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ON THE REFINEMENT OF THE UNIT AND UNITARY CAYLEY GRAPHS OF RINGS

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ABSTRACT. Let R be a ring (not necessarily commutative) with nonzero identity. We define $\Gamma(R)$ to be the graph with vertex set R in which two distinct vertices x and y are adjacent if and only if there exist unit elements u, v of R such that $x + uyv$ is a unit of R. In this paper, basic properties of $\Gamma(R)$ are studied. We investigate connectivity and the girth of $\Gamma(R)$, where R is a left Artinian ring. We also determine when the graph $\Gamma(R)$ is a cycle graph. We prove that if $\Gamma(R) \cong \Gamma(M_n(F))$ then $R \cong M_n(F)$, where R is a ring and F is a finite field. We show that if R is a finite commutative semisimple ring and S is a commutative ring such that $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$. Finally, we obtain the spectrum of $\Gamma(R)$, where R is a finite commutative ring.

1. Introduction

Throughout this paper, R is a ring (not necessarily commutative) with nonzero identity. We denote the group of units of R, the Jacobson radical of R and the set of $n \times n$ matrices with entries in R by $U(R)$, $J(R)$ and $M_n(R)$, respectively. As usual, $\mathbb{Z}_n$ will denote the integers modulo n and for a set X, $|X|$ will denote the cardinal of X.

The *unit graph* $G(R)$ is the graph with vertex set R in which two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. The unit graph was first investigated by Grimaldi for $\mathbb{Z}_n$ (see [11]). The

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unit graphs for an arbitrary ring R were introduced in [4] and their properties were investigated in [7, 12, 22, 23, 28].

The *unitary Cayley graph* G_R is the graph with vertex set R such that two distinct vertices x and y are adjacent if and only if $x - y \in U(R)$. Unitary Cayley graphs were introduced in [10] and their properties were investigated in [2, 15, 16, 17, 21, 25].

In [14], Khashyarmanesh and Khorsandi provided a generalization of the unit and unitary Cayley graphs as follows: Let R be a commutative ring and let G be a multiplicative subgroup of $U(R)$ and S be a non-empty subset of G such that $S^{-1} = \{s^{-1} | s \in S\} \subseteq S$. Then $\Gamma(R, G, S)$ is the (simple) graph with vertex set R in which two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x+sy \in G$. As a special case of $\Gamma(R, G, S)$, the graph $\Gamma(R, U(R), U(R))$ was first introduced and studied in [26]. In this paper, we extend the definition of the graph $\Gamma(R, U(R), U(R))$ for an arbitrary ring R (not necessary commutative).

Definition. Let R be a ring. Then $\Gamma(R)$ is the (simple) graph with vertex set R in which two distinct elements $x, y \in R$ are adjacent if and only if there exist $u, v \in U(R)$ such that $x + uyv \in U(R)$.

If we omit the word "distinct", we obtain the graph $\overline{\Gamma}(R)$; this graph may have loops (see Figure 1).

For the sake of completeness, first we state some definitions and notions used throughout to keep this paper as self contained as possible. For a graph G, let $V(G)$ denotes the set of vertices, and let $E(G)$ denotes the set of edges. For $x \in V(G)$ we denote by $N_G(x)$ the set of all vertices of G adjacent to x. Also, the degree of x, denoted $\deg_G(x)$, is the size of $N_G(x)$. For two vertices x and y of G, a *walk* between x and y is an ordered list of vertices (not necessarily distinct) $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that x_{i-1} is adjacent to x_i for $i = 1, \ldots, n$. We denote this walk by $x — x_1 — \cdots — x_{n-1} — y$. Also a *path* between x and y is a walk between x and y without repeated vertices. A *cycle* is a path $x_0 — x_1 — \cdots — x_{n-1} — x_n$ with an extra edge $x_0 — x_n$. The *length* of a walk, path or cycle is the number of edges (counting repeats for walks). We denote the cycle graph with n vertices by C_n.

