \sigma o \rho$ for any two congruences ρ and σ on T.*

Definition 2.18. Let R be a relation on a ternary semigroup T. Then, the smallest congruence on T containing R (the intersection of all congruences on T containing R) is called the congruence generated by R and it is denoted by $R^{\#}$.

For making a congruence on a ternary semigroup T from a relation R on T, we need to adjoin an extra element 1 to T as identity element. Although every ordinary semigroup can be extended to a monoid by adjoining an extra element as identity but it is important to note that not every ternary semigroup extend to a ternary monoid. However, in the following, we will mention an equivalent condition in which ternary semigroup T can be extended to a ternary monoid.

Remark 2.19. Let $(T, .)$ be a ternary monoid with the identity e. Then T can be reduced to an ordinary semigroup.

Proof. It is enough to define the binary operation $* : T \times T \longrightarrow T$ by $a * b = a.b.e$ for every $a, b \in T$. Then, $(T, *)$ is an ordinary semigroup and $a.b.c = (a * b) * c$, for every $a, b, c \in T$. $\square$

Proposition 2.20. *Let $(T, .)$ be a ternary semigroup. Then T can be extended to a ternary monoid if and only if it can be reduced to an ordinary semigroup.*

Proof. Suppose that T can be extended to ternary monoid $(T^1, .)$ with identity 1. Then, T can be reduced to the ordinary semigroup $(T, *)$ with $a * b = a.b.1$, for every $a, b \in T$. Conversely, let $(T, .)$ be a ternary semigroup that can be reduced to an ordinary semigroup $(T, *)$. Then, ordinary semigroup $(T, *)$ is extended to monoid T^1. Now, T^1 with ternary operation $[\,]$ defined as following is a ternary monoid.

$$
\begin{aligned}
&[abc] = a.b.c \qquad \forall a, b, c \in T;\\
&[a11] = [1a1] = [11a] = a \qquad \forall a \in T^1;\\
&[a1b] = [ab1] = [1ab] = a * b \qquad \forall a, b \in T.
\end{aligned}
$$

Therefore, T can be extended to ternary monoid T^1. $\square$

From now on, we suppose that we can adjoin an element 1 to T as identity.

Lemma 2.21. *Let T be a ternary semigroup and R be a relation on T. Then, $R^c = \{(xay, xby) \mid x, y \in T^1, (a, b) \in R\}$ is the smallest left, right and lateral compatible relation on T containing R.*

Proof. It is clear that $R \subseteq R^c$ (let $x = y = 1$ in R^c). Suppose that $(xay, xby) \in R^c$ and $s, t \in T$. Then, $(a, b) \in R$ and therefore $(st(xay), st(xby)) = ((stx)ay, (stx)by) \in R^c$. Hence, R^c is left compatible. Right compatibility follows in a similar way. Also, we get $(s(xay)t, s(xby)t) = (1(sxayt)1, 1(sxbyt)1) = ((1sx)a(yt1), (1sx)b(yt1))$ in R^c. Hence, R^c is lateral compatible. Now, Let ρ be a left, right and lateral compatible relation on T containing R and $(xay, xby) \in R^c$. Then, $(a, b) \in R$ and consequently $(a, b) \in \rho$. Since ρ is lateral compatible, then $(xay, xby) \in \rho$. Therefore, $R^c \subseteq \rho$, as desired. $\square$

Lemma 2.22. *Let R and S be two relations on a ternary semigroup T. Then, we have:*
(1) $R \subseteq S \Rightarrow R^c \subseteq S^c$.
(2) $(R^{-1})^c = (R^c)^{-1}$.
(3) $(R \cup S)^c = R^c \cup S^c$.

Proof. (1). Suppose that $(xay, xby) \in R^c$. Then, $(a, b) \in R \subseteq S$ and $x, y \in T^1$. Hence, $(xay, xby) \in S^c$.

(2) Suppose that $(xay, xby) \in (R^{-1})^c$. Then, $(a, b) \in R^{-1}$ and $x, y \in T^1$. Hence, $(b, a) \in R$ and $(xby, xay) \in R^c$. Therefore, $(xay, xby) \in (R^c)^{-1}$. Consequently, $(R^{-1})^c \subseteq (R^c)^{-1}$. We can prove that $(R^c)^{-1} \subseteq (R^{-1})^c$ in a similar way.

