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ON STRONGLY ASSOCIATIVE HYPERRINGS

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Abstract. This paper generalizes the idea of strongly associative hyperoperation introduced in [7] to the class of hyperrings. We introduce and investigate hyperrings of type 1, type 2 and SDIS. Moreover, we study some examples of these hyperrings and give a new kind of hyperrings called totally hyperrings. Totally hyperrings give us a characterization of Krasner hyperrings. Also, we investigate these strongly hyperoperations in hyperring of series.

1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [12] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini, Leoreanu, Mittas, Stratigopoulos, Vougiouklis, Davvaz [1, 2, 5, 17, 22, 25] and by various authors [6, 3, 18, 21, 24]. Basic definitions and propositions about the hyperstructures are found in [1, 2, 5, 23]. Krasner [11] studied the notions of hyperfield, hyperring and then some other researchers did, for example see [3, 4, 9, 18, 19]. Hyperrings are basically rings with approximately modified axioms. There are different types of hyperrings. If the addition + is a hyperoperation and the multiplication is a binary operation, then the hyperring is called additive hyperring. A special case of this type is the Krasner hyperring [11]. Rota [20] introduced a multiplicative hyperring, where + is a binary operation and the multiplication is a hyperoperation. De Salvo [8] studied hyperrings in which the additions

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and the multiplications were hyperoperations. Surveys of hyperrings theory and its applications can be found in the book of Davvaz and Leoreanu-Fotea [5]. In particular, the relationships between the fuzzy sets, soft sets and hyperrings have been considered by many researchers for example see [4, 13, 14, 15]. In this paper, we first generalize the idea of strongly associative hyperoperation (SASS) introduced in [7] to the class of hyperrings. Next, using SASS and SDIS notions we introduce some types of hyperrings which are called *type1, type2* and *totally*, respectively. Finally, a characterization of Krasner hyperrings by strongly distributive hyperrings are investigated. In the following, we give the preliminaries which will be used throughout this article.

Definition 1.1. Let H be a non-empty set and $\circ\colon H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. Then, the couple $(H, \circ)$ is called a hypergroupoid. For non-empty subsets A and B of H and $x \in H$, we define:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b\ , \quad A \circ x = A \circ \{x\}.$$

Definition 1.2. A hypergroupoid $(H, \circ)$ is called hypergroup if for all a, b, c in H, it satisfies the following conditions:

(1) $(a \circ b) \circ c = a \circ (b \circ c)$,
(2) $a \circ H = H = H \circ a$.

A semihypergroup H is *complete* if it satisfies one of the following equivalent conditions:

(1) $\forall (x, y) \in H^2, \forall a \in x \circ y, \mathcal{C}(a) = x \circ y$, where $\mathcal{C}(a)$ denotes the complete closure of a,
(2) $\forall (x, y) \in H^2, \mathcal{C}(x \circ y) = x \circ y$,
(3) $\forall (n, m) \in \mathbf{N}^2$, $m, n \geq 2$, $\forall (x_1, ..., x_n) \in H^n, \forall (y_1, ..., y_m) \in H^m$, the following implication is valid:

$$\Pi_{i=1}^n x_i \cap \Pi_{j=1}^m y_j \neq \emptyset \Rightarrow \Pi_{i=1}^n x_i = \Pi_{j=1}^m y_j.$$

Theorem 1.3. ([1]) *A semihypergroup $(H, \circ)$ is complete if it can be written as a union $H = \cup_{s \in S} A_s$ of its subsets, where S and A_s satisfy the conditions:*

(1) *$(S, \cdot)$ is a semigroup,*
(2) *for all $(s, t) \in S^2$, where $s \neq t$, we have $A_s \cap A_t = \emptyset$,*
(3) *if $(a, b) \in A_s \times A_t$, then $a \circ b = A_{s \cdot t}$.*

Definition 1.4. Let $(H, \circ)$ be a hypergroupoid. Then,

(1) An element $e \in H$ is called an *identity* if $x \in e \circ x \cap x \circ e$, for every $x \in H$.

(2) Let e be an identity element in H. Then, the element $x' \in H$ is called an *inverse* of $x \in H$ if $e \in x \circ x' \cap x' \circ x$.

Definition 1.5. A semihypergroup $(H, \circ)$ is *regular* if it has at least one identity and each element has at least one inverse.