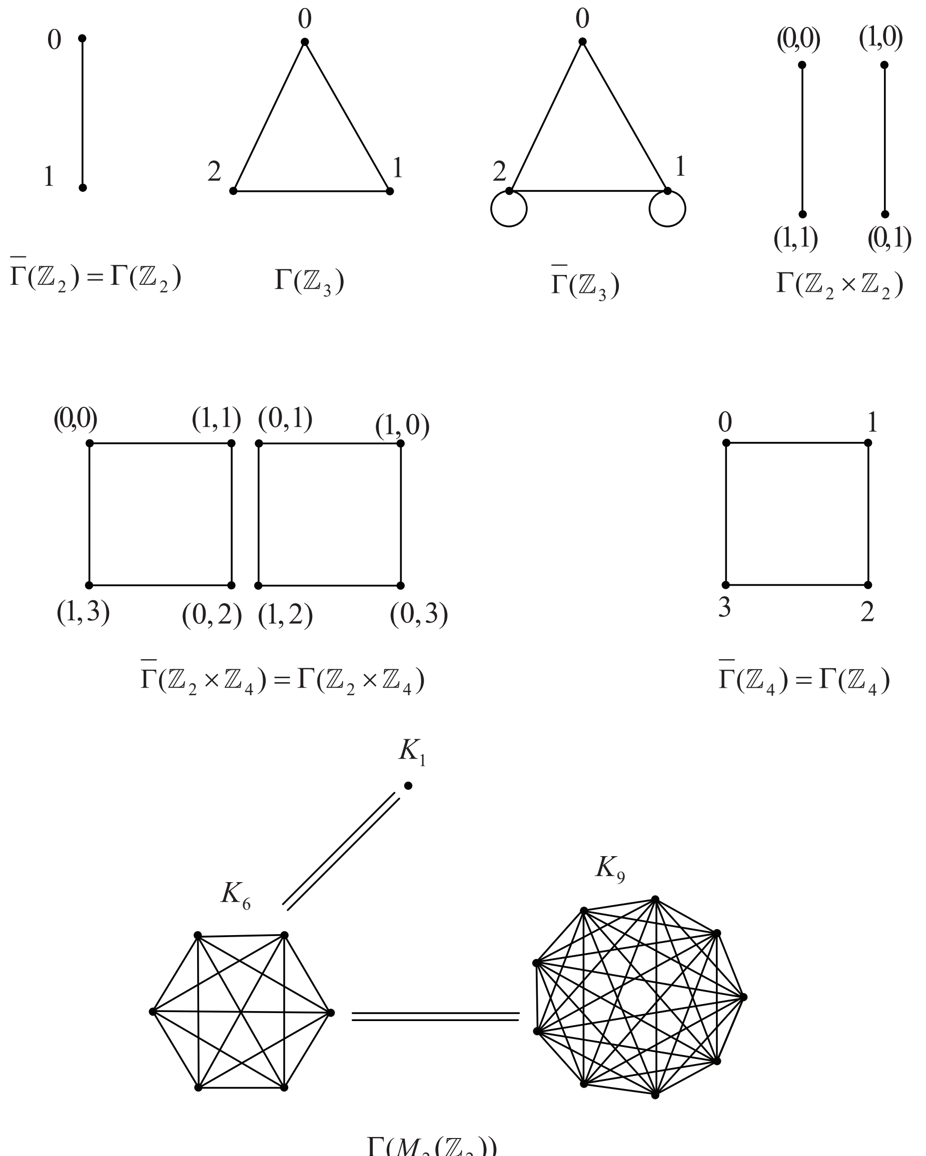

Figure 1. The graphs $\Gamma(R)$ of the specific rings R (for two graphs G and H, the notation $G = H$, means that every vertex of G is connected to every vertex of H).

The *girth* of G, denoted by $\mathrm{gr}(G)$ is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G has no cycles). A graph G is called *connected* if for any two distinct vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. A graph in which each pair of distinct vertices is joined by an edge is called *complete graph*. We denote the complete graph on n vertices by K_n. A *complete bipartite graph* is a simple graph in which the vertices can be partitioned into two disjoint sets V and W such that each vertex in V is adjacent to each vertex in W. If $|V| = m$ and $|W| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

A *clique* (resp. *coclique*) in G is a set of pairwise adjacent (resp. nonadjacent) vertices of G. A *maximum clique* is a clique of the largest possible size in G. The clique number $w(G)$ of a graph G is the number of vertices in a maximum clique in G. A *coloring* of G is a labeling of the vertices with colors such that no two adjacent vertices have the same color. The smallest number of colors needed to color the vertices of a graph G is called its chromatic number, and denoted by $\chi(G)$.

The *union* of two graphs G and H is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, and denote it by $G + H$. The disjoint union of n copies of G is denoted by nG.

Any unexplained notation in this paper will be as in [13, 18, 29].

The plan of this paper is as follows: In Section 2, we give some basic properties of $\Gamma(R)$. We determine when $\Gamma(R)$ is a connected graph (see Theorem 2.2). We also determine when $\Gamma(R)$ is a cycle graph (see Theorem 2.4). For an Artinian ring R, we completely characterize the girth of $\Gamma(R)$ (see Theorem 2.5). For two finite rings R and S, the question of when $\Gamma(R) \cong \Gamma(S)$ implies $R \cong S$ is very interesting and this kind of question has been studied extensively in [1, 2, 3, 15, 24]. In Section 3, we show that if $\Gamma(R) \cong \Gamma(M_n(F))$ then $R \cong M_n(F)$, where R is a ring and F is a finite field (see Theorem 3.5). We show that if R is finite commutative semisimple ring and S is a commutative ring such that $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$ (see Theorem 3.9). Finally, we find the spectrum of $\Gamma(R)$, where R is a finite commutative ring.

2. Basic Properties of $\Gamma(R)$

In this section we study some basic properties of unit graphs. The following lemma immediately follows from [18, Proposition 4.8].

Lemma 2.1. *Let R be a ring and let $x, y \in R$. Then the following statements hold:*

(1) *If $x + J(R)$ and $y + J(R)$ are adjacent in $\Gamma(\frac{R}{J(R)})$, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$ in $\Gamma(R)$.*

(2) *If x and y are adjacent in $\Gamma(R)$, then $x + J(R)$ is adjacent to $y + J(R)$ in $\Gamma(\frac{R}{J(R)})$.*

The following theorem contains a necessary and sufficient condition for $\Gamma(R)$ to be connected.