(3) Suppose that $(xay, xby) \in (R \cup S)^c$. Then, $(a, b) \in R \cup S$ and $x, y \in T^1$. Therefore, $(xay, xby) \in R^c$ or $(xay, xby) \in S^c$. Consequently, $(R \cup S)^c \subseteq R^c \cup S^c$. We can prove that $R^c \cup S^c \subseteq (R \cup S)^c$ in a similar way. □

Theorem 2.23. *For every relation R on a ternary semigroup T, we have $R^{\#} = (R^c)^e$.*

Proof. We know that $(R^c)^e$ is an equivalence relation on T containing R^c and so certainly containing R. To show that $(R^c)^e$ is a congruence, we must show that it is left, right and lateral compatible by Proposition 2.10. So, suppose that $(a, b) \in (R^c)^e$. Then, by Proposition 1.11, $(a, b) \in (R^c \cup (R^c)^{-1} \cup 1_T)^\infty = ((R \cup R^{-1} \cup 1_T)^c)^\infty$. Therefore, $(a, b) \in ((R \cup R^{-1} \cup 1_T)^c)^n$ for some $n \geq 1$. Now, by Proposition 2.8 and Lemma 2.21, $((R \cup R^{-1} \cup 1_T)^c)^n$ is a left, right and lateral compatible. Hence, $(sta, stb), (ast, bst), (sat, sbt) \in ((R \cup R^{-1} \cup 1_T)^c)^n$ for every $s, t \in T$. Therefore, $(sta, stb), (ast, bst), (sat, sbt) \in (R^c)^e$ for every $s, t \in T$ and consequently $(R^c)^e$ is a congruence on T containing R. Now, consider a congruence ρ on T containing R. Then, $R^c \subseteq \rho^c$. On the other hand, $\rho^c = \rho$ since ρ is a congruence. Thus, ρ is an equivalence on T containing R^c. Therefore, $(R^c)^e \subseteq \rho$ and consequently $(R^c)^e$ is the smallest congruence on T containing R. □

Corollary 2.24. *Let R be a relation on a ternary semigroup T and $a, b \in T$. Then, $(a, b) \in R^{\#}$ if and only if either $a = b$ or for some $n \in \mathbb{N}$, there is a sequence $a = c_1, c_2, \ldots, c_{n-1}, c_n = b$ of elements of T such that, for each $i \in \{1, 2, \ldots, n-1\}$, either $(c_i, c_{i+1}) \in R^c$ or $(c_{i+1}, c_i) \in R^c$.*

Proposition 2.25. *Let T be a ternary semigroup and let E be an equivalence on T. Then,*

$$E^\flat = \{(a, b) \in T \times T \mid (xay, xby) \in E \; for\ all\ x, y \in T^1\}$$

is the largest congruence on T contained in E.

Proof. It is clear that $E^\flat \subseteq E$ (let $x = y = 1$ in $E^\flat$) and since E is an equivalent relation on T, so $E^\flat$ is an equivalent relation on T. Now, suppose that $(a, b) \in E^\flat$ and $s, t \in T$ are arbitrary elements. Then, for every $x, y \in T^1$, we have

$$((xst)ay, (xst)by), (xa(sty), xb(sty)), ((1xs)a(ty1), (1xs)b(ty1)) \in E.$$

Hence, for every $x, y \in T^1$, we have

$$(x(sta)y, x(stb)y), (x(ast)y), x(bst)y), (x(sat)y, x(sbt)y) \in E.$$

Therefore, $(sta, stb), (ast, bst), (sat, sbt) \in E^\flat$ and consequently $E^\flat$ is a congruence on T. Finally, Let ρ be a congruence on T contained in E and $(a, b) \in \rho$. Since ρ is a congruence on T, then $(xay, xby) \in \rho \subseteq E$ for every $x, y \in T^1$ and consequently $(a, b) \in E^\flat$. Thus $\rho \subseteq E^\flat$. $\square$

Example 2.26. Let T be a ternary semigroup and A be a subset of T. Also let π_A be an equivalence on T whose classes are A and $T \setminus A$. Then, $\pi_A^\flat = \{(a, b) \in T \times T \mid xay \in A \Leftrightarrow xby \in A \; for \; all \; x, y \in T^1\}$.

