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BAER AND QUASI-BAER PROPERTIES OF SKEW PBW EXTENSIONS

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ABSTRACT. A ring R with an automorphism σ and a σ-derivation δ is called *δ-quasi-Baer* (resp., *σ-invariant quasi-Baer*) if the right annihilator of every δ-ideal (resp., σ-invariant ideal) of R is generated by an idempotent, as a right ideal. In this paper, we study Baer and quasi-Baer properties of skew PBW extensions. More exactly, let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type of a ring R. (i) It is shown that R is Δ-quasi-Baer if and only if A is quasi-Baer. (ii) R is Δ-Baer if and only if A is Baer, when R has IFP. Also, let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension of a ring R. (iii) If R is a Σ-quasi-Baer ring, then A is a quasi-Baer ring. (iv) If A is a quasi-Baer ring, then R is a Σ-invariant quasi-Baer ring. (v) If R is a Σ-Baer ring, then A is a Baer ring, when R has IFP. (vi) If A is a Baer ring, then R is a Σ-invariant Baer ring. Finally, we show that if $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension of a quasi-Baer ring R, then A is a quasi-Baer ring.

1. INTRODUCTION

Throughout this paper, R denotes an associative ring with unity. Recall from Kaplansky [21] and Clark [11] that R is a *Baer* (resp., *quasi-Baer*) ring if the right annihilator of every nonempty subset (resp., ideal) of R is generated, as a right ideal, by an idempotent. Baer rings are introduced by Kaplansky (1965) to abstract various properties of von Neumann algebras and complete $*$-regular rings. Clark uses the

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quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Another generalization of Baer rings are the p.p.-rings. A ring B is called right (resp., left) *p.p.* if the right (resp., left) annihilator of each element of B is generated by an idempotent (or equivalently, rings in which each principal right (resp., left) ideal is projective). In [8], Birkenmeier et al. defined a ring to be called a right (resp., left) *principally quasi-Baer* (or simply *right* (resp., *left*) *p.q.-Baer*) ring if the right annihilator of each principal right (resp., left) ideal of R is generated by an idempotent.

Pollingher and Zaks [28], showed that the class of quasi-Baer rings is closed under $n \times n$ matrix rings and under $n \times n$ upper (or lower) triangular matrix rings. It follows from this results that quasi-Baer condition is a Morita invariant property. Further works on quasi-Baer rings appeared in [6, 7, 8, 9, 10, 11, 13, 14, 15, 26, 27, 28].

There is considerable interest in studying if and how certain properties of rings are preserved under various ring-theoretic extensions. Armendariz [1] seems to be the first to consider the behavior of a polynomial rings over a Baer ring by obtaining the following result (recall that a ring R is called *reduced* if it has no nonzero nilpotent elements): For a reduced ring R, $R[x]$ is a Baer ring if and only if R is a Baer ring [1, Theorem B]. Armendariz provided an example to show that the reduced condition was not superfluous. Note that in a reduced ring R, R is Baer if and only if R is quasi-Baer. A generalization of Armendariz's result for several types of polynomial extensions over Baer and quasi-Baer rings, are obtained by various authors, [6, 7, 9, 13, 15, 16, 20]. In [9], Birkenmeier et al. showed that the quasi-Baer condition is preserved by many polynomial extensions. Also, Birkenmeier et al. [6] showed that a ring R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer.

Let σ be an endomorphism and δ be a σ-derivation of a ring R (so δ is an additive map satisfying $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$). The general (left) Ore extension $R[x; \sigma, \delta]$ is the ring of polynomials over R in the variable x, with coefficients written on the left of x and with termwise addition, subject to the skew-multiplication rule $xr = \sigma(r)x + \delta(r)$ for $r \in R$. If σ is an injective endomorphism of R, then we say $R[x; \sigma, \delta]$ is an Ore extension of injective type. If σ is an identity map on R or $\delta = 0$, then we denote $R[x; \sigma, \delta]$ by $R[x; \delta]$ and $R[x; \sigma]$, respectively.

According to Krempa [23], an endomorphism σ of a ring R is called to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R. Note that any rigid endomorphism of a ring is a monomorphism and σ-rigid rings are reduced by Hong et al. [20]. Properties of σ-rigid rings have been studied in Krempa [23], Hirano [19] and Hong et al. [20]. In [20], Hong et al. studied Ore extensions of quasi-Baer rings over σ-rigid rings. Further work on Ore extensions over Baer and quasi-Baer rings appeared in [13, 14, 15, 23, 26, 27].

Other ring-theoretic extensions of a ring R, which were defined by Bell and Goodearl [4], are the Poincaré-Birkhoff-Witt (PBW for short) extensions. The skew Poincaré-Birkhoff-Witt (skew PBW for short) extensions, introduced by Gallego and Lezama [12] as a generalization of PBW extensions, are more general than Ore extensions of injective type. These extensions include several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, etc.) More exactly, it has been shown that skew PBW extensions contain various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. (see [12, 29]).

It is natural to ask if these properties (Baer, quasi-Baer, p.q.-Baer and p.p.) can be extended from a ring R to the skew PBW extensions. Reyes [31], studied the behavior of skew PBW extensions over a Baer, quasi-Baer, p.p. and p.q.-Baer ring, where R is a rigid ring.

A ring R with an automorphism σ and a σ-derivation δ is called *δ-quasi-Baer* (resp., *σ-invariant quasi-Baer*) if the right annihilator of every δ-ideal (resp., σ-invariant ideal) of R is generated by an idempotent, as a right ideal.

In this paper, we further study the Baer and quasi-Baer properties of skew PBW extensions. More exactly, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of derivation type of a ring R. (i) It is shown that R is Δ-quasi-Baer if and only if A is quasi-Baer. (ii) R is Δ-Baer if and only if A is Baer, when R has IFP. Also, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of a ring R. (iii) If R is a Σ-quasi-Baer ring, then A is a quasi-Baer ring. (iv) If A is a quasi-Baer ring, then R is a Σ-invariant quasi-Baer ring. (v) If R is a Σ-Baer ring, then A is a Baer ring, when R has IFP. (vi) If A is a Baer ring, then R is a Σ-invariant Baer ring. Finally, we show that if $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of a quasi-Baer ring R, then A is a quasi-Baer ring.

For a nonempty subset U of R, $r_R(U)$ and $\ell_R(U)$ denote respectively the right and the left annihilator of U in R (if it is clear from the context, the subscript will be omitted).

2. Definitions and basic properties of skew PBW extensions

We start by recalling the definition of (skew) PBW extensions and present some key properties of these rings.

Let R and A be rings. According to Bell and Goodearl [4], we say that A is a Poincaré-Birkhoff-Witt extension (also called a *PBW extension*) of R, denoted by $A := R\langle x_1, \ldots, x_n\rangle$, if the following conditions hold:

(1) $R \subseteq A$;
(2) There exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\}$.
(3) $x_i r - r x_i \in R$ for each $r \in R$ and $1 \leq i \leq n$.
(4) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$, for any $1 \leq i, j \leq n$.

Definition 2.1. [12, Definition 1] Let R and A be rings. We say that A is a *skew PBW extension of* R (also called a *σ-PBW extension*) if the following conditions hold:

(1) $R \subseteq A$;
(2) There exist elements $x_1, \ldots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\}$.
(3) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.
(4) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

It is clear that any PBW extension is a skew PBW extension. Observe that if σ is an injective endomorphism of the ring R and δ is a σ-derivation, then the skew polynomial ring $R[x; \sigma, \delta]$ is a trivial skew PBW extension in only one variable, $\sigma(R)\langle x\rangle$. Many important classes of rings and algebras are skew PBW extensions, for example:

Example 2.2. Habitual polynomail rings, skew polynomial rings of injective type, Ore extensions of bijective type, Weyl algebras, enveloping algebras of finite dimensional Lie algebras (and its quantization), quantum n-space, n^{th} quantized Weyl algebra, quantum polynomials, diffusion algebras, Manin algebra of quantum matrices, are particular examples of skew PBW extensions. A detailed list of examples of skew PBW extensions is presented in [12, 24, 29, 30].

Now, we give some examples of skew PBW extensions which can not be expressed as Ore extensions (a more complete list can be found in [24, 29]).

Example 2.3.

(1) Let k be a commutative ring and $\mathfrak{g}$ a finite dimensional Lie algebra over k with basis $\{x_1, \dots, x_n\}$; the *universal enveloping algebra* of $\mathfrak{g}$, denoted by $\mathcal{U}(\mathfrak{g})$, is a PBW extension of k (see [24]). In this case, $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = [x_i, x_j] \in \mathfrak{g} = k + kx_1 + \cdots + kx_n$, for any $r \in k$ and $1 \leq i, j \leq n$.