Theorem 2.2. *Let R be an Artinian ring. Then the following three condition are equivalent:*

(1) *The graph $\Gamma(R)$ is connected.*
(2) *The factor ring $\frac{R}{J(R)}$ has at most one summand isomorphic to $\mathbb{Z}_2$.*
(3) *Every element of R is a sum of two or three units.*

Proof. (1) $\Longrightarrow$ (2) Assume to the contrary that $\frac{R}{J(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\frac{R}{J(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times S$, where S is a subring of $\frac{R}{J(R)}$. If $\frac{R}{J(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then there is not any path between $(0,0)$ and $(0,1)$ (see Figure 1). Similarly if $\frac{R}{J(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times S$, then there is not any path between $(0,0,0)$ and $(0,1,0)$. So $\Gamma(\frac{R}{J(R)})$ is disconnected and therefore $\Gamma(R)$ is disconnected, by Lemma 2.1(2), which is a contradiction.

(2) $\Longrightarrow$ (3) By [18, Proposition 4.8], it is enough to show that every element of $\frac{R}{J(R)}$ is a sum of two or three units. It is easy to see that if S and T are rings in which every element can be expressed as the sum of two units, then the ring $S \times T$ has this property. Therefore, if $\frac{R}{J(R)}$ has no summand isomorphic to $\mathbb{Z}_2$, then we are done by [20, Theorem 1]. If $\frac{R}{J(R)} \cong \mathbb{Z}_2$, then $0 = 1 + 1$ and $1 = 1 + 1 + 1$ and (3) holds for $\frac{R}{J(R)}$. If $\frac{R}{J(R)} \cong \mathbb{Z}_2 \times S$, where S is a subring of $\frac{R}{J(R)}$ which does not contain a summand isomorphic to $\mathbb{Z}_2$. Let $s \in S$. By [20, Theorem 1] there are unit elements $u_1, u_2 \in U(S)$ such that $s = u_1 + u_2$. Also there are unit elements $v_1, v_2 \in U(S)$ such that $u_1 = v_1 + v_2$. Therefore, we have

$$\begin{aligned} (0,s) &= (1,u_1) + (1,u_2), \\ (1,s) &= (1,v_1) + (1,v_2) + (1,u_2). \end{aligned}$$

Hence (3) holds for $\frac{R}{J(R)}$.

(3) $\Longrightarrow$ (1) Let x be a nonzero element of R. If $x = u_1 + u_2$, where $u_1, u_2 \in U(R)$, then we have the walk 0—u_1—$u_1 + u_2$ between 0 and x. If $x = u_1 + u_2 + u_3$, where $u_1, u_2, u_3 \in U(R)$, then we have the walk 0—u_1—$u_1 + u_2$—$u_1 + u_2 + u_3$ between 0 and x. Hence $\Gamma(R)$ is connected. $\square$

The following theorem determines when $\Gamma(R)$ is a complete bipartite graph.

Theorem 2.3. *Let R be a ring with a maximal ideal $\mathfrak{m}$ such that $|\frac{R}{\mathfrak{m}}| = 2$. Then $\Gamma(R)$ is a complete bipartite graph if and only if R is a local ring.*

Proof. Let $\Gamma(R)$ be a complete bipartite graph with bipartition $\{V_1, V_2\}$. Let $x, y \in R$ such that $x+y \in U(R)$. Since x and y are adjacent, without loss of generality, we may assume that $x \in V_1$ and $y \in V_2$. If $0 \in V_1$, then $y \in U(R)$. If $0 \in V_2$, then $x \in U(R)$. Therefore $x+y \in U(R)$ implies that $x \in U(R)$ or $y \in U(R)$. It follows from [18, Theorem 19.1] that R is a local ring.

Conversely, suppose that R is a ring with a maximal ideal $\mathfrak{m}$. Set $V_1 := \mathfrak{m}$ and $V_2 := 1+\mathfrak{m}$. Then $V(\Gamma(R)) = V_1 \cup V_2$. Since 0 and 1 are adjacent in $\frac{R}{\mathfrak{m}} \cong \mathbb{Z}_2$, then Lemma 2.1(1) implies that every elements of $\mathfrak{m}$ is adjacent to every elements of $1+\mathfrak{m}$. It easy to see that $V_1 =: \mathfrak{m}$ is coclique. Now let $x, y \in \mathfrak{m}$ and let $1+x$ and $1+y$ are two adjacent elements of $1+\mathfrak{m}$. Then there exist $u, v \in U(R)$ and $z \in \mathfrak{m}$ such $(1+x)+u(1+y)v = 1+z$. It follows that $uv = z-x-uyv \in \mathfrak{m}$, which is a contradiction. Therefore $\Gamma(R)$ is a complete bipartite graph. $\square$

In the following theorem, we determine when $\Gamma(R)$ is a cycle graph.

Theorem 2.4. *Let R be a ring. Then $\Gamma(R)$ is a cycle graph if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_3, \mathbb{Z}_4, \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}.$$

Proof. Let $\Gamma(R)$ be a cycle graph. Then we have $|R| = |V(\Gamma(R))| < \infty$. If $|U(R)| \geq 3$, then $\deg_{\Gamma(R)}(0) = 3$ and hence $\Gamma(R)$ is not a cycle graph. We show that $|U(R)| \neq 1$. Suppose on the contrary that $U(R) = \{1\}$. Since $\Gamma(R)$ is a cycle graph, it has a path of length 2. Let x—y—z be a path of length 2 in $\Gamma(R)$. Then $x+y = 1$ and $y+z = 1$. Hence $x = z$, which is a contradiction. So $|U(R)| \neq 1$ and hence $|U(R)| = 2$. It follows from [8, Corollary 4.5] that R is isomorphic to one of the following rings.