Definition 1.6. A canonical hypergroup $(H, +)$ is a non-empty set H equipped with a hyperoperation $+$ with the following properties:

(1) $x + y = y + z, \quad \forall x, y \in H,$
(2) $(x + y) + z = x + (y + z), \quad \forall x, y, z \in H,$
(3) $\exists ! 0_H \in H$ such that $0_H + x = x = x + 0_H, \quad \forall x \in H,$
(4) $\forall x \in H \quad \exists ! y \in H$ such that $0_H \in x + y$. We denote $y = -x$,
(5) $x \in y + z \Longleftrightarrow z \in x - y, \quad \forall x, y, z \in H.$

Definition 1.7. ([7]) Let H be a non-empty set and $\circ \colon H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation. Then,

(1) $\circ$ is called left strongly associative if for all $x, y, z \in H$ and for all $t \in y \circ z$, there exists $s \in x \circ y$ such that $x \circ t = s \circ z$,
(2) $\circ$ is called right strongly associative if for all $x, y, z \in H$ and for all $t \in x \circ y$, there exists $s \in y \circ z$ such that $t \circ z = x \circ s$,
(3) $\circ$ is strongly associative or for simplicity SASS if it is left and right strongly associative.

Definition 1.8. ([7]) A hypergroupoid $(H, \circ)$ is called left(right) strongly associative if the hyperoperation $\circ$ is left(right) strongly associative. $(H, \circ)$ is called strongly associative if $\circ$ is strongly associative.

Remark 1.9. The group operation is strongly associative.

2. Strongly hyperoperations in hyperrings

Definition 2.1. A general hyperring $(R, +, *)$ is a non-empty set R such that $(R, +)$ is a hypergroup, $(R, *)$ is an associative hyperoperation and hyperoperation $+$ is distributive with respect to the hyperoperation $*$, i.e., $x * (y + z) = (x * y) + (x * z)$ which means that:

$$\bigcup_{t \in y+z} x * t = \bigcup_{u \in x*y, v \in x*z} u + v$$

and $(y + z) * x = (y * x) + (z * x)$.

The above Definition was given by Vougiouklis [24] and then used by Spartalis [21].

Definition 2.2. A subhyperring of a hyperring $(R, +, *)$ is a non-empty subset S of R that preserves the structure of the hyperring R, i.e. a hyperring $(S, +, *)$ with $S \subseteq R$.

In the following, we present a definition to obtain a new class of hyperrings.

Definition 2.3. Let R be a non-empty set and $\circ, *\colon R \times R \longrightarrow \mathcal{P}^*(R)$ be two hyperoperations. Then,

(1) $*$ is called left strongly distributive with respect to $\circ$ if for all $x, y, z \in R$ and for all $t \in y \circ z$, there exists $u \in x * y$ and $v \in x * z$ such that $x * t = u \circ v$, also for all $s \in x * y$ and $t \in x * z$, there exists $w \in y \circ z$ such that $x * w = s \circ t$,

(2) $*$ is called right strongly distributive with respect to $\circ$ if for all $x, y, z \in R$ and for all $t \in y \circ z$, there exists $u \in y * x$ and $v \in z * x$ such that $t * x = u \circ v$, also for all $s \in y * x$ and $t \in z * x$, there exists $w \in y \circ z$ such that $w * x = s \circ t$,

(3) $*$ is strongly distributive with respect to $\circ$ if it is left and right strongly distributive.

Remark 2.4. It is clear that if $*$ is a commutative hyperoperation, then left and right strongly distributive coincide.

Lemma 2.5. *If $*$ is strongly distributive with respect to $\circ$, then $*$ is distributive with respect to $\circ$.*

Proof. Let $*$ be strongly distributive with respect to $\circ$ and $(x, y, z) \in R^3$. Then, we have $\{x * t | t \in y \circ z\} = \{u \circ v | u \in x * y, v \in x * z\}$. Hence $x * (y \circ z) = (x * y) \circ (x * z)$. Similarly $(y \circ z) * x = (y * x) \circ (z * x)$. $\square$

Definition 2.6. Let $\mathbf{R} = (R, +, *)$ be a hyperring. Then,

(1) $\mathbf{R}$ is called of type 1 if the hyperoperation $+$ is strongly associative,

(2) $\mathbf{R}$ is called of type 2 if the hyperoperation $*$ is strongly associative,

(3) $\mathbf{R}$ is called SDIS if the hyperoperation $*$ is strongly distributive with respect to $+$.