(2) Let $k, \mathfrak{g}, \{x_1, \dots, x_n\}$ and $\mathcal{U}(\mathfrak{g})$ be as in the previous example; let R be a k-algebra containing k. The *tensor product* $A := R \otimes_k \mathcal{U}(\mathfrak{g})$ is a PBW extension of R, and it is a particular case of a more general construction, the *crossed product* $R * \mathcal{U}(\mathfrak{g})$ of R by $\mathcal{U}(\mathfrak{g})$, that is also a PBW extension of R (see [25]).

(3) The *twisted or smash product differential operator ring* $k\#_\sigma\mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra acting on k by derivations, and σ is Lie 2-cocycle with values in k.

Proposition 2.4. [12, Proposition 3] *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_ir = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.*

According to the properties of $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$ as mentioned in the Proposition 2.4, we need to introduce some special classes of skew PBW extensions.

Definition 2.5. Let A be a skew PBW extension of a ring R with a set of endomorphisms Σ and set of derivations Δ.

(1) If $\sigma_i = id_R$ for every $1 \leq i \leq n$, we say that A is a skew PBW extension of derivation type.

(2) If $\delta_i = 0$ for every $1 \leq i \leq n$, we say that A is a skew PBW extension of endomorphism type. In addition, if every σ_i is bijective, A is a skew PBW extension of automorphism type.

(3) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Let A be a skew PBW extension of R. According to [12, Definition 4], A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by (3$'$): for each $1 \leq i \leq n$ and all $r \in R\backslash\{0\}$ there exists $c_{i,r} \in R\backslash\{0\}$ such that $x_ir = c_{i,r}x_i$; (4$'$): for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_jx_i = c_{i,j}x_ix_j$.

Definition 2.6. [12, Definition 6] Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$ and σ_i-derivations δ_i as in Proposition 2.4.

(1) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n, \sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}, \delta^\alpha := \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(2) For $X = x^\alpha \in \mathrm{Mon}(A), \exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \mathrm{Mon}(A), \exp(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$.
Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R \backslash \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $lm(f) := X_m$, the *leading monomial* of $f; lc(f) := a_m$, the *leading coefficient* of f; $lt(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $lm(0) := 0, lc(0) := 0, lt(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$.

Remark 2.7. [12, Remark 2]

(1) Since $Mon(A)$ is a R-basis for A, the elements $c_{i,r}$ and $c_{i,j}$ in the Definition 2.1 are unique.
(2) If $r = 0$, then $c_{i,0} = 0$. Moreover, in Definition 2.1(4), $c_{i,i} = 1$.
(3) Let $i < j$. Then there exist $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, but since $Mon(A)$ is a R-basis, then $1 = c_{j,i} c_{i,j}$, i.e., for every $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.
(4) Each element $f \in A \setminus \{0\}$ has a unique representation in the form $f = a_1 X_1 + \cdots + a_t X_t$, with $a_i \in R \backslash \{0\}$ and $X_i \in Mon(A)$, $1 \leq i \leq t$.

Skew PBW extensions can be characterized in the following way.

Theorem 2.8. [12, Theorem 7] *Let A be a polynomial ring over R with respect to $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(1) *For each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(2) *For each* $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$ *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that* $x^{\alpha}x^{\beta} = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}$, *where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha+\beta|$ *if* $p_{\alpha,\beta} \neq 0$.

We remember also the following facts [12, Remark 8].

Remark 2.9.

(1) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c^{'}_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c^{'}_{\alpha,\beta} = 1$.
(2) We observe if A is a skew PBW extension quasi-commutative, then from Theorem 2.8, we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$, for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}_0^n$.
(3) From Theorem 2.8, we get also that if A is a bijective skew PBW extension, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}_0^n$.

In the next remark, we will look more closely at the form of the polynomials $p_{\alpha,r}$ and $p_{\alpha,\beta}$ which appears in Theorem 2.8.

Remark 2.10. [31, Remark 2.10]

(1) Let x_n be a variable and α_n an element of $\mathbb{N}_0$. Then we have

$$x_n^{\alpha_n} r = \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + \sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}, \quad \sigma_n^0 := id_R \tag{2.1}$$

and so

$$\begin{aligned} x_n^{\alpha_n} r &= \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + x_n^{\alpha_n-1}\delta_n(r) + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2 \\ &+ \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1}, \quad \sigma_n^0 := id_R. \end{aligned}$$

Note that

$$\begin{aligned} p_{\alpha_n,r} &= x_n^{\alpha_n-1}\delta_n(r) + x_n^{\alpha_n-2}\delta_n(\sigma_n(r))x_n + x_n^{\alpha_n-3}\delta_n(\sigma_n^2(r))x_n^2 \\ &+ \cdots + x_n\delta_n(\sigma_n^{\alpha_n-2}(r))x_n^{\alpha_n-2} + \delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1}, \end{aligned}$$

where $p_{\alpha_n,r} = 0$ or $\deg(p_{\alpha_n,r}) < |\alpha_n|$ if $p_{\alpha_n,r} \neq 0$. It is clear that $\exp(p_{\alpha_n,r}) \prec \alpha_n$. Again, using (2.1) in every term of the product $x_n^{\alpha_n} r$ above, we obtain

$$\begin{aligned} x_n^{\alpha_n} r &= \sigma_n^{\alpha_n}(r)x_n^{\alpha_n} + \sigma_n^{\alpha_n-1}(\delta_n(r))x_n^{\alpha_n-1} \\ &+ \sum_{j=1}^{\alpha_n-1} x_n^{\alpha_n-1-j}\delta_n(\sigma_n^{j-1}(\delta_n(r)))x_n^{j-1} \\ &+ \left[\sigma_n^{\alpha_n-2}(\delta_n(\sigma_n(r)))x_n^{\alpha_n-2} + \sum_{j=1}^{\alpha_n-2} x_n^{\alpha_n-2-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n(r))))x_n^{j-1}\right] x_n \end{aligned}$$

$$
\begin{aligned}
&+\left[\sigma_n^{\alpha_n-3}(\delta_n(\sigma_n^2(r)))x_n^{\alpha_n-3}+\sum_{j=1}^{\alpha_n-3}x_n^{\alpha_n-3-j}\delta_n(\sigma_n^{j-1}(\delta_n(\sigma_n^2(r))))x_n^{j-1}\right]x_n^2\\
&+\cdots+\left[\sigma_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))x_n+\delta_n(\delta_n(\sigma_n^{\alpha_n-2}(r)))\right]x_n^{\alpha_n-2}\\
&+\delta_n(\sigma_n^{\alpha_n-1}(r))x_n^{\alpha_n-1},
\end{aligned}
$$

which shows that

$$
lc(p_{\alpha_n,r})=\sum_{p=1}^{\alpha_n}\sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r))).
$$

In this way, we can see that $lc(p_{\alpha_n,r})$ involves elements obtained evaluating σ_n and δ_n in the element r of R.

(2) Let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n, r\in R$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Then

$$
\begin{aligned}
x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r &= \sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\
&+p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\
&+x_1^{\alpha_1}p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\\
&+x_1^{\alpha_1}x_2^{\alpha_2}p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}x_4^{\alpha_4}\cdots x_n^{\alpha_n}\\
&+\cdots+x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-2}^{\alpha_{n-2}}p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}x_n^{\alpha_n}\\
&+x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}p_{\alpha_n,r}.
\end{aligned}
$$

Considering the leading coefficients of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}r$, we can write this term as

$$
\begin{aligned}
&=\sigma_1^{\alpha_1}(\cdots(\sigma_n^{\alpha_n}(r)))x_1^{\alpha_1}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_1}\sigma_1^{\alpha_1-p}(\delta_1(\sigma_1^{p-1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})}\\
&\quad x_2^{\alpha_2}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_2}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2-p}(\delta_2(\sigma_2^{p-1}(\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\alpha_1}\\
&\quad x_2^{\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})}x_3^{\alpha_3}\cdots x_n^{\alpha_n}\\
&+\left[\sum_{p=1}^{\alpha_3}\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\sigma_3^{\alpha_3-p}(\delta_3(\sigma_3^{p-1}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))))))\right]x_1^{\alpha_1}x_2^{\alpha_2}\\
&\quad x_3^{\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})}x_4^{\alpha_4}\cdots x_n^{\alpha_n}+\cdots\\
&+\left[\sum_{p=1}^{\alpha_{n-1}}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-2}^{\alpha_{n-2}}(\sigma_{n-1}^{\alpha_{n-1}-p}(\delta_{n-1}(\sigma_{n-1}^{p-1}(\sigma_n^{\alpha_n}(r)))))))\right]x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}
\end{aligned}
$$

$$x_{n-1}^{\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})}x_n^{\alpha_n}$$
$$+\left[\sum_{p=1}^{\alpha_n}\sigma_1^{\alpha_1}(\cdots(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n-p}(\delta_n(\sigma_n^{p-1}(r)))))))\right]x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\deg(p_{\alpha_n},r)}$$
$$+\text{other terms of degree less than } \deg(p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))})$$
$$+\alpha_2+\cdots+\alpha_n$$
$$+\text{other terms of degree less than } \alpha_1+\deg(p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))})$$
$$+\alpha_3+\cdots+\alpha_n$$
$$+\text{other terms of degree less than } \alpha_1+\alpha_2+\deg(p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))})$$
$$+\alpha_4+\cdots+\alpha_n$$
$$\vdots$$
$$+\text{other terms of degree less than } \alpha_1+\cdots+\alpha_{n-2}+\deg(p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)})$$
$$+\alpha_n$$
$$+\text{other terms of degree less than } \alpha_1+\cdots+\alpha_{n-1}+\deg(p_{\alpha_n,r}).$$

Therefore, we can see that the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}, p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \dots, p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$ in the expression above for the term $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}r$, involve elements obtained evaluating σ's and δ's in the element r of R.