(1) $R_1 = \mathbb{Z}_3$.
(2) $R_2 = \mathbb{Z}_4$.
(3) $R_3 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.
(4) $R_4 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$.
(5) $S_i = \mathbb{Z}_2 \times R_i, 1 \leq i \leq 4$.

The graphs $\Gamma(\mathbb{Z}_3)$, $\Gamma(\mathbb{Z}_4)$ and $\Gamma(R_3)$ are cycle graphs and the graphs $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and $\Gamma(R_4)$ are not cycle graphs (see Figures 1 and 2). We have $N_{\Gamma(\mathbb{Z}_2\times\mathbb{Z}_3)}(1,1) = \{(0,0), (0,1), (0,2)\}$, and so $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is not cycle. Also, it is easy to see that $\Gamma(\mathbb{Z}_2 \times R_3) \cong 2C_4$ and $\Gamma(\mathbb{Z}_2 \times R_4) \cong$

$4C_4$. So the graphs $\Gamma(\mathbb{Z}_2 \times R_3)$ and $\Gamma(\mathbb{Z}_2 \times R_4)$ are not cycle graphs. This completes the proof. $\square$

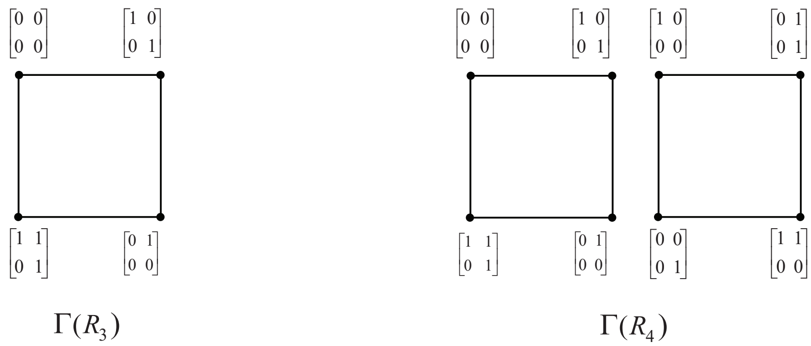

Figure 2. The graphs $\Gamma(R_3)$ and $\Gamma(R_4)$.

Theorem 2.5. *Let R be an Artinian ring. Then* $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$.

Proof. First, suppose that $J(R) \neq 0$ and x, y are two distinct elements of $J(R)$. Since every element of $J(R)$ is adjacent to every element of $U(R)$, x—$(1+x)$—y—$(1+y)$ is a cycle in $\mathrm{gr}(\Gamma(R))$. Therefore $\mathrm{gr}(\Gamma(R)) = 3$. Now assume that $J(R) = 0$. So the Wedderburn-Artin Theorem [18, Theorem 3.5] implies that $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$, where $D_1, \ldots, D_t$ are division rings and $n_1, \ldots, n_t$ are positive integers. If $R \cong \mathbb{Z}_2$, then $\mathrm{gr}(\Gamma(R)) = \infty$. If R is a division ring and $|R| \geq 3$, then, for any two nonzero distinct elements x and y of R, 0—x—y form a triangle in $\Gamma(R)$. So $\mathrm{gr}(\Gamma(R)) = 3$. Now assume that $R \cong M_n(D)$, where $n \geq 2$ and D is a division ring. If $\mathrm{Char}(D) = 2$, then by [20, Theorem 1], we have $I = U + V$, where I is the identity matrix and U, V are two invertible (unit) matrices. Hence the vertices $\{0, U, V\}$ form a triangle in $\Gamma(R)$. If $\mathrm{Char}(D) \neq 2$, then the vertices $\{0, I, U\}$ form a triangle in $\Gamma(R)$, where $U = (u_{ij})$ is a lower triangular matrix such $u_{ii} = u_{n1} = 1$ for $i = 1, 2, \ldots, n$ and the other vertices are zero. So $\mathrm{gr}(\Gamma(R)) = 3$. Now we consider the following three cases:

Case 1: $R \cong \prod_{i=1}^{t} \mathbb{Z}_2$, where $t \geq 2$. Then $\Gamma(R) \cong 2^{t-1}K_2$. Therefore $\Gamma(R)$ is disconnected and $\mathrm{gr}(\Gamma(R)) = \infty$.

Case 2: $R \cong \prod_{i=1}^{t} M_{n_i}(D_i)$, where $t \geq 2$ and $M_{n_i}(D_i)$ is not isomorphic to $\mathbb{Z}_2$. Assume that the vertices $\{A_i, B_i, C_i\}$ form a triangle in $M_{n_i}(D_i)$ for i with $1 \leq i \leq t$. Then the vertices $\{(A_1, \ldots, A_t), (B_1, \ldots, B_t), (C_1, \ldots, C_t)\}$ form a triangle in $\Gamma(R)$ and so $\mathrm{gr}(\Gamma(R)) = 3$.