Definition 2.7. A hyperring $(R, +, *)$ is called totally hyperring if it is of type 1, 2 and SDIS.

Remark 2.8. Every ring is a totally hyperring.

Remark 2.9. If S is a subhyperring of R and R is totally, then S is a totally hyperring.

Example 2.10. Let $(R = \{e, a\}, +, *)$, where

$+$	e	a
e	e	a
a	a	$\{e, a\}$

$*$	e	a
e	e	$\{e, a\}$
a	$\{e, a\}$	$\{e, a\}$

be a hyperring. Then, R is a general hyperring that is neither type 1 nor type 2 and it is not SDIS, too.

There exist hyperrings of type 1 and 2, which are not SDIS. The next example is such a hyperring.

Example 2.11. Let $(R = \{e, a, b, c\}, +, *)$, where

$+$	e	a	b	c
e	$\{e,a\}$	$\{a,b\}$	$\{b,c\}$	$\{e,c\}$
a	$\{a,b\}$	$\{b,c\}$	$\{e,c\}$	$\{e,a\}$
b	$\{b,c\}$	$\{e,c\}$	$\{e,a\}$	$\{a,b\}$
c	$\{e,c\}$	$\{e,a\}$	$\{a,b\}$	$\{b,c\}$

$*$	e	a	b	c
e	$\{e,a\}$	$\{b,c\}$	$\{e,a\}$	$\{b,c\}$
a	$\{b,c\}$	$\{e,a\}$	$\{b,c\}$	$\{e,a\}$
b	$\{e,a\}$	$\{b,c\}$	$\{e,a\}$	$\{b,c\}$
c	$\{b,c\}$	$\{e,a\}$	$\{b,c\}$	$\{e,a\}$

R is a hyperring of type 1 and 2 but it is not SDIS, since

$$e*(e+e) = \{e*t \mid t \in e+e\} \neq \{u+v \mid u \in e*e, v \in e*e\} = (e*e)+(e*e).$$

Also, there exists SDIS hyperring which is not of type 1 and type 2. The next example is such a hyperring.

Example 2.12. Let $R = (\{a, b\}, +, *)$, where

$+$	a	b
a	a	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$

$*$	a	b
a	a	$\{a,b\}$
b	a	$\{a,b\}$

be a hyperring. Then, R is neither of type 1 nor type 2 but it is SDIS.

Example 2.13. Let $(R = \{e, a, b\}, +, *)$, where

$+$	e	a	b
e	$\{e,a\}$	H	H
a	H	$\{e,a\}$	$\{e,a\}$
b	H	$\{e,a\}$	$\{e,a\}$

$*$	e	a	b
e	$\{e,a\}$	H	$\{e,a\}$
a	H	$\{e,a\}$	H
b	$\{e,a\}$	H	$\{e,a\}$

R is a hyperring of type 1, 2 and SDIS. Then, R is a totally hyperring.

Proposition 2.14. *Let $(F, +, \cdot)$ be a field and $\mid F \mid \geq 3$. Then, $(F, \oplus, \odot)$ is a hyperring of type 1, where $x \oplus y = F - \{x + y\}$ and*

$$x \odot y = \begin{cases} F - \{x \cdot y\} & \text{if} \quad x \neq e \quad \text{and} \quad y \neq e \\ F & \text{if} \quad x = e \quad \text{or} \quad y = e \end{cases}$$

for all $x, y \in F$ and "e" is the identity element of $(F, +)$.

Proof. $(F, \oplus)$ is a strongly associative hypergroup (see [7]). Moreover, $(F, \odot)$ is a semihypergroup. It would be sufficient to check the distributivity. Let $\{x, y, z\} \subseteq F$. If $x = e$, then $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z) = F$. Let $x \neq e, y \neq e$ and $z = e$. Then, $y + z \neq e$ and $x \cdot y \neq e$, so we have

$$x \odot (y \oplus z) = x \odot (F - (y + z)) \supseteq x \odot e = F.$$

In this case, $(x \odot y) \oplus (x \odot z) = (F - \{x \cdot y\}) \oplus F \supseteq e \oplus t_1 \cup e \oplus t_2$, where $t_1, t_2 \in F$ and $t_1 \neq t_2$. Since $(F, +)$ is a group, $e \oplus t_1 \cup e \oplus t_2 = F$. Hence $(x \odot y) \oplus (x \odot z) = F$. Now let $x \neq e, y \neq e, z \neq e$. If $y + z \neq e$, then

$$x \odot (y \oplus z) = \bigcup_{t \in F - \{y+z\}} x \odot t \supseteq x \odot e = F.$$