(3) Let $X_i := x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$ and $a_i, b_j \in R$. Then

$$\begin{aligned}
a_iX_ib_jY_j &= a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\
&+ \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j}\\
&+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}.
\end{aligned}$$

As we saw in above, the polynomials $p_{\alpha_1,\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2,\sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3,\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \dots, p_{\alpha_{n-1},\sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n,r}$, involve elements of R obtained evaluating σ_j and δ_j in the element r of R. So, when we compute every summand of $a_iX_ib_jY_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

3. Baer and quasi-Baer skew PBW extensions of derivation type

Let δ be a derivation on a ring R. A subset I of R is called δ-*subset* if $\delta(I) \subseteq I$. According to Han et al. [15], a ring R is called δ-*quasi-Baer* (resp., δ-Baer) if the right annihilator of every δ-ideal (resp., δ-subset) of R is generated, as a right ideal, by an idempotent. Recall from Bell [3], that R is said to satisfy the *insertion of factors property* (IFP) if $r_R(x)$ is an ideal for all $x \in R$. An idempotent $e \in R$ is left (resp., right) *semicentral* in R if $Re = eRe$ (resp., $eR = eRe$) by [5].

Throughout this section, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of derivation type of a ring R. Let $\Delta := \{\delta_1, \ldots, \delta_n\}$ be the derivations as mentioned in the Proposition 2.4. We say a subset I of R is Δ-*subset* if $\delta^\alpha(I) \subseteq I$ for each $\alpha \in \mathbb{N}_0^n$. Moreover, R is said to be Δ-*quasi-Baer* (resp., Δ-*Baer*) if the right annihilator of every Δ-ideal (resp., Δ-*subset*) of R is generated by an idempotent, as a right ideal.

Let I be an ideal of R. We denote the set of all elements of A with coefficients in I by $I\langle x_1, \ldots, x_n\rangle$. If A is a skew PBW extension of derivation type of a ring R and I is an Δ-ideal of R, then by using Remark 2.10 one can show that $I\langle x_1, \ldots, x_n\rangle$ is an ideal of A.

Our main aim in this section is to provide a necessary and sufficient conditions to the skew PBW extension of derivation type to be (quasi-) Baer.

Lemma 3.1. *Let I be a Δ-ideal and U a Δ-subset of R. Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of derivation type.*

(1) *If $r_A(I\langle x_1, \ldots, x_n\rangle) = eA$ for some idempotent $e = e_0 + e_1X_1 + \cdots + e_mX_m \in A$, then $r_A(I\langle x_1, \ldots, x_n\rangle) = e_0A$.*
(2) *If $r_A(U) = eA$ for some idempotent $e = e_0 + e_1X_1 + \cdots + e_mX_m \in A$, then $r_A(U) = e_0A$.*

Proof. We only prove (1), as the proof of (2) is similar to that of (1). Since $Ie = 0$, we have $Ie_i = 0$ for each $i = 0, \ldots, m$. Hence for each $c \in I$, we have $0 = \delta_{t_1}(ce_i) = \delta_{t_1}(c)e_i + c\delta_{t_1}(e_i)$ for all $i = 0, \ldots, m$ and $t_1 = 1, \ldots, n$. Since I is Δ-ideal and $Ie_i = 0$, we have $c\delta_{t_1}(e_i) = 0$ and hence $c\delta_{t_1}^{k_1}(e_i) = 0$ for each $i = 0, \ldots, m$, $t_1 = 1, \ldots, n$ and $k_1 \geq 0$. By the same method, we can see that $c\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\ldots\delta_{t_j}^{k_j}(e_i) = 0$ for each $i = 0, \ldots, m$, $\{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. We claim that $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) \in r_A(I\langle x_1, \ldots, x_n\rangle)$ for each $i = 0, \ldots, m$, $\{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in I\langle x_1, \ldots, x_n\rangle$, where $a_i \in I$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, and $X_m \succ X_{m-1} \succ \cdots \succ X_1$. Then we have

$f\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = a_0\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) + \cdots + a_m x^{\alpha_m}\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$. When we compute every summand of $a_i x^{\alpha_i}\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$, we obtain products of the coefficient a_i with several evaluations of $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$ in δ's depending of the coordinates of α_i. Since I is Δ-ideal and $c\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = 0$ for each $c \in I$, $i = 0,\ldots,m$, $\{t_1,\ldots,t_j\} \subseteq \{1,\ldots,n\}$ and $k_s \geq 0$, we obtain $f\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\ldots\delta_{t_j}^{k_j}(e_i) = 0$. Thus $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = e\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$ and that $e_m\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = 0$, which implies that $e_m X_m(\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)) = 0$ for each $i = 0,\ldots,m$, $\{t_1,\ldots,t_j\} \subseteq \{1,\ldots,n\}$ and $k_s \geq 0$. Hence $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = (e_0 + e_1X_1 + \cdots + e_{m-1}X_{m-1})\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$ and that $e_{m-1}\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = 0$ for each $i = 0,\ldots,m$, $\{t_1,\ldots,t_j\} \subseteq \{1,\ldots,n\}$ and $k_s \geq 0$. Continuing in this way, we have $e_jX_j(\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)) = 0$ for each $i = 0,\ldots,m$, $j = 1,\ldots,m$, $\{t_1,\ldots,t_j\} \subseteq \{1,\ldots,n\}$ and $k_s \geq 0$. Thus $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i) = e_0\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_i)$ for each $i = 0,\ldots,m$, $\{t_1,\ldots,t_j\} \subseteq \{1,\ldots,n\}$ and $k_s \geq 0$. Thus $e_i = e_0e_i$ for each i and so $e = e_0e$. Therefore $r_A(I\langle x_1,\ldots,x_n\rangle) = eA \subseteq e_0A$. On the other hand, since $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e_0) \in r_R(I)$, so $e_0 \in r_A(I\langle x_1,\ldots,x_n\rangle)$ and that $e_0A \subseteq r_A(I\langle x_1,\ldots,x_n\rangle)$. Therefore $r_A(I\langle x_1,\ldots,x_n\rangle) = e_0A$. □

Proposition 3.2. *Let R be a ring and $A = \sigma(R)\langle x_1,\ldots,x_n\rangle$ be a skew PBW extension of derivation type. If R is a Δ-quasi-Baer ring, then A is a quasi-Baer ring.*