3. The lattice of congruences on a ternary semigroup

In this section, we study the lattice of congruences on a ternary semigroup and we prove that this lattice is a complete lattice but it is not necessarily modular and not even semimodular.

Let T be a ternary semigroup. Then we denote the set of all congruences on T by $C(T)$.

Theorem 3.1. *Let T be a ternary semigroup. Then $C(T)$ is a complete lattice.*

Proof. It is clear that $C(T)$ is a partially ordered set by inclusion. Since $\cap_{i \in I}\rho_i$ and $(\cup_{i \in I}\rho_i)^\#$ are congruences on T for every non-empty family $\{\rho_i\}_{i \in I}$ of congruences on T, so it is enough to take $\wedge_{i \in I}\rho_i = \cap_{i \in I}\rho_i$ and $\vee_{i \in I}\rho_i = (\cup_{i \in I}\rho_i)^\#$. $\square$

Proposition 3.2. *Let T be a ternary semigroup. Then for every $\rho, \sigma \in C(T)$, we have $(\rho \cup \sigma)^\# = (\rho \cup \sigma)^e$.*

Proof. By the Theorem 2.23 and Lemma 2.22, we get that $(\rho \cup \sigma)^\# = ((\rho \cup \sigma)^c)^e = (\rho^c \cup \sigma^c)^e$. But $\rho^c = \rho$ and $\sigma^c = \sigma$, since ρ and σ are congruences on T. Therefore, $(\rho \cup \sigma)^\# = (\rho \cup \sigma)^e$. $\square$

Let T be a ternary semigroup and $\rho, \sigma \in C(T)$. Then we denote $(\rho \cup \sigma)^\# = (\rho \cup \sigma)^e$ by $\rho \vee \sigma$.

Proposition 3.3. *Let T be a ternary semigroup. Then for every $\rho, \sigma \in C(T)$, we have $\rho \vee \sigma = (\rho o \sigma)^\infty$.*

Proof. Let $\rho, \sigma \in C(T)$. Then,

$$(\rho \vee \sigma) = (\rho \cup \sigma)^e = ((\rho \cup \sigma) \cup (\rho \cup \sigma)^{-1} \cup 1_T)^\infty = (\rho \cup \sigma \cup \rho^{-1} \cup \sigma^{-1} \cup 1_T)^\infty.$$

Therefore, $(\rho \vee \sigma) = (\rho \cup \sigma)^{\infty}$, since ρ and σ are equivalence relations. It is enough to show that $(\rho \cup \sigma)^{\infty} = (\rho o \sigma)^{\infty}$. Since $\rho, \sigma \subseteq \rho \cup \sigma$, so $(\rho o \sigma)^{n} \subseteq (\rho \cup \sigma)^{2n}$ for every $n \geq 1$. Consequently, $(\rho o \sigma)^{\infty} \subseteq (\rho \cup \sigma)^{\infty}$. On the other hand, ρ and σ are reflexive. So, $\rho \subseteq \rho o \sigma$ and $\sigma \subseteq \rho o \sigma$. Hence, $\rho \cup \sigma \subseteq \rho o \sigma$ and consequently $(\rho \cup \sigma)^{\infty} \subseteq (\rho o \sigma)^{\infty}$. $\square$

Corollary 3.4. *Let T be a ternary semigroup and $\rho, \sigma \in C(T)$. Then $(a, b) \in \rho \vee \sigma$ if and only if for some $n \in \mathbb{N}$, there exist $x_1, x_2, \ldots, x_{2n-1}$ in T such that, $(a, x_1) \in \rho, (x_1, x_2) \in \sigma, \ldots, (x_{2n-2}, x_{2n-1}) \in \rho, (x_{2n-1}, b) \in \sigma$.*

Corollary 3.5. *Let T be a ternary semigroup and $\rho, \sigma \in C(T)$ such that $\rho o \sigma = \sigma o \rho$. Then, $\rho \vee \sigma = \rho o \sigma$.*

Proof. Since $\rho o \sigma = \sigma o \rho$, then $(\rho o \sigma)^{n} = \rho^{n} o \sigma^{n} = \rho o \sigma$ for every $n \geq 1$. Hence, $(\rho o \sigma)^{\infty} = \rho o \sigma$ and the result follows from Proposition 3.3. $\square$

Corollary 3.6. *Let T be a ternary semigroup whose elements are invertible. Then, $\rho \vee \sigma = \rho o \sigma$ for every $\rho, \sigma \in C(T)$.*

Corollary 3.7. *Let T be a ternary group. Then $\rho \vee \sigma = \rho o \sigma$ for every $\rho, \sigma \in C(T)$.*

Every sublattice of a modular lattice is a modular lattice but a sublattice of a semimodular lattice may not be a semimodular lattice. For example for any ternary semigroup T, $C(T)$ is a sublattice of semimodular lattice $E(T)$, while $C(T)$ may not be a modular lattice. It is not generally even a semimodular lattice.