If $y + z = e$, then $x \odot (y \oplus z) = \bigcup_{t \in F - \{e\}} x \odot t \supseteq x \odot t_1 \cup x \odot t_2$, where $t_1, t_2 \in F$ and $e \neq t_1 \neq t_2 \neq e$. Since $(F - \{e\}, \cdot)$ is a group, $x \odot t_1 \cup x \odot t_2 = F$. Hence $x \odot (y \oplus z) = F$. Moreover, we have $(x \odot y) \oplus (x \odot z) = F$. Hence $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z) = F$. Therefore $(F, \oplus, \odot)$ is a hyperring of type 1. $\square$

Now, we present a way to obtain a class of hyperrings that is of type 2. Let $(H, \circ, *)$ be a hyperring and $\{A_i\}_{i \in R}$ be a family of non-empty sets such that

(1) $(R, +, \cdot)$ is a ring,
(2) $A_{0_R} = H$,
(3) $\forall i, j \in R, A_i \cap A_j = \emptyset$.

Let $K = \bigcup_{i \in R} A_i$ and define the following hyperoperations on K

(1) $\forall x, y \in H$, $x \oplus y = x \circ y$, $x \odot y = H$,
(2) $\forall x \in A_i$, $\forall y \in A_j$, such that $A_i \times A_j \neq H \times H$, $x \oplus y = A_{i+j}$, $x \odot y = A_{i \cdot j}$.

The structure $(K, \oplus, \odot)$ is a hyperring which is called $(H, R)-$*hyperring*.

Theorem 2.15. *$K = \bigcup_{i \in R} A_i$ is a hyperring of type 2. We shall say that K is an $(H, R) -$ hyperring of type 2.*

Proof. We show that $\odot$ is a left strongly associative hyperoperation. Let $\{x, y, z\} \subseteq K$. If $x, y, z \in H$ then it is clear that $x \odot (y \odot z) = \{H\} = (x \odot y) \odot z$. If $x, y \in H = A_0, z \in A_k$ such that $k \neq 0$, then for all $t \in y \odot z = A_{0 \cdot k} = H$, we have $x \in H = x \odot y$, and $x \odot t = H = x \odot z$. If $x \in A_m, y \in A_n, z \in H = A_0$ such that $m, n \neq 0$, then for all $t \in y \odot z = A_0 = H$ and for all $s \in x \odot y = A_{m \cdot n}$ we have

$$x \odot t = H = s \odot z.$$

Now let $x \in A_i, y \in A_j, z \in A_k$, such that $A_i \times A_j \times A_k \neq H \times H \times H$ and let $t \in y \odot z = A_m$, such that $j \cdot k = m$. If $s \in x \odot y = A_n$, such that $i \cdot j = n$, then we have

$$x \odot t = A_{i \cdot m} = A_{i \cdot (j \cdot k)} = A_{(i \cdot j) \cdot k} = A_{n \cdot k} = s \odot z.$$

Thus $(K, \odot)$ is a left strongly associative hyperoperation. Similarly $\odot$ is a right strongly associative hyperoperation. Consequently K is a hyperring of type 2.

□

Remark 2.16. In structure $(K, \oplus, \odot)$ although H is totally hyperring, K is not totally. The next example is such a hyperring.

Example 2.17. Consider $H = (\{e, a, b\}, \circ, *)$ and $R = (\mathbf{Z}_2, +, \cdot)$ be as follow:

$\circ$	e	a	b
e	$\{e,a\}$	H	H
a	H	$\{e,a\}$	$\{e,a\}$
b	H	$\{e,a\}$	$\{e,a\}$

$*$	e	a	b
e	$\{e,a\}$	H	$\{e,a\}$
a	H	$\{e,a\}$	H
b	$\{e,a\}$	H	$\{e,a\}$

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Set $A_0 = \{e, a, b\}, A_1 = \{c\}$. Then we have the following hyperoperations:

$\oplus$	e	a	b	c
e	$\{e,a\}$	H	H	c
a	H	$\{e,a\}$	$\{e,a\}$	c
b	H	$\{e,a\}$	$\{e,a\}$	c
c	c	c	c	H

$\odot$	e	a	b	c
e	H	H	H	H
a	H	H	H	H
b	H	H	H	H
c	H	H	H	c

$(K, \oplus)$ is not a strongly associative hypergroup and $(K, \oplus, \odot)$ is not also SDIS. Indeed, $c \oplus (c \oplus e) \neq (c \oplus c) \oplus e$ and $c \odot (a \oplus b) \neq (c \odot a) \oplus (c \odot b)$. So $(K, \oplus, \odot)$ is not a totally hyperring, but H is totally.