Proof. Let I be an arbitrary ideal of A. Denote by I_0 the set of leading coefficients of elements of I. First, we show that I_0 is an ideal of R. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in I$, where $a_i \in I$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in I$, $1 \leq j \leq t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}}\cdots x_n^{\alpha_{jn}}$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Then by Theorem 2.8,

$$fx^{\alpha_t} = \cdots + a_mx^{\alpha_m}x^{\alpha_t} = \cdots + a_mp_{\alpha_m,\alpha_t} + a_mx^{\alpha_m+\alpha_t} \in I,$$

where $p_{\alpha_m,\alpha_t} = 0$ or $\deg(p_{\alpha_m,\alpha_t}) < |\alpha_m + \alpha_t|$ if $p_{\alpha_m,\alpha_t} \neq 0$. Similarly,

$$gx^{\alpha_m} = \cdots + b_tx^{\alpha_t}x^{\alpha_m} = \cdots + b_tp_{\alpha_t,\alpha_m} + b_tx^{\alpha_t+\alpha_m} \in I,$$

where $p_{\alpha_t,\alpha_m} = 0$ or $\deg(p_{\alpha_t,\alpha_m}) < |\alpha_t + \alpha_m|$ if $p_{\alpha_t,\alpha_m} \neq 0$. Since I is an ideal of A, $fx^{\alpha_t} + gx^{\alpha_m} = \cdots + (a_m + b_t)x^{\alpha_t+\alpha_m} \in I$ and hence $a_m + b_t \in I_0$. Now, suppose that $f \in I$ and $r \in R$. Then

$$fr = \cdots + a_mx^{\alpha_m}r = \cdots + a_mp_{\alpha_m,r} + a_mrx^{\alpha_m} \in I,$$

where $p_{\alpha_m,r}=0$ or $\deg(p_{\alpha_m,r})<|\alpha_m|$ if $p_{\alpha_m,r}\neq 0$. Hence $a_m r\in I_0$. To show that I_0 is Δ-ideal, let $f=a_0+a_1X_1+\cdots+a_mX_m\in I$. Then by Remark 2.10 (2), we have

$$x^\alpha f-fx^\alpha=\cdots+[a_mx_1+\delta_1(a_m)x_1^{\deg(p_{1,a_m})}+a_{m-1}x^{\alpha_{m-1}}]x^{\alpha_m}-fx^\alpha\in I,$$

where $\alpha=(1,0,\ldots,0)\in\mathbb{N}_0^n$. Thus $\delta_1(a_m)\in I_0$, and that I_0 is Δ-ideal of R. Since R is Δ-quasi-Baer, $r_R(I_0)=eR$ for some idempotent $e\in R$. On the other hand, since $I_0e=0$ and I_0 is Δ-ideal of R, we have $I_0\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)=0$ for each $\{t_1,\ldots,t_j\}\subseteq\{1,\ldots,n\}$ and $k_s\geq 0$. Hence $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)=e\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)$ and $eA\subseteq r_A(I_0\langle x_1,\ldots,x_n\rangle)$. Now, we prove that $r_A(I_0\langle x_1,\ldots,x_n\rangle)\subseteq eA$. Let $g=b_0+b_1Y_1+\cdots+b_tY_t\in r_A(I_0\langle x_1,\ldots,x_n\rangle)$ and $f=a_0+a_1X_1+\cdots+a_mX_m\in I_0\langle x_1,\ldots,x_n\rangle$. Since $fg=0$ and $lc(fg)=a_mb_t$, we have $a_mb_t=0$. Hence $b_t\in r_R(I_0)=eR$ and so $b_t=eb_t$. Therefore $f(b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1})+feb_tY_t=0$. Since $fe=0$, hence $feb_tY_t=0$, and so $f(b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1})=0$. A similar argument shows that $b_{t-1}=eb_{t-1}$. Continuing in this way, we get $b_j=eb_j$ for each $0\leq j\leq t$, and that $g=eb_0+eb_1Y_1+\cdots+eb_tY_t$. Hence $g=eg$. Then $r_A(I_0\langle x_1,\ldots,x_n\rangle)\subseteq eA$. Therefore, $r_A(I_0\langle x_1,\ldots,x_n\rangle)=eA$. We claim that $r_A(I)=eA$. Let $f=a_0+a_1X_1+\cdots+a_mX_m\in I$. Then $a_m\in I_0$ and that $a_m\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)=0$ for each $\{t_1,\ldots,t_j\}\subseteq\{1,\ldots,n\}$ and $k_s\geq 0$. Hence, by Remark 2.10, we get

$$fe=(a_0+a_1X_1+\cdots+a_{m-1}X_{m-1})e=\cdots+a_{m-1}p_{\alpha_{m-1},e}+a_{m-1}eX_{m-1}.$$

Thus $a_{m-1}e\in I_0$, and then $a_{m-1}\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)=a_{m-1}e\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)$ for each $\{t_1,\ldots,t_j\}\subseteq\{1,\ldots,n\}$ and $k_s\geq 0$. Hence $a_{m-1}X_{m-1}e=0$. Continuing in this way, we can show that $a_iX_ie=0$, for each $i=0,\ldots,m$. Hence $fe=0$ and so $eA\subseteq r_A(I)$. Now, let $g=b_0+b_1Y_1+\cdots+b_tY_t\in r_A(I)$ and $f=a_0+a_1X_1+\cdots+a_mX_m\in I$. First, we will show that $a_iX_ib_jY_j=0$ for each $i=0,\ldots,m,\ j=0,\ldots,t$. Since $fg=0$ we obtain $a_mb_t=0$. Hence $b_t\in r_R(I_0)$. Since I_0 is Δ-ideal of R, then $\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(b_t)\in r_R(I_0)$ for each $\{t_1,\ldots,t_j\}\subseteq\{1,\ldots,n\}$ and $k_s\geq 0$ and that $b_t\in r_A(I_0\langle x_1,\ldots,x_n\rangle)$. Thus $b_t=eb_t$ and $a_mX_mb_tY_t=0$. Since $fe=(a_0+a_1X_1+\cdots+a_mX_m)e=(a_0+a_1X_1+\cdots+a_{m-1}X_{m-1})e$, we have $a_{m-1}e\in I_0$ and $a_{m-1}\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)=a_{m-1}e\delta_{t_1}^{k_1}\delta_{t_2}^{k_2}\cdots\delta_{t_j}^{k_j}(e)$ for each $\{t_1,\ldots,t_j\}\subseteq\{1,\ldots,n\}$ and $k_s\geq 0$. Thus $a_{m-1}X_{m-1}b_tY_t=0$. Continuing in this way, we have $a_iX_ib_jY_j=0$ for each $i,\ j$. Therefore $b_j\in r_A(I_0\langle x_1,\ldots,x_n\rangle)=eA$, for each $j\geq 0$. Consequently, $g=eg$ and $r_A(I)=eA$. Therefore, A is a quasi-Baer ring. $\square$

Since each quasi-Baer ring is Δ-quasi-Baer, Proposition 3.2 immediately implies the following corollary.

Corollary 3.3. *Let R be a quasi-Baer ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type. Then A is a quasi-Baer ring.*

Theorem 3.4. *Let R be a ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type. Then the following are equivalent:*

(1) *R is Δ-quasi-Baer;*
(2) *A is quasi-Baer.*

Proof. (1) $\Rightarrow$ (2). It follows from Proposition 3.2.

(2) $\Rightarrow$ (1). Suppose that A is quasi-Baer and I be any Δ-ideal of R. Then $I\langle x_1, \dots, x_n\rangle$ is an ideal of A. Since A is quasi-Baer, $r_A(I\langle x_1, \dots, x_n\rangle) = eA$ for some idempotent $e \in A$. Hence, by Lemma 3.1, we get $r_A(I\langle x_1, \dots, x_n\rangle) = e_0A$, for some idempotent $e_0 \in R$. Since $r_R(I) = r_A(I\langle x_1, \dots, x_n\rangle) \cap R = e_0A \cap R = e_0R$, hence R is Δ-quasi-Baer. □

The following is an example of a Δ-quasi-Baer ring R which is not quasi-Baer, but a skew PBW extension $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ of derivation type is quasi-Baer.

Example 3.5. [15, Example] Let p be a prime and $R = \mathbb{Z}_p[t]/(t^p)$ with the derivation δ such that $\delta(\bar{t}) = 1$ where $\bar{t} = t+(t^p)$ in R and $\mathbb{Z}_p[t]$ is the polynomial ring over a field $\mathbb{Z}_p$ of p elements. Consider the skew PBW extension of derivation type $R[x;\delta]$. Then $R[x;\delta] \cong Mat_p(\mathbb{Z}_p[x^p]) \cong Mat_p(\mathbb{Z}_p)[y]$, where $Mat_p(\mathbb{Z}_p)[y]$ is the polynomial ring over $Mat_p(\mathbb{Z}_p)$. Thus $R[x;\delta]$ is a quasi-Baer ring, but R is not quasi-Baer. Also, by Theorem 3.4, R is a δ-quasi-Baer ring.

In the following results, we show that the Baer property is inherited by skew PBW extensions of derivation type. The next lemma will be used in Theorem 3.7.