Case 3: $R \cong \prod_{i=1}^{k} \mathbb{Z}_2 \times \prod_{j=1}^{l} M_{n_j}(D_j)$, where $k, l \geq 1$ and $M_{n_j}(D_j)$ is not isomorphic to $\mathbb{Z}_2$. In this case, it is easy to see that $\Gamma(R)$ is a

bipartite graph and hence $\operatorname{gr}(\Gamma(R)) \geq 4$. We consider the following two cases:
Subcase 1: $\operatorname{Char}(D_l) = 2$. Let $I = U + V$, where I is the identity matrix and U, V are two distinct invertible matrices in $M_{n_l}(D_l)$. Then we have the following cycle

$$(0,\dots,0,0,\dots,0)\text{---}\ (1,\dots,1,I,\dots,I)\text{---}\ (0,\dots,0,0,\dots,0,V)\text{---}\ (1,\dots,1,I,\dots,I,U).$$

Subcase 2: $\operatorname{Char}(D_l) \neq 2$. Let $U = (u_{ij})$ be a lower triangular matrix in $M_{n_l}(D_l)$ such $u_{ii} = u_{n1} = 1$ for $i = 1, 2, \dots, n$ and 0 otherwise. Then we have the following cycle

$$(0,\dots,0,0,\dots,0)\text{---}\ (1,\dots,1,I,\dots,I,U)\text{---}\ (0,\dots,0,0,\dots,0,U)\text{---}\ (1,\dots,1,I,\dots,I).$$

So $\operatorname{gr}(\Gamma(R)) = 4$. □

The maximum (respectively minimum) vertex degree in a graph G is denoted by $\Delta(G)$ (respectively $\delta(G)$). We denote by $\Delta_2(G)$, the second greatest degree of G. We end this section by the following theorems which is used in the next section.

Theorem 2.6. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Let $|\frac{R_i}{\mathfrak{m}_i}| > 2$ for every i. Then the following are hold:*
(1) $\Delta(\Gamma(R)) = |R| - 1$ and $\delta(\Gamma(R)) = |U(R)|$.
(2) $\deg_{\Gamma(R)}(x) = \Delta(\Gamma(R))$ if and only if $x \in U(R)$.
(3) $\deg_{\Gamma(R)}(x) = \delta(\Gamma(R))$ if and only if $x \in J(R)$.

Proof. The assertions follow from [26, Theorems 2.2 and 2.3] □

Theorem 2.7. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Assume that there exists t with $1 \leq t \leq n$ such that $|\frac{R_i}{\mathfrak{m}_i}| = 2$ for every $i \leq t$ and $|R_i/\mathfrak{m}_i| > 2$ for every $i > t$. Then the following statements hold:*
(1) $\Delta(\Gamma(R)) = \frac{|R_1|}{2}\frac{|R_2|}{2}\cdots\frac{|R_t|}{2}|R_{t+1}|\cdots|R_n|$ and
$\delta(\Gamma(R)) = \frac{|R_1|}{2}\frac{|R_2|}{2}\cdots\frac{|R_t|}{2}|U(R_{t+1})|\cdots|U(R_n)|$.
(2) $\deg_{\Gamma(R)}(x) = \Delta(\Gamma(R))$ if and only if $x \in R_1 \times \cdots \times R_t \times U(R_{t+1}) \times \cdots \times U(R_n)$.
(3) $\deg_{\Gamma(R)}(x) = \delta(\Gamma(R))$ if and only if $x \in R_1 \times \cdots \times R_t \times \mathfrak{m}_{t+1} \times \cdots \times \mathfrak{m}_n$.

Proof. The assertions follow from [26, Theorems 2.2 and 2.3]. □

3. Isomorphisms

We begin this section by the following remark.

Remark 3.1. Let R be a ring and $x, y \in R$. Then, in $\Gamma(R)$, the following are equivalent:
(1) x is adjacent to y.
(2) x is adjacent to uyv for some unit elements $u, v \in U(R)$.
(3) x is adjacent to uyv for all unit elements $u, v \in U(R)$.

Notation. Let E_{ij} the $n \times n$ matrix that has 1 in the (i, j)-th entry and zero elsewhere. For each $2 \leq t \leq n$, we set

$$J^{n,t} := E_{21} + E_{32} + \cdots + E_{t(t-1)}.$$

Theorem 3.2. *Let $R = M_n(F)$, where F is a field and $(n, |F|) \neq (1, 2)$ and let $A, B \in R$. Then, A is adjacent to B if and only if $\operatorname{rank}(A) + \operatorname{rank}(B) \geq n$.*

Proof. Let $\operatorname{rank}(A) + \operatorname{rank}(B) \geq n$. By [20, Theorem 1], there are unit elements U_1 and U_2 such that $A = U_1 + U_2$. Therefore A is adjacent to U_1. It follows from Remark 3.1 that A is adjacent to every unit element of R. So, if A or B is unit, then A is adjacent to B. Now suppose that A and B be nonunits of R. Let $n_1 = \operatorname{rank}(A)$ and $n_2 = \operatorname{rank}(B)$. Then by [13, Proposition 2.11], there are unit elements U_1, U_2, V_1, V_2 of R such that

$$A = U_1 \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} V_1, \quad B = U_2 \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} V_2.$$

We consider two cases:
Case 1: $\operatorname{rank}(A) + \operatorname{rank}(B) = n$. In this situation, again by using Remark 3.1, we have that A is adjacent to B.
Case 2: $\operatorname{rank}(A) + \operatorname{rank}(B) > n$. There are unit elements U_3, V_3 of R such that

$$B = U_3 \begin{bmatrix} J^{n_1,t} & 0 \\ 0 & I_{n-n_1} \end{bmatrix} V_3,$$

where $t = (n_1 + n_2) - n$. We have

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} J^{n_1,t} & 0 \\ 0 & I_{n-n_1} \end{bmatrix} = \begin{bmatrix} J^{n_1,t} + I_{n_1} & 0 \\ 0 & I_{n-n_1} \end{bmatrix} \in U(R).$$

It follows that A and B are adjacent.