Now, we construct a totally hyperring. Let $(R, +, \cdot)$ be a ring and $\{A(g)\}_{g \in R}$ be a family of non-empty sets such that

(1) $\forall g, g^{'} \in R, g \neq g^{'} \Rightarrow A(g) \cap A(g^{'}) = \emptyset$,

(2) $g \notin R \cdot R \Rightarrow \mid A(g) \mid = 1$.

Set $H_R = \bigcup_{g \in R} A(g)$ and define the following hyperoperations $\oplus$ and $\odot$ on H_R: $\forall a, b \in H_R$, $\exists g, g^{'} \in R$ such that $a \in A(g), b \in A(g')$, set $a \oplus b = A(g + g^{'})$, $a \odot b = A(g \cdot g^{'})$. For all $g, g^{'} \in R$, and $u \in A(g), v \in A(g^{'})$, we have:

(1) $u \oplus v = A(g + g^{'}) = A(g) \oplus A(g^{'})$,

(2) $u \odot v = A(g \cdot g') = A(g) \odot A(g')$.

Theorem 2.18. *H_R is a totally hyperring.*

Proof. According to the results of [7], the hyperoperations of H_R are strongly associative and so H_R is a hyperring of type 1 and type 2. We show that H_R is a SDIS hyperring. Let $\{x, y, z\} \subseteq H_R$. Then, there exists $\{g, g', g''\} \subseteq R$ such that $x \in A(g), y \in A(g'), z \in A(g'')$. Let $t \in y \oplus z = A(g' + g'')$, $u \in A(g \cdot g')$ and $v \in A(g \cdot g'')$, so $u \in x \odot y$ and $v \in x \odot z$. Then, we have:

$$x \odot t = A(g \cdot (g' + g'')) = A(g \cdot g' + g \cdot g'') = A(g \cdot g') \oplus A(g \cdot g'') = u \oplus v.$$

Moreover, let $s \in x \odot y = A(g \cdot g')$, $t \in x \odot z = A(g \cdot g'')$ and $w \in A(g' + g'')$, so $w \in y \oplus z$. Then we have

$$x \odot w = A(g \cdot (g' + g'')) = A(g \cdot g' + g \cdot g'') = A(g \cdot g') \oplus A(g \cdot g'') = s \oplus t.$$

Hence H_R is left strongly distributive. Similarly it is right strongly distributive. Therefore H_R is SDIS and so it is a totally hyperring. □

Definition 2.19. Let $(R, \oplus, *)$ be a hyperring. If $(R, \oplus)$ is complete, then we say that R is $\oplus -$ *complete*. If $(R, *)$ is complete, then we say that R is $* -$ *complete* and if both $(R, \oplus), (R, *)$ are complete, then we say that R is complete.

In hypergroups every complete semihypergroup is strongly associative (see [7]), but complete hyperrings are not totally. The next example is such a hyperring.

Example 2.20. Let $R = (\{a, b\}, +, *)$, where

$+$	a	b
a	a	b
b	b	a

$*$	a	b
a	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$

be a hyperring. Then, R is a complete hyperring that is not a totally hyperring.

Proposition 2.21. *Every complete hyperring is a hyperring of type 1 and type 2.*

Proof. Using ([7], Corollary 2.13). □

Remark 2.22. Every SDIS complete hyperring is totally hyperring.

Remark 2.23. According to the Theorem 2.18 and ([5], 5.2.21), H_R is a totally complete hyperring.

Let $(R_1, +_1, *_1)$ and $(R_2, +_2, *_2)$ be two hyperrings. On $R_1 \times R_2$, we can define hyperproducts as follow:
$(x_1, x_2) \oplus (y_1, y_2) = \{(x, y) | x \in x_1 +_1 y_1, y \in x_2 +_2 y_2\}$,
$(x_1, x_2) \odot (y_1, y_2) = \{(x, y) | x \in x_1 *_1 y_1, y \in x_2 *_2 y_2\}$.
The structure $(R_1 \times R_2, \oplus, \odot)$ is called the direct product of R_1 and R_2.