Lemma 3.6. *Let R be a ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type. If e is a right semicentral idempotent element of a ring R, then e is also a right semicentral idempotent element of the ring $A = \sigma(R)\langle x_1, \dots, x_n\rangle$.*

Proof. Let e be a right semicentral idempotent element of R. We have $\delta_t(e) = \delta_t(e^2) = \delta_t(e)e + e\delta_t(e)$ for each $1 \leq t \leq n$. Then $e\delta_t(e) = e\delta_t(e)e + e\delta_t(e)$. Since e is right semicentral, hence $e\delta_t(e) = e\delta_t(e)e$. Then $e\delta_t(e) = 0$ and hence $\delta_t(e) = \delta_t(e)e$ for each $1 \leq t \leq n$. So, from

$$0 = \delta_t(e\delta_t(e)) = \delta_t(e)\delta_t(e) + e\delta_t^2(e)$$

$$= \delta_t(e)e\delta_t(e)e + e\delta_t^2(e),$$

we have $e\delta_t^2(e) = 0$, since $e\delta_t(e) = 0$. Continuing in this process, we get $e\delta_t^k(e) = 0$ for each $k \geq 0$. Now, we prove that for each element $f \in A, efe = ef$. We work by induction on $\deg(f)$. If $\deg(f) = 0$, then the assertion is clear. Now, assume that the assertion holds for elements of A with degree less than $|\alpha_m|$ and let $f = q + a_m x^{\alpha_m}$, where $\deg(q) < |\alpha_m|$. Then by using Remark 2.10, we have

$$\begin{aligned} fe &= qe + a_m x^{\alpha_m} e \\ &= qe + a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} (x_n^{\alpha_{mn}} e) \\ &= qe + a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} (e x_n^{\alpha_{mn}} + p_{\alpha_{mn},e}) \\ &= qe + a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e x_n^{\alpha_{mn}} \\ &\quad + a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} p_{\alpha_{mn},e}. \end{aligned}$$

Note that

$$\begin{aligned} p_{\alpha_{mn},e} &= x_n^{\alpha_{mn}-1}\delta_n(e) + x_n^{\alpha_{mn}-2}\delta_n(e)x_n + x_n^{\alpha_{mn}-3}\delta_n(e)x_n^2 \\ &\quad + \cdots + x_n\delta_n(e)x_n^{\alpha_{mn}-2} + \delta_n(e)x_n^{\alpha_{mn}-1}, \end{aligned}$$

where $p_{\alpha_{mn},e} = 0$ or $\deg(p_{\alpha_{mn},e}) < |\alpha_{mn}|$ if $p_{\alpha_{mn},e} \neq 0$, by Remark 2.10. By induction hypothesis, we have

$$e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} = e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e.$$

Thus

$$\begin{aligned} efe &= eqe + e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e x_n^{\alpha_{mn}} \\ &\quad + e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e p_{\alpha_{mn},e}. \end{aligned}$$

Applying Theorem 2.8 for every term of $p_{\alpha_{mn},e}$, and since $e\delta_t^k(e) = 0$ for each $k \geq 0$, we obtain $ep_{\alpha_{mn},e} = 0$. Hence

$$efe = eqe + e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e x_n^{\alpha_{mn}}.$$

By induction hypothesis, $eqe = eq$ and

$$e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} e = e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}}.$$

Thus

$$\begin{aligned} efe &= eq + e a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} x_n^{\alpha_{mn}} \\ &= e(q + a_m x_1^{\alpha_{m1}} x_2^{\alpha_{m2}} \cdots x_{n-1}^{\alpha_{m(n-1)}} x_n^{\alpha_{mn}}) \\ &= e(q + a_m x^{\alpha_m}) \\ &= ef, \end{aligned}$$

completing the induction step, and the proof. $\square$

Theorem 3.7. *Let R be a ring with IFP and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type. Then the following are equivalent:*

(1) *R is Δ-Baer;*
(2) *A is Baer.*

Proof. (1) $\Rightarrow$ (2). Let $U \subseteq A$ and I_0 be the set of leading coefficients of elements of AUA. Then, by a similar way as used in the proof of Proposition 3.2, we can prove that I_0 is a Δ-ideal of R. Since R is Δ-Baer, $\ell_R(I_0) = Re$ for some right semicentral idempotent $e \in R$. We will show that $\ell_A(U) = Ae$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in U$. So $ea_m = 0$. Since $ef = ea_0 + ea_1X_1 + \cdots + ea_{m-1}X_{m-1} \in AUA$ and $ea_{m-1} \in I_0$, hence $ea_{m-1} = 0$. By this method we see that $ea_i = 0$ for each $0 \leq i \leq m$. Thus $ef = 0$ and so $Ae \subseteq \ell_A(U)$. Let $g = b_0+b_1Y_1+\cdots+b_tY_t \in \ell_A(U)$ and $f = a_0+a_1X_1+\cdots+a_mX_m \in U$. Since $gf = 0$ hence $b_ta_m = 0$. Since R satisfies IFP, $b_tRa_m = 0$. Thus for each $c \in I_0$, we get $b_tc = 0$. Whence $b_t \in \ell_R(I_0) = Re$ and so $b_t = b_te$. Since $gf = 0$, then $(b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1})f + b_teY_tf = 0$. But e is right semicentral in R, so from Lemma 3.6, e is right semicentral in A. Thus $b_teY_tf = b_teY_tef = 0$. Since $ef = 0$, hence $b_teY_tf = 0$ and then $(b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1})f = 0$. Similarly, we have $b_{t-1} = b_{t-1}e$, and by this method we see that $b_j = b_je$, for each $0 \leq j \leq t$. Hence $g = b_0e + b_1eY_1 + \cdots + b_teY_t$. Since e is right semicentral in A, we get $g = b_0e + b_1eY_1e + \cdots + b_teY_te$ and therefore $g = ge$. Thus $\ell_A(U) = Ae$.

(2) $\Rightarrow$ (1). Suppose that A is a Baer ring and U be any Δ-subset of R. Since $U \subseteq A$ and A is Baer, then $r_A(U) = eA$ for some idempotent $e \in A$. Hence $r_A(U) = e_0A$ for some idempotent $e_0 \in R$, by Lemma 3.1. Since $r_R(U) = r_A(U) \cap R = e_0A \cap R = e_0R$, hence R is Δ-Baer. □

Recall from Han et al. [15], that a ring R with a derivation δ is called *δ-semiprime* if for each δ-ideal I of R, $I^2 = 0$ implies that $I = 0$. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type of a ring R. We say that R is *Δ-semiprime* if for each Δ-ideal I of R, $I^2 = 0$ implies that $I = 0$. Since reduced rings are Δ-semiprime, then Theorem 3.7 immediately implies the following corollary.

Corollary 3.8. *Let R be a reduced ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of derivation type. Then the following are equivalent:*

(1) *R is Δ-Baer;*
(2) *R is Baer;*
(3) *A is Baer.*

Corollary 3.9. *Let R be a ring with IFP and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of derivation type. Then R is a Baer ring if and only if so is A.*

Corollary 3.10. [26, Corollary 3.4] *Let R be a ring with IFP. Then R is a Baer ring if and only if so is $R[x;\delta]$.*

The following example shows that, in Theorem 3.7, the IFP condition on R is not superfluous.

Example 3.11. [9, Example 1.1] Let $R = Mat_2(\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers. It is shown in [21], that R is a Baer ring, but $R[x]$ is not Baer. Observe that the right annihilator

$$r_{R[x]}\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x\right)$$

contains no nonzero idempotent element in $R[x]$, and that R does not satisfy IFP.

Since *abelian* (i.e., every idempotent is central) Baer rings are reduced and reduced rings satisfy IFP, we deduce the following result.

Corollary 3.12. [26, Corollary 3.8] *Let R be an abelian ring. Then R is a Baer ring if and only if so is $R[x;\delta]$.*

A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. As observed by Armendariz [1], reduced rings are Armendariz. Moreover, following [22], Armendariz rings are abelian.

Corollary 3.13. [26, Corollary 3.9] *Let R be an Armendariz ring. Then R is a Baer ring if and only if so is $R[x;\delta]$.*

4. Baer and quasi-Baer quasi-commutative skew PBW extensions

In this section, let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of a ring R, unless otherwise stated. Let σ be an automorphism of a ring R. A subset I of R is called *σ-subset* (resp., *σ-invariant subset*) if $\sigma(I) \subseteq I$ (resp., $\sigma(I) = I$). According to Hirano [17], a ring R is called *σ-quasi-Baer* (resp., *σ-invariant quasi-Baer*) if the right annihilator of every σ-ideal (resp., σ-invariant ideal) of R is generated by an idempotent. Also a ring R is a *σ-Baer* (resp., *σ-invariant Baer*) ring if the right annihilator of every nonempty σ-subset (resp., σ-invariant) of R is generated, as a right ideal, by an idempotent.