Example 3.8. Let $T = \{e, f, a, b\}$. Define a ternary operation $[\ ]$ on T as $[abc] = a \cdot b \cdot c$, where $\cdot$ is the binary operation defined as following:

$\cdot$	e	f	a	b
e	e	f	a	b
f	f	f	b	b
a	a	b	e	f
b	b	b	f	f

Then, $(T, [\ ])$ is a ternary semigroup. Consider the relations
$\alpha = \{(e, e), (f, f), (a, a), (b, b), (e, f), (f, e), (a, b), (b, a)\}$,
$\beta = \{(e, e), (f, f), (a, a), (b, b), (a, e), (e, a), (f, b), (b, f)\}$
and $\gamma = \{(e, e), (f, f), (a, a), (b, b), (f, b), (b, f)\}$ on T. Then $\alpha, \beta, \gamma \in C(T)$. Furthermore, $\alpha \cap \gamma = 1_T$ and $\alpha \vee \gamma = T \times T$. It is clear that $\alpha \succ \alpha \wedge \gamma$ and $\gamma \succ \alpha \wedge \gamma$ but $\alpha \vee \gamma \succ \beta \succ \gamma$. Therefore, $C(T)$ is not semimodular lattice and consequently $C(T)$ is not a modular lattice.

We showed that $C(T)$ is not generally semimodular, while it is modular lattice when T satisfies in the condition of the following proposition.

Proposition 3.9. *Let T be a ternary semigroup and let K be a sublattice of the lattice $(C(T), \subseteq, \cap, \vee)$ such that $\rho o \sigma = \sigma o \rho$ for all $\rho, \sigma \in K$. Then, K is a modular lattice.*

Proof. Let $\alpha, \beta, \gamma \in K$ such that $\alpha \subseteq \gamma$ and let $(x, y) \in (\alpha \vee \beta) \cap \gamma$. Then, $(x, y) \in \alpha o \beta$ by Corollary 3.5. Therefore, there exists $z \in T$ such that $(x, z) \in \alpha$ and $(z, y) \in \beta$. Since $\alpha \subseteq \gamma$ and γ is an equivalence relation, we deduce that $(z, y) \in \gamma$ and consequently $(x, y) \in \alpha o (\beta \cap \gamma) = \alpha \vee (\beta \cap \gamma)$. We have shown that $(\alpha \vee \beta) \cap \gamma \subseteq \alpha \vee (\beta \cap \gamma)$, and so K is modular. □

Corollary 3.10. *Let T be a ternary semigroup whose elements are invertible. Then $C(T)$ is a modular lattice.*

Corollary 3.11. *Let T be a ternary group. Then $C(T)$ is a modular lattice.*

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The lattice of congruences on a ternary semigroup

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مشبکه‌ی هم‌نهشتی‌ها روی یک نیم‌گروه سه‌تایی

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در این مقاله برخی از خواص هم‌نهشتی‌ها روی یک نیم‌گروه سه‌تایی را مورد بررسی قرار می‌دهیم. همچنین هم‌نهشتی تولید شده توسط یک رابطه روی یک نیم‌گروه سه‌تایی را تعریف کرده، روش به دست آوردن آن را از رابطه مورد نظر تعیین می‌کنیم. به‌علاوه، مشبکه‌ی هم‌نهشتی‌ها روی یک نیم‌گروه سه‌تایی را مطالعه می‌نماییم و نشان می‌دهیم در حالت کلی این مشبکه، هنگی و یا حتّی نیم‌هنگی نیست. سپس به بیان شرایطی می‌پردازیم که تحت آن این مشبکه، هنگی خواهد بود.

کلمات کلیدی: نیم‌گروه سه‌تایی، هم‌نهشتی، مشبکه.