Theorem 2.24. *Let $(R_1, +_1, *_1)$ and $(R_2, +_2, *_2)$ be two hyperrings. Then,*

(1) *If R_1 and R_2 are two hyperrings of type 1, then $(R_1 \times R_2, \oplus, \odot)$ is a hyperring of type 1.*
(2) *If R_1 and R_2 are two hyperrings of type 2, then $(R_1 \times R_2, \oplus, \odot)$ is a hyperring of type 2.*
(3) *If R_1 and R_2 are two SDIS hyperrings, then $(R_1 \times R_2, \oplus, \odot)$ is a SDIS hyperring.*

Proof. By ([7], Theorem 2.14) we have both (1) and (2).
(3) Let $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\} \subseteq R_1 \times R_2$. If $(t_1, t_2) \in (y_1, y_2) \odot (z_1, z_2)$, then $t_1 \in y_1 *_1 z_1, t_2 \in y_2 *_2 z_2$. Since R_1 and R_2 are SDIS hyperrings, there exist $u_1 \in x_1 +_1 y_1$ and $v_1 \in x_1 +_1 z_1$ also $u_2 \in x_2 +_2 y_2$ and $v_2 \in x_2 +_2 z_2$ such that $x_1 *_1 t_1 = u_1 +_1 v_1$ and $x_2 *_2 t_2 = u_2 +_2 v_2$. Thus $(t_1, t_2) \in (y_1, y_2) \odot (z_1, z_2)$ and $(x_1, x_2) \odot (t_1, t_2) = (u_1, u_2) \oplus (v_1, v_2)$. Now let $(s_1, s_2) \in (x_1, x_2) \odot (y_1, y_2)$ and $(t_1, t_2) \in (x_1, x_2) \odot (z_1, z_2)$. Therefore $s_1 \in x_1 *_1 y_1, t_1 \in x_1 *_1 z_1, s_2 \in x_2 *_2 y_2, t_2 \in x_2 *_2 z_2$. Since R_1 and R_2 are SDIS hyperrings, there exist $w_1 \in y_1 +_1 z_1$ and $w_2 \in y_2 +_2 z_2$ such that $x_1 *_1 w_1 = s_1 +_1 t_1$ and $x_2 *_2 w_2 = s_2 +_2 t_2$. Thus $(x_1, x_2) \odot (w_1, w_2) = (s_1, s_2) \oplus (t_1, t_2)$. Therefore $R_1 \times R_2$ is a left strongly distributive. Similarly, it can be checked that $R_1 \times R_2$ is a right strongly distributive and this completes the proof. □

Corollary 2.25. *If R_1 and R_2 are two totally hyperrings, then $(R_1 \times R_2, \oplus, \odot)$ is a totally hyperring.*

Example 2.26. $(\mathbf{Z}_2 \times R, \oplus, \odot)$ is a totally hyperring, where $\mathbf{Z}_2$ and R are hyperrings in the following tables, respectively:

$+_1$	x	y
x	x	y
y	y	x

$*_1$	x	y
x	x	x
y	x	y

$+_2$	e	a	b
e	$\{e, a\}$	H	H
a	H	$\{e, a\}$	$\{e, a\}$
b	H	$\{e, a\}$	$\{e, a\}$

$*_2$	e	a	b
e	$\{e, a\}$	H	$\{e, a\}$
a	H	$\{e, a\}$	H
b	$\{e, a\}$	H	$\{e, a\}$

$\oplus$	$1=(x,e)$	$2=(x,a)$	$3=(x,b)$	$4=(y,e)$	$5=(y,a)$	$6=(y,b)$
$1=(x,e)$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{4,5\}$	$\{4,5,6\}$	$\{4,5,6\}$
$2=(x,a)$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\{4,5,6\}$	$\{4,5\}$	$\{4,5\}$
$3=(x,b)$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\{4,5,6\}$	$\{4,5\}$	$\{4,5\}$
$4=(y,e)$	$\{4,5\}$	$\{4,5,6\}$	$\{4,5,6\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$
$5=(y,a)$	$\{4,5,6\}$	$\{4,5\}$	$\{4,5\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$
$6=(y,b)$	$\{4,5,6\}$	$\{4,5\}$	$\{4,5\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$