Let $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ be as mentioned in the Proposition 2.4. We say that a subset I of R is Σ-*subset* (resp., Σ-*invariant subset*), if $\sigma^{\alpha}(I) \subseteq I$ (resp., $\sigma^{\alpha}(I) = I$) for each $\alpha \in \mathbb{N}_0^n$, where σ^{α} is as mentioned in the Definition 2.6. Moreover, R is said to be Σ-*quasi-Baer* (resp., Σ-*invariant quasi-Baer*), if the right annihilator of every Σ-ideal (resp., Σ-invariant ideal) of R is generated by an idempotent, as a right ideal. Also a ring R is a Σ-*Baer* (resp., Σ-*invariant Baer*) ring if the right annihilator of every nonempty Σ-subset (resp., Σ-invariant subset) of R is generated, as a right ideal, by an idempotent.

Our main result in this section states that the (quasi)-Baer condition on R is preserved by quasi-commutative skew PBW extensions of R.

Proposition 4.1. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension of a ring R. If R is a Σ-quasi-Baer ring, then A is a quasi-Baer ring.*

Proof. Suppose that R is a Σ-quasi-Baer ring and I be an arbitrary ideal of A. Denote by I_0 the set of leading coefficients of elements of I. First, we will show that I_0 is an ideal of R. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in I$, where $a_i \in I$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in I$, $1 \leq j \leq t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}}$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Then by Remark 2.10, we have

$$\begin{aligned} f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m}) &= \cdots + a_m x^{\alpha_m}(\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})) \\ &= \cdots + a_m \sigma^{\alpha_m}(\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m}))x^{\alpha_m}. \end{aligned}$$

Multiplying x^{α_t} from right-hand side and applying Remark 2.7, we get $f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})x^{\alpha_t} = a_m x^{\alpha_m+\alpha_t} \in I$. By a similar way as above, we get $g\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})x^{\alpha_m} = b_t x^{\alpha_t+\alpha_m} \in I$. Since I is an ideal of A, $f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})x^{\alpha_t} + g\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})x^{\alpha_m} = (a_m + b_t)x^{\alpha_t+\alpha_m} \in I$, and hence $a_m + b_t \in I_0$. Now, suppose that $f \in I$ and $r \in R$. Then

$$\begin{aligned} f\sigma^{-\alpha_m}(r) &= \cdots + a_m x^{\alpha_m}(\sigma^{-\alpha_m}(r)) \\ &= \cdots + a_m r x^{\alpha_m} \in I. \end{aligned}$$

Hence $a_m r \in I_0$. To show that I_0 is Σ-ideal, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Then

$$\begin{aligned} x^{\alpha} f &= x^{\alpha}(a_0 + a_1X_1 + \cdots + a_mX_m) \\ &= \sigma^{\alpha}(a_0)x^{\alpha} + \sigma^{\alpha}(a_1)x^{\alpha}x^{\alpha_1} + \cdots + \sigma^{\alpha}(a_m)x^{\alpha}x^{\alpha_m} \\ &= \sigma^{\alpha}(a_0)x^{\alpha} + \sigma^{\alpha}(a_1)x^{\alpha}x^{\alpha_1} + \cdots + \sigma^{\alpha}(a_m)c_{\alpha,\alpha_m}x^{\alpha+\alpha_m} \in I. \end{aligned}$$

So that $\sigma^{\alpha}(a_m)c_{\alpha,\alpha_m} \in I_0$ and hence $\sigma^{\alpha}(a_m) \in I_0$, since c_{α,α_m} is invertible. Therefore I_0 is a Σ-ideal of R. Hence, there exists an idempotent

$e \in R$ such that $\ell_R(I_0) = Re$. First, to see that $Ae \subseteq \ell_A(I)$, take $0 \neq f = a_0+a_1X_1+\cdots+a_mX_m \in I$. Since $a_m \in I_0$, hence $ea_m = 0$. But $ef = ea_0 + ea_1X_1 + \cdots + ea_{m-1}X_{m-1} \in I$, hence $ea_{m-1} = eea_{m-1} = 0$, since $ea_{m-1} \in I_0$. Similarly, we can get that $ea_i = 0$ for each $0 \leq i \leq m$. So $ef = 0$ and hence $e \in \ell_A(I)$. Therefore $Ae \subseteq \ell_A(I)$. Now, we claim that $\ell_A(I) \subseteq Ae$. Let $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in \ell_A(I)$. We shall show that $g = ge$. The proof is by induction on the degree of elements of $\ell_A(I)$. Assume that $g \in \ell_A(I)$ with $deg(g) = 0$. Let $c \in I_0$ and $f = c_0 + c_1X_1 + \cdots + c_{m-1}X_{m-1} + cX_m \in I$. Then from $gf = 0$, we get $gc = 0$. Thus $g \in \ell_R(I_0) = Re$ and hence $g = ge$. Now, assume inductively that the assertion is true for all $g \in \ell_A(I)$ with $\deg(g) \leq |\alpha_t|$. Now, let $g = b_0 + b_1X_1 + \cdots + b_tX_t \in \ell_A(I)$. Since σ_i is surjective for each $1 \leq i \leq n$, $b_t = \sigma^{\alpha_t}(c)$ for some $c \in R$. Now let $a \in I_0$ and $f = a_0 + a_1X_1 + \cdots + aX_m \in I$. Then $gf = 0$. So, from the equality $lc(gf) = b_t\sigma^{\alpha_t}(a)c_{\alpha_t,\alpha_m} = 0$, we obtain $b_t\sigma^{\alpha_t}(a) = 0$ and hence $\sigma^{\alpha_t}(c)\sigma^{\alpha_t}(a) = 0$, since σ_i is surjective for each $1 \leq i \leq n$. Then $ca = 0$ and hence $c \in \ell_R(I_0)$, so $c = ce$. Therefore, $b_t = \sigma^{\alpha_t}(c) = \sigma^{\alpha_t}(ce) = b_t\sigma^{\alpha_t}(e)$. Thus we have $g = b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1}+b_t\sigma^{\alpha_t}(e)Y_t = b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1}+b_tY_te$. Now, we have $gI = (b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + b_tY_te)I = 0$ and $eI = 0$, so $b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} \in \ell_A(I)$. By induction hypothesis, $(b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1}) = (b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1})e$. So, we get $g = b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1}+b_tY_te = (b_0+b_1Y_1+\cdots+b_{t-1}Y_{t-1})e+b_tY_te = ge$. Consequently, $g = ge$ and $\ell_A(I) = eA$. Therefore, A is a quasi-Baer ring. $\square$

Since each quasi-Baer ring is Σ-quasi-Baer, Proposition 4.1 immediately implies the following corollary.

Corollary 4.2. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension of a ring R. If R is a quasi-Baer ring, then so is A.*

Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of a ring R and I be an Σ-invariant ideal of R. Then, by Remark 2.10, $I\langle x_1, \ldots, x_n\rangle$ is an ideal of A.

Lemma 4.3. *Let I be a Σ-invariant ideal and U a Σ-invariant subset of R. Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type.*

(1) *If $r_A(I\langle x_1, \ldots, x_n\rangle) = eA$ for some idempotent $e = e_0 + e_1X_1 + \cdots + e_mX_m \in A$, then $r_A(I\langle x_1, \ldots, x_n\rangle) = e_0A$.*
(2) *If $r_A(U) = eA$ for some idempotent $e = e_0+e_1X_1+\cdots+e_mX_m \in A$, then $r_A(U) = e_0A$.*

Proof. (1) Since $Ie = 0$, we have $Ie_i = 0$ for each $i = 0, \ldots, m$. Since I is Σ-invariant ideal, then for each $c \in I$, we get $0 = \sigma_{t_1}^{-1}(c) \in I$. So $\sigma_{t_1}^{-1}(c)e_i = 0$ and hence $c\sigma_{t_1}(e_i) = 0$, for each $i = 0, \ldots, m,\ t_1 = 1, \ldots, n$. By the same method, we see that $c\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$ for each $i = 0, \ldots, m,\ \{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. We claim that $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) \in r_A(I\langle x_1, \ldots, x_n\rangle)$ for each $i = 0, \ldots, m$, $\{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in I\langle x_1, \ldots, x_n\rangle$, $a_i \in I$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$, and $X_m \succ X_{m-1} \succ \cdots \succ X_1$. Then we have $f\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = a_0\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) + \cdots + a_mx^{\alpha_m}\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$. When we compute every summand of $a_ix^{\alpha_i}\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)$, we obtain products of the coefficient a_i with several evaluations of $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)$ in σ's depending of the coordinates of α_i. Since $c\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$ for each $c \in I,\ i = 0, \ldots, m,\ \{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$, we obtain $f\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$. Thus $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = e\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)$ and that $e_m\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$ for each $i = 0, \ldots, m,\ \{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Hence $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = (e_0 + e_1X_1 + \cdots + e_{m-1}X_{m-1})\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)$ and that $e_{m-1}\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = 0$ for each $i = 0, \ldots, m,\ \{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Continuing in this way, we have $e_jX_j(\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)) = 0$ for each $i = 0, \ldots, m,\ j = 1, \ldots, m,\ \{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Thus $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i) = e_0\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_i)$ for each $i = 0, \ldots, m$, $\{t_1, \ldots, t_j\} \subseteq \{1, \ldots, n\}$ and $k_s \geq 0$. Hence, $e_i = e_0e_i$ for each i and so $e = e_0e$. Therefore, $r_A(I\langle x_1, \ldots, x_n\rangle) = eA \subseteq e_0A$. On the other hand, since $\sigma_{t_1}^{k_1}\sigma_{t_2}^{k_2}\cdots\sigma_{t_j}^{k_j}(e_0) \in r_R(I)$, so $e_0 \in r_A(I\langle x_1, \ldots, x_n\rangle)$ and that $e_0A \subseteq r_A(I\langle x_1, \ldots, x_n\rangle)$. Therefore, $r_A(I\langle x_1, \ldots, x_n\rangle) = e_0A$.