Conversely, suppose that $\operatorname{rank}(A) + \operatorname{rank}(B) < n$. There are unit elements U_0 and V_0 such that

$$A = U_0 \begin{bmatrix} 0 & 0 \\ 0 & I_{n_1} \end{bmatrix} V_0.$$

It is easy to see that $\begin{bmatrix} 0 & 0 \\ 0 & I_{n_1} \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$ are not adjacent and hence A and B are not adjacent. This completes the proof. □

Let $A, B \in M_n(F)$, where F is a field. Recall from [13, Definition 1.8] that the matrices A and B are called *equivalent* if there exist two invertible matrices $U, V \in M_n(F)$ such that $A = UBV$. It is easy is see this definition of "equivalent" gives an equivalence relation on $M_n(F)$. By [13, Theorem 2.6(ii)], the matrices A and B are equivalent matrices if and only if $\mathrm{rank}(A) = \mathrm{rank}(B)$. Let R_k be the set of all matrices of rank k, for $0 \leq k \leq n$. The number of $n \times n$ matrices of rank k over a finite field of order q is given by

$$r_k = |R_k| = \frac{((q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}))^2}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

This result was established by Landsberg in [19].

Theorem 3.3. *Let $R = M_n(F)$, where F is a field. Then*

$$\chi(\Gamma(R)) = \omega(\Gamma(R)) = \begin{cases} r_n + r_{n-1} + \cdots + r_{\frac{n}{2}} & \text{if } n \text{ is even} \\ r_n + r_{n-1} + \cdots + r_{\frac{n+1}{2}} + 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. We consider the partition $V(\Gamma(R)) = R_0 \cup R_1 \cup \cdots \cup R_n$. Let n be an even number. By Theorem 3.2 the set $R_n \cup R_{n-1} \cup R_{n-1} \cup \cdots \cup R_{\frac{n}{2}}$ is a clique. So $\chi(\Gamma(R)) \geq \omega(\Gamma(R)) \geq r_n + r_{n-1} + \cdots + r_{\frac{n}{2}}$. On the other hand, $R_0 \cup R_1 \cup \cdots \cup R_{\frac{n}{2}-1}$ is a coclique and every vertex of $R_0 \cup R_1 \cup \cdots \cup R_{\frac{n}{2}-1}$ is not adjacent to every vertex of $R_{\frac{n}{2}}$. So $r_n + r_{n-1} + \cdots + r_{\frac{n}{2}}$ colors provide a proper coloring for $\Gamma(R)$. It follows that $\chi(\Gamma(R)) = \omega(\Gamma(R)) = r_n + r_{n-1} + \cdots + r_{\frac{n}{2}}$. Now let n be an odd number. Again by Theorem 3.2 the set $R_n \cup R_{n-1} \cup R_{n-1} \cup \cdots \cup R_{\frac{n+1}{2}} \cup \{x\}$ is a clique, where $x \in R_{\frac{n-1}{2}}$. So $\chi(\Gamma(R)) \geq \omega(\Gamma(R)) \geq r_n + r_{n-1} + \cdots + r_{\frac{n+1}{2}} + 1$. On the other hand, $R_0 \cup R_1 \cup \cdots \cup R_{\frac{n-1}{2}}$ is a coclique and every vertex of $R_{\frac{n-1}{2}}$ is adjacent to every vertex of $R_0 \cup R_1 \cup \cdots \cup R_{\frac{n-1}{2}}$. So $r_n + r_{n-1} + \cdots + r_{\frac{n+1}{2}} + 1$ colors provide a proper coloring for $\Gamma(R)$. It follows that $\chi(\Gamma(R)) = \omega(\Gamma(R)) = r_n + r_{n-1} + \cdots + r_{\frac{n+1}{2}} + 1$. □

Theorem 3.4. *Let F and E be two finite fields and m, n be two natural numbers. If $\Gamma(M_n(F)) \cong \Gamma(M_m(E))$, then $m = n$ and $F \cong E$.*

Proof. Let $|F| = p^r$ and $|E| = q^s$, for some prime numbers p, q and natural numbers r, s. Since $|\Gamma(M_n(F))| = |\Gamma(M_m(E))|$, we have $p^{rn^2} = q^{sm^2}$. So $p = q$ and $rn^2 = sm^2$. On the other hand,

$$\begin{aligned} p^{r\frac{n(n-1)}{2}} \prod_{i=1}^{n}(p^{ir}-1) &= \Delta(\Gamma(M_n(F))) = \Delta(\Gamma(M_m(E))) \\ &= p^{s\frac{m(m-1)}{2}} \prod_{i=1}^{m}(p^{is}-1). \end{aligned}$$

It follows that $rn(n-1) = sm(m-1)$ and hence $rn = sm$. So $n = m$ and $r = s$. □

Now we are in position to give one of the main results of this paper.