$\odot$	$1=(x,e)$	$2=(x,a)$	$3=(x,b)$	$4=(y,e)$	$5=(y,a)$	$6=(y,b)$
$1=(x,e)$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$
$2=(x,a)$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$3=(x,b)$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$
$4=(y,e)$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{4,5\}$	$\{4,5,6\}$	$\{4,5\}$
$5=(y,a)$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\{4,5,6\}$	$\{4,5\}$	$\{4,5,6\}$
$6=(y,b)$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{4,5\}$	$\{4,5,6\}$	$\{4,5\}$

3. On totally Krasner hyperrings

Definition 3.1. (See [11]). A Krasner hyperring $(R,+,\cdot,0,1)$ is a non-empty set R such that $(R,+,0)$ is a canonical hypergroup and $(R,\cdot,1)$ is a commutative monoid which satisfies the following conditions:

(1) $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$, $(y+z)\cdot x=(y\cdot x)+(z\cdot x)$, for all $x,y,z\in R$,
(2) $x\cdot 0=0=0\cdot x$, for all $x\in R$,
(3) $0\neq 1$.

Remark 3.2. Every Krasner hyperring is a hyperring of type 2.

Example 3.3. (See [10], 2.7). Let $\mathbf{R}=(R,+,\cdot)$, the underlying set is the set of real numbers and the multiplication is the usual multiplication of real numbers. The (hyper)addition is given as follows:

$$x+y=\begin{cases}\{x\} & \text{if} \quad \mid x\mid>\mid y\mid\\ \{y\} & \text{if} \quad \mid x\mid<\mid y\mid\\ \{x\} & \text{if} \qquad x=y\\ [-|x|,|x|] & \text{if} \qquad x=-y\end{cases}$$

Then, $\mathbf{R}$ is a Krasner hyperring.

Theorem 3.4. *A Krasner hyperring R is a SDIS hyperring if and only if it is a ring.*

Proof. Suppose that $(R,+,\cdot)$ is SDIS Krasner hyperring. We show that for all $0\neq x\in R,(x+(-x))=0$. Let $a\in R$. Since R is SDIS, for every $t\in x+(-x)$ there exist $u=a\cdot x$ and $v=a\cdot(-x)$ such that

$a \cdot t = u + v = a \cdot x + a \cdot (-x)$. Set $t = 0, a = 1$. Then, we have $0 = 1 \cdot 0 = (1 \cdot x) + (1 \cdot (-x)) = x + (-x)$, thus $(x + (-x)) = 0$. Now let $b, c \in R$ and $\alpha, \beta \in b + c$. We have

$$\begin{aligned}
&\alpha \in b + c \\
\Rightarrow &c \in \alpha - b \\
\Rightarrow &\beta \in b + c = c + b \subseteq \alpha - b + b = \alpha \\
\Rightarrow &\alpha = \beta.
\end{aligned}$$

Thus, for all $(b, c) \in R^2$, $|b + c| = 1$ and hence $(R, +)$ is a group. Therefore $(R, +, \cdot)$ is a ring. The converse is obvious. □

Corollary 3.5. *Every Krasner hyperring is a ring if and only if it is totally.*

4. Strongly hyperoperations in hyperring of series

In this section we construct a hyperring of series over hyperrings. Let $(R, +, \cdot)$ be a commutative hyperring such that $(R, +)$ is regular hypergroup. A series with coefficients in R is an infinite sequence $(a_0, a_1, ..., a_n, ...)$ in which all a_i belong to R. The set of all series with coefficients in R will be denoted as usual by $R[[x]]$. Two series $(a_0, a_1, ..., a_n, ...)$ and $(b_0, b_1, ..., b_n, ...)$ are equal if and only if $a_i = b_i$, for all $i \in \mathbb{N} \cup \{0\}$. For all $(a_0, a_1, ..., a_n, ...), (b_0, b_1, ..., b_n, ...) \in R[[x]]$ we define:

$$(a_0, a_1, ..., a_n, ...) \oplus (b_0, b_1, ..., b_n, ...) = \{(c_0, c_1, ..., c_n, ...) \mid c_i \in a_i + b_i\}$$

and

$$(a_0, a_1, ..., a_n, ...) \odot (b_0, b_1, ..., b_n, ...) = \{(c_0, c_1, ..., c_n, ...) \mid c_i \in \sum_{k+l=i} a_k \cdot b_l\}.$$

More about the hyperring of series can be found in the original article of Jančić-Rašović [9].