(2) The proof is similar to that of (1). □

Proposition 4.4. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type. If A is a quasi-Baer ring, then R is an Σ-invariant quasi-Baer ring.*

Proof. Suppose that A is quasi-Baer and I be any Σ-invariant ideal of R. Then $I\langle x_1, \ldots, x_n\rangle$ is an ideal of A. Since A is quasi-Baer, $r_A(I\langle x_1, \ldots, x_n\rangle) = eA$, for some idempotent $e \in A$. Hence, by Lemma 4.3, $r_A(I\langle x_1, \ldots, x_n\rangle) = e_0A$, for some idempotent $e_0 \in R$. Since $r_R(I) = r_A(I\langle x_1, \ldots, x_n\rangle) \cap R = e_0A \cap R = e_0R$, then R is Σ-invariant quasi-Baer. □

In the following, we investigate the relationship between the Baer property of a ring R and that of the quasi-commutative skew PBW extension of automorphism type $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Proposition 4.5. *Let R be a ring with IFP and $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type. If R is a Σ-Baer ring, then A is a Baer ring.*

Proof. Let $U \subseteq A$ with $\sigma(U) = U$ and I_0 be the set of leading coefficients of elements of AUA. Then by a similar way as used in the proof of the Proposition 4.1, one can prove that I_0 is a Σ-ideal of R. Since R is a Σ-Baer ring, hence $\ell_R(I_0) = Re$ for some idempotent $e \in R$. We will show that $\ell_A(U) = Ae$. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m \in U$. So $ea_m = 0$. Since $ef = ea_0 + ea_1X_1 + \cdots + ea_{m-1}X_{m-1} \in AUA$, $ea_{m-1} \in I_0$, so $ea_{m-1} = eea_{m-1} = 0$. Applying this method, we can see that $ea_i = 0$ for each $0 \leq i \leq m$. Thus $ef = 0$, and so $Ae \subseteq \ell_A(U)$. Now, we will show that $\ell_A(U) \subseteq Ae$. Assume that $g = d_0 + d_1X_1 + \cdots + d_pX_p \in \ell_A(U)$. By induction on the degree of g, we want to show that $g = ge$. Assume that $\deg(g) = 0$, and $a \in I_0$. Suppose that for $h = a_0 + a_1X_1 + \cdots + a_mX_m,\ l = c_0 + c_1X_1 + \cdots + c_tX_t \in A$ and $k = b_0 + b_1X_1 + \cdots + b_nX_n \in U$, $a = a_m\sigma^{\alpha_m}(b_n)c_{\alpha_m,\alpha_n}\sigma^{\alpha_m+\alpha_n}(c_t)c_{\alpha_m+\alpha_n,\alpha_t}$ is the leading coefficient of hkl . Since $gU = 0$ and R has IFP, we have $ga = 0$. This is because $ga = ga_m\sigma^{\alpha_m}(b_n)c_{\alpha_m,\alpha_n}\sigma^{\alpha_m+\alpha_n}(c_t)c_{\alpha_m+\alpha_n,\alpha_t}$. Since $\sigma^{\alpha_m}(b_0) + \cdots + \sigma^{\alpha_m}(b_n)X_n \in U$, we obtain $g\sigma^{\alpha_m}(b_n) = 0$. Since R has IFP, we have $ga_m\sigma^{\alpha_m}(b_n) = 0$, hence $ga = 0$. Thus $gI_0 = 0$ and hence $g \in \ell_R(I_0) = Re$, so $g = ge$. Now, assume inductively that the assertion holds for elements of $\ell_A(U)$ with degree less than $|\alpha_p|$, and let $\deg(g) = \alpha_p$ and $g = d_0 + d_1X_1 + \cdots + d_pX_p$. Since σ_i is surjective for each $1 \leq i \leq n$, $d_p = \sigma^{\alpha_p}(c)$ for some $c \in R$. Assume that $a \in I_0$, similarly as above $a = a_m\sigma^{\alpha_m}(b_n)\sigma^{\alpha_m+\alpha_n}(c_t)$. We have $d_p\sigma^{\alpha_p}(a) = 0$ so $\sigma^{\alpha_p}(c)\sigma^{\alpha_p}(a) = 0$ and hence $ca = 0$. Thus $c \in \ell_R(I_0) = Re$ so $c = ce$. We have $d_p = \sigma^{\alpha_p}(c) = \sigma^{\alpha_p}(c)\sigma^{\alpha_p}(e) = d_p\sigma^{\alpha_p}(e)$. We have

$$
\begin{aligned}
g &= d_0 + \cdots + d_{p-1}X_{p-1} + d_pX_p \\
&= d_0 + \cdots + d_{p-1}X_{p-1} + d_p\sigma^{\alpha_p}(e)X_p \\
&= d_0 + \cdots + d_{p-1}X_{p-1} + d_pX_pe,
\end{aligned}
$$

where $\deg(d_0 + \cdots + d_{p-1}X_{p-1}) \leq |\alpha_p|$. So we have $gU = 0$, and hence $(d_0 + \cdots + d_{p-1}X_{p-1})U + d_pX_peU = 0$. So $(d_0 + \cdots + d_{p-1}X_{p-1})U = 0$. By induction hypothesis $d_0 + \cdots + d_{p-1}X_{p-1} = (d_0 + \cdots + d_{p-1}X_{p-1})e$. Thus, we have $g = (d_0 + \cdots + d_{p-1}X_{p-1})e + d_pX_pe = ge$. Therefore, $g \in Ae$ and hence $\ell_A(U) \subseteq Ae$. $\square$

Since each Baer ring is Σ-Baer, Proposition 4.5 immediately implies the following corollary.

Corollary 4.6. *Let R be a ring with IFP and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type. If R is a Baer ring, then so is A.*

Proposition 4.7. *Let R be a ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type. If A is a Baer ring, then R is a Σ-invariant Baer ring.*

Proof. The proof is similar to that of Proposition 4.4. □

Definition 4.8. A right ideal I of A is called Σ-*invariant ideal* if

$$a_0 + a_1X_1 + \cdots + a_mX_m \in I \Leftrightarrow \sigma(a_0) + \sigma(a_1)X_1 + \cdots + \sigma(a_m)X_m \in I.$$

Clearly, if I is a Σ-invariant ideal of R, then $I\langle x_1, \dots, x_n\rangle$ is Σ-invariant ideal of A.

By a similar proof as employed in Propositions 4.1, 4.4, 4.5 and 4.7, one can prove the following.