Theorem 3.5. *Let $R = M_n(F)$, where F is a finite field and S is a ring. If $\Gamma(R) \cong \Gamma(S)$, then $S \cong M_n(F)$.*

Proof. It is clear that S is a finite ring. If $R \cong \mathbb{Z}_2$, then $S \cong \mathbb{Z}_2$ and we are done. So assume that $R \ncong \mathbb{Z}_2$. We show that S is semisimple. First we note that if $x, y \in S$ and $x - y \in J(R)$, then by [18, Lemma 4.3], we have $N_{\Gamma(S)}(x) = N_{\Gamma(S)}(y)$. Let $f : \Gamma(R) \to \Gamma(S)$ be an isomorphism and let $a = f(0)$. Then

$$a + J(S) \subseteq \{x \in S | \deg_{\Gamma(S)}(x) = \deg_{\Gamma(S)}(a)\}.$$

On the other hand, by Theorem 3.2, we have

$$\begin{aligned} 1 &= |\{x \in R | \deg_{\Gamma(R)}(x) = \deg_{\Gamma(R)}(0)\}| \\ &= |\{x \in S | \deg_{\Gamma(S)}(x) = \deg_{\Gamma(R)}(a)\}|. \end{aligned}$$

Hence $J(R) = 0$, and so S is a semisimple ring. Let $|F| = p^r$ and $S \cong M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_k)$ and $|F_i| = p_i^{r_i}$ such that $p_1^{r_1 n_1^2} \leq p_2^{r_2 n_1^2} \leq \cdots \leq p_k^{r_k n_k^2}$. Since $|R| = |S|$, we have

$$p^{rn^2} = p_1^{r_1 n_1^2} \times \cdots \times p_k^{r_k n_k^2}.$$

It follows that $p = p_1 = p_2 = \ldots = p_k$ and $rn^2 = \sum_{i=1}^{k} r_i n_i^2$. We have

$$p^{rn^2} - 2 = \Delta_2(\Gamma(R)) = \Delta_2(\Gamma(R)) = p^{r_1 n_1^2} p^{r_2 n_2^2} \ldots (p^{r_k n_k^2} - 1) - 1.$$

It follows that $\sum_{i=1} r_i n_i^2 = 0$. So $S = M_{n_k}(F_k)$ and Theorem 3.4 completes the proof. □

Let G and H be two graphs. The *tensor product* (sometimes called *category product*) of G and H, $G \otimes H$, is a graph with the vertex set $V(G) \times V(H)$, such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if x_1 is adjacent to x_2 in G and y_1 is adjacent to y_2 in H.

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a direct product of rings and $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in R$, where x and y are distinct. Now, according to our definition, it is not hard to see that x is adjacent to y in $\Gamma(R)$, if and only if x_i is adjacent to y_i in $\overline{\Gamma}(R_i)$, for all $1 \leq i \leq n$. Hence we have the following immediate lemma.

Lemma 3.6. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a direct product of rings. Then $\overline{\Gamma}(R) \cong \bigotimes_{i=1}^{n} \overline{\Gamma}(R_i)$.*

It is well known that every Artinian commutative ring can be expressed as a direct product of Artinian local rings, and this decomposition is unique up to permutations of such local rings (see [5, Theorem 8.7]).

For a finite commutative ring R, we have the following result about the loops of $\overline{\Gamma}(R)$.

Theorem 3.7. *Let $R = R_1 \times \cdots \times R_n$ be a finite commutative ring, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Then*
(1) If $|\frac{R_i}{\mathfrak{m}_i}| = 2$ for some $1 \leq i \leq n$, then $\overline{\Gamma}(R) = \Gamma(R)$.
(2) If $|\frac{R_i}{\mathfrak{m}_i}| \neq 2$ for every $1 \leq i \leq n$, then only the elements of $U(R)$ has a loop in $\overline{\Gamma}(R)$.

Proof. Follows easily from [26, Proposision 1.1]. □

Lemma 3.8. *Let R and S be two finite commutative rings. Then $\overline{\Gamma}(R) \cong \overline{\Gamma}(S)$ if and only if $\Gamma(R) \cong \Gamma(S)$.*

Proof. It is easy to see that if $\overline{\Gamma}(R) \cong \overline{\Gamma}(S)$ then $\Gamma(R) \cong \Gamma(S)$. Conversely, suppose that $\Gamma(R) \cong \Gamma(S)$. Let

$$R \cong R_1 \times R_2 \times \cdots \times R_n,$$
$$S \cong S_1 \times S_2 \times \cdots \times S_m,$$

where R_i and S_j are local rings with maximal ideals $\mathfrak{m}_i$ and $\mathfrak{n}_i$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. We consider the following cases:
Case 1: There exists $1 \leq i \leq n$ such that $\frac{|R_i|}{|\mathfrak{m}_i|} = 2$. In this case, we claim that there exists $1 \leq j \leq m$ such that $\frac{|S_j|}{|\mathfrak{n}_j|} = 2$. Suppose on the contrary that $\frac{|S_j|}{|\mathfrak{n}_j|} \neq 2$ for every $1 \leq j \leq m$. Then by [26, Theorem 3.1], we have

$$2 = w(\Gamma(R)) = w(\Gamma(S)) = |U(S)| + m.$$

It follows that $m = 1$ and $|U(S)| = 1$. It is not hard to see that $S \cong \mathbb{Z}_2$, which is a contradiction. Now Theorem 3.7 implies that $\overline{\Gamma}(R) \cong \overline{\Gamma}(S)$.
Case 2: There exists $1 \leq i \leq m$ such that $\frac{|S_i|}{|\mathfrak{n}_i|} = 2$. This case is exactly similar to Case 1.