Theorem 4.1. *(See [5], 5.6.2).* $(R[[x]], \oplus, \odot)$ *is a general hyperring.*

Theorem 4.2. *If R is a hyperring of type 1, then $R[[x]]$ is of type 1.*

Proof. Let $(a_0, a_1, ..., a_n, ...), (b_0, b_1, ..., b_n, ...), (c_0, c_1, ..., c_n, ...) \in R[[x]]$ and

$$(t_0, t_1, ..., t_n, ...) \in (a_0, a_1, ..., a_n, ...) \oplus (b_0, b_1, ..., b_n, ...).$$

Since $(R, +)$ is strongly associative, so for all $t_i \in a_i + b_i$, there exist $s_i \in b_i + c_i$ such that $t_i + c_i = a_i + s_i$. Thus

$$(t_0, t_1, ..., t_n, ...) \oplus (c_0, c_1, ..., c_n, ...) = (a_0, a_1, ..., a_n, ...) \oplus (s_0, s_1, ..., s_n, ...)$$

and

$$(s_0, s_1, ..., s_n, ...) \in (b_0, b_1, ..., b_n, ...) \oplus (c_0, c_1, ..., c_n, ...).$$

Therefore $R[[x]]$ is a right strongly associative hyperoperation. Similarly it is a left strongly associative hyperoperation. So $(R[[x]], \oplus, \odot)$ is a hyperring of type 1. $\square$

In the following, by example we show that if $(R, \cdot)$ be strongly associative, then $(R[[x]], \odot)$ is not strongly associative necessarily.

Example 4.3. Let $(R = \{0, 1\}, +, \cdot)$, where

$+$	0	1
0	0	1
1	1	$\{0,1\}$

$\cdot$	0	1
0	0	0
1	0	1

Consider $X = (0, 1, 1, ...), Y = (1, 1, 0, ...), Z = (1, 0, 0, ...) \in R[[x]]$. We have

$$T = (0, 1, 0, ...) \in X \odot Y = \{(0, 1, \{0, 1\}, \{0, 1\}, ...)\}$$

and $T \odot Z = (0, 1, 0, ...)$. But there is not an element like $S = (s_0, s_1, ...)$ belongs to $Y \odot Z = (1, 1, 1, 0, ...)$ such that $T \odot Z = X \odot S$.

Corollary 4.4. *If R is a hyperring of type 2, then $R[[x]]$ is not of type 2 necessarily.*

Example 4.5. The following hyperring is a SDIS hyperring but $R[[x]]$ is not a SDIS hyperring.

$+$	0	1
0	0	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

$\cdot$	0	1
0	0	$\{0,1\}$
1	0	$\{0,1\}$

Now, let $a = (1, 0, 0, 0, ...)$, $b = (1, 1, 0, 0, 0, ...)$ and $c = (0, 0, 1, 0, 0, 0, ...)$. If $x = (0, 0, 1, 0, 0, 0, 0, ...) \in a \odot (b \oplus c)$, then $a \odot u = \{(0, i, j, ...) | i, j, ... \in \{0, 1\}\}$, for every $u \in b \oplus c$. Moreover, $y = (1, 0, 0, 0, ...) \in a \odot b \oplus a \odot c$, but $y \notin a \odot u$.

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ON STRONGLY ASSOCIATIVE HYPERRINGS

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مطالعه ابرحلقه‌های به‌طور قوی شرکت‌پذیر

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این مقاله ایده‌ی ابرعمل به‌طور قوی شرکت‌پذیر که در مرجع [۷] معرفی شده را برای رده‌ی ابرحلقه‌ها تعمیم می‌دهد. ما ابرحلقه‌های نوع ۱ ، نوع ۲ و به‌طور قوی شرکت‌پذیر را معرفی و مورد بررسی قرار می‌دهیم. علاوه بر این، مثال‌هایی از این نوع ابرحلقه‌ها را مطالعه می‌کنیم و یک رده جدید از ابرحلقه ها که ابرحلقه‌های کلی نامیده می‌شوند، ارائه می‌دهیم. ابرحلقه‌های کلی یک رده‌بندی از ابرحلقه‌های از نوع کراسنر را مشخص می‌سازد. همچنین ابرعمل‌های بطور قوی را در ابرحلقه چندجمله‌ای‌ها مورد تحقیق قرار می‌دهیم.

کلمات کلیدی: ابرعمل بطور قوی، ابرحلقه اس دی آی اس، ابرحلقه کراسنر، ابرحلقه کلی، ابرحلقه چندجمله‌ای‌ها.