Corollary 4.9. *Let R be a ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative bijective skew PBW extension of a ring R. Then R is Σ-invariant quasi-Baer if and only if A is Σ-invariant quasi-Baer.*

Corollary 4.10. *Let R be a ring with IFP and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a quasi-commutative skew PBW extension of automorphism type. Then R is Σ-invariant Baer if and only if A is Σ-invariant Baer.*

Proposition 4.11. *Let R be a ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R. If R is a quasi-Baer ring, then so ia A.*

Proof. Suppose that R is quasi-Baer and I be an arbitrary ideal of A. Denote by I_0 the set of all leading coefficients of elements of I. First, we will show that I_0 is an ideal of R. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in I$, where $a_i \in I$, $1 \leq i \leq m$, $a_m \neq 0$, with $X_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $X_m \succ X_{m-1} \succ \cdots \succ X_1$, and $b_j \in I$, $1 \leq j \leq t$, $b_t \neq 0$, with $Y_j = x^{\alpha_j} = x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}}$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$. Then

$$\begin{aligned} f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m}) &= \cdots + a_m x^{\alpha_m}(\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})) \\ &= \cdots + a_m p_{\alpha_m,\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})} + a_m\sigma^{\alpha_m}(\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m}))x^{\alpha_m}, \end{aligned}$$

where $p_{\alpha_m,\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})} = 0$ or $\deg(p_{\alpha_m,\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})}) < |\alpha_m|$ if $p_{\alpha_m,\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})} \neq 0$. Multiplying x^{α_t} from right-hand side of the above, we get

$$f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})x^{\alpha_t} = \cdots + a_m p_{\alpha_m,\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})}x^{\alpha_t} + a_m p_{\alpha_m,\alpha_t} + a_m x^{\alpha_m+\alpha_t},$$

is an element of I, where $p_{\alpha_m,\alpha_t} = 0$ or $\deg(p_{\alpha_m,\alpha_t}) < |\alpha_m + \alpha_t|$ if $p_{\alpha_m,\alpha_t} \neq 0$. By a similar way as above, we get

$$g\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})x^{\alpha_m} = \cdots + b_t p_{\alpha_t,\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})}x^{\alpha_m} + b_t p_{\alpha_t,\alpha_m} + b_t x^{\alpha_m+\alpha_t} \in I,$$

where $p_{\alpha_t,\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})} = 0$ or $\deg(p_{\alpha_t,\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})}) < |\alpha_t|$ if $p_{\alpha_t,\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})} \neq 0$. Since I is an ideal of A, we get

$$f\sigma^{-\alpha_m}(c_{\alpha_t,\alpha_m})x^{\alpha_t} + g\sigma^{-\alpha_t}(c_{\alpha_m,\alpha_t})x^{\alpha_m} = \cdots + (a_m + b_t)x^{\alpha_t+\alpha_m} \in I,$$

and hence $a_m + b_t \in I_0$. Now, suppose that $f \in I,\ r \in R$. Then

$$\begin{aligned} f\sigma^{-\alpha_m}(r) &= \cdots + a_m x^{\alpha_m}(\sigma^{-\alpha_m}(r)) \\ &= \cdots + a_m p_{\alpha_m,\sigma^{-\alpha_m}(r)} + a_m r x^{\alpha_m} \in I, \end{aligned}$$

where $p_{\alpha_m,\sigma^{-\alpha_m}(r)} = 0$ or $\deg(p_{\alpha_m,\sigma^{-\alpha_m}(r)}) < |\alpha_m|$ if $p_{\alpha_m,\sigma^{-\alpha_m}(r)} \neq 0$. Hence $a_m r \in I_0$. Therefore, I_0 is an ideal of R. Hence there exists an idempotent $e \in R$ such that $\ell_R(I_0) = Re$. First, to see that $Ae \subseteq \ell_A(I)$, take $0 \neq f = a_0 + a_1X_1 + \cdots + a_mX_m \in I$. Since $a_m \in I_0$, $ea_m = 0$. But $ef = ea_0 + ea_1X_1 + \cdots + ea_{m-1}X_{m-1} \in I$, hence $ea_{m-1} = eea_{m-1} = 0$, since $ea_{m-1} \in I_0$. Similarly, we can get that $ea_i = 0$ for each $0 \leq i \leq m$. So $ef = 0$ and hence $e \in \ell_A(I)$. Therefore $Ae \subseteq \ell_A(I)$. Now, we claim that $\ell_A(I) \subseteq Ae$. Let $g = b_0 + b_1Y_1 + \cdots + b_tY_t \in \ell_A(I)$. We shall show that $g = ge$. The proof is by induction on the degree of elements of $\ell_A(I)$. Assume that $g \in \ell_A(I)$ with $deg(g) = 0$. Let $c \in I_0$ and $f = c_0 + \cdots + c_{m-1}X_{m-1} + cX_m \in I$. Then from $gf = 0$, we get $gc = 0$. Thus $g \in \ell_R(I_0) = Re$ and hence $g = ge$. Now, assume inductively that the assertion is true for all $g \in \ell_A(I)$ with $\deg(g) \leq |\alpha_t|$. Now let $g = b_0 + b_1X_1 + \cdots + b_tX_t \in \ell_A(I)$. Since σ_i is surjective for each $1 \leq i \leq n$, $b_t = \sigma^{\alpha_t}(c)$ for some $c \in R$. Now, let $a \in I_0$ and $f = a_0 + a_1X_1 + \cdots + aX_m \in I$. Then $gf = 0$. So from the equality $lc(gf) = b_t\sigma^{\alpha_t}(a)c_{\alpha_t,\alpha_m} = 0$ we obtain $b_t\sigma^{\alpha_t}(a) = 0$ and hence $\sigma^{\alpha_t}(c)\sigma^{\alpha_t}(a) = 0$, since c_{α_t,α_m} is invertible. Then $ca = 0$ and hence $c \in \ell_R(I_0)$ so $c = ce$. So $b_t = \sigma^{\alpha_t}(c) = \sigma^{\alpha_t}(ce) = b_t\sigma^{\alpha_t}(e)$. Thus we have $g = b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + b_t\sigma^{\alpha_t}(e)Y_t = b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h + b_tY_te$ for some $h \in A$ with $\deg(h) \leq |\alpha_{t-1}|$. Now we have $gI = (b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h + b_tY_te)I = 0$ and $eI = 0$ so $b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h \in \ell_A(I)$. By induction hypothesis, $(b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h) = (b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h)e$. So we get $g = b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h + b_tY_te = (b_0 + b_1Y_1 + \cdots + b_{t-1}Y_{t-1} + h)e + b_tY_te = ge$. Consequently, $g = ge$ and $\ell_A(I) = eA$. Therefore, A is a quasi-Baer ring, and the proof is complete. $\square$

Corollary 4.12. [9, Theorem 1.2] *Let R be a quasi-Baer ring. Then $A = R[x;\sigma]$ is a quasi-Baer ring.*

In the following, we give an example of a ring R which is not quasi-Baer, but the skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is quasi-Baer. So, the converse of Proposition 4.11 is not true in general.

Example 4.13. [27, Example 2.1] Let F be a field and for each positive integer i, $R_i = F[t_i]$ be the polynomial ring with indeterminate t_i. Let $R = \prod_{i=1}^{\infty} R_i$. Then R is a reduced Baer (hence quasi-Baer) ring. Consider skew PBW extension $A = R[x;\sigma]$. Define $\sigma : R \to R$ given by $\sigma(f_1(t_1), f_2(t_2), f_3(t_3), \ldots) = (f_1(0), f_1(t_2), f_2(t_3), f_3(t_4), \ldots)$. It is easy to see that σ is a non-surjective monomorphism of R. Also, one can see that $A = R[x;\sigma]$ is not a right p.q.-Baer ring, and hence it is not quasi-Baer.

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BAER AND QUASI-BAER PROPERTIES OF SKEW PBW EXTENSIONS

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توسیع‌های PBW اریب حلقه‌های بئر و شبه‌بئر

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فرض کنید σ یک خودریختی و δ یک تابع σ-مشتق برحلقه R است. حلقه R را δ-شبه‌بئر (متناظراً σ-شبه‌بئر پایا) نامیم هرگاه پوچ‌ساز راست هر δ-ایده‌آل (متناظراً σ-ایده‌آل پایا) از R، به عنوان ایده‌آل راست، توسط یک خودتوان تولید گردد. در این مقاله ما توسیع‌های PBW اریب حلقه‌های بئر و شبه‌بئر را مورد بررسی قرار می‌دهیم. به بیان دقیق‌تر، فرض کنید $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ یک توسیع PBW اریب از نوع مشتق بر حلقه R است. در این‌صورت ثابت می‌شود: (۱) R، حلقه‌ای Δ-شبه‌بئر است اگر و تنها اگر A شبه‌بئر باشد. (۲) اگر حلقه R خاصیت IFP داشته باشد آن‌گاه R، Δ-بئر است اگر و تنها اگر A بئر باشد. هم‌چنین فرض کنید $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ یک توسیع PBW اریب شبه‌جابجایی R است، در این‌صورت ثابت می‌شود: (۳) اگر R، Σ-شبه‌بئر باشد آن گاه A شبه‌بئر است. (۴) اگر A، شبه‌بئر باشد آن‌گاه R Σ-شبه‌بئر پایاست. (۵) فرض کنید R خاصیت IFP داشته باشد. در این‌صورت اگر R، Σ-بئر باشد آن‌گاه A بئر است. (۶) اگر A بئر باشد آن‌گاه R، Σ-بئر پایاست. در پایان هم ثابت می‌شود هر توسیع PBW اریب دوسویی بر حلقه شبه‌بئر R، ویژگی شبه‌بئر را داراست.

کلمات کلیدی: حلقه‌های Δ-شبه‌بئر، حلقه‌های Σ-شبه‌بئر، توسیع‌های PBW اریب.