Case 3: $\frac{|R_i|}{|\mathfrak{m}_i|} \neq 2$ and $\frac{|S_j|}{|\mathfrak{n}_j|} \neq 2$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. By Theorem 2.6 and [26, Theorem 3.1], we have

$$|U(R)| = \delta(\Gamma(R)) = \delta(\Gamma(S)) = |U(S)|,$$

$$|U(R)| + n = w(\Gamma(R)) = w(\Gamma(S)) = |U(S)| + m.$$

So we have $n = m$. Now we consider two subcases:
Subcase 1: S is a field. In this case, we have

$$|U(R)| = \delta(\Gamma(R)) = \delta(\Gamma(S)) = |S| - 1.$$

It follows that $|U(R)| = |R| - 1$. Therefore R is also a field. Since two finite fields are isomorphic if and only if they have the same number of elements, we must have $R \cong S$ and hence $\overline{\Gamma}(R) \cong \overline{\Gamma}(S)$.
Subcase 2: S is not a field. Let $f : \Gamma(R) \to \Gamma(S)$ be a graph isomorphism. By Lemma 3.7(2), it is enough to show that $f(U(R)) \subseteq U(S)$. Suppose on the contrary that $f(u) = (x_1, x_2, \ldots, x_n) \notin U(S)$, for some $u \in U(R)$. Without loss of generality, we may assume that there exists $2 \leq k \leq n$ such that $x_i \in \mathfrak{n}_i$ for every $1 \leq i \leq k$ and $x_i \in U(S_i)$ for every $k+1 \leq i \leq n$. We have

$$\begin{aligned} |R| - 1 &= \deg_{\Gamma(R)}(u) = \deg_{\Gamma(S)}(f(u)) \\ &= |U(S_1)||U(S_2)| \ldots |U(S_k)||S_{k+1}| \ldots |S_n|. \end{aligned}$$

By Theorem 2.6(1), we have $|R| - 1 = \Delta(\Gamma(R)) = \Delta(\Gamma(S)) = |S| - 1$ and so

$$|S_1||S_2| \cdots |S_n| - 1 = |U(S_1)||U(S_2)| \cdots |U(S_k)||S_{k+1}| \cdots |S_n|.$$

Hence

$$|S_{k+1}| \cdots |S_n|(|S_1||S_2| \cdots |S_k| - |U(S_1)||U(S_2)| \cdots |U(S_k)|) = 1.$$

By [2, Proposition 2.1], we must have $x_i \in \mathfrak{n}_i$ for every $1 \leq i \leq n$. Hence

$$|S_1||S_2| \cdots |S_n| - |U(S_1)||U(S_2)| \cdots |U(S_n)| = 1.$$

So $|U(S)| = |S| - 1$ and hence S is a field, which is a contradiction. $\square$

Now we are ready to state another main result of this section.

Theorem 3.9. *Let R and S be two finite commutative rings such that R is semisimple. If $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$.*

Proof. First we claim that S is a semisimple ring. By [18, Page 41], we may assume

$$R \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{r \text{ times}} \times F_1 \times F_2 \times \cdots \times F_n,$$

where F_i is a field for all $1 \leq i \leq n$ and $3 \leq f_i = |F_i| \leq f_{i+1} = |F_{i+1}|$ for all $1 \leq i \leq n-1$, and

$$S \cong R_1 \times R_2 \times \cdots \times R_t \times R_{t+1} \times \cdots \times R_m,$$

where R_i is a local ring with maximal ideal $\mathfrak{m}_i$ and $1 \leq t \leq m$ is an integer number such that $|\frac{R_i}{\mathfrak{m}_i}| = 2$ for every $i \leq t$ and $|\frac{R_i}{\mathfrak{m}_i}| > 2$ for every $i > t$. Since $\Gamma(R) \cong \Gamma(S)$, hence the number of connected components of $\Gamma(R)$ should be equal to the number of connected components of $\Gamma(S)$. Therefore, by [9, Corollary 5.10], we have $2^{r-1} = 2^{t-1}$ and so $r = t$. On the other hand,

$$\begin{aligned}&|\{x \in \Gamma(R)|\deg_{\Gamma(R)}(x) = \delta(\Gamma(R))\}| = \\ &|\{x \in \Gamma(S)|\deg_{\Gamma(S)}(x) = \delta(\Gamma(S))\}|.\end{aligned}$$

It follows that

$$|\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{t \text{ times}} \times \{0\} \times \cdots \times \{0\}| = |R_1 \times \cdots \times R_t \times \mathfrak{m}_{t+1} \times \cdots \times \mathfrak{m}_m|.$$

By [2, Proposition 2.1], we have $R_1 \cong R_1 \cong \ldots \cong R_t \cong \mathbb{Z}_2$ and $\mathfrak{m}_{t+1} = \ldots = \mathfrak{m}_m = \{0\}$. Hence S is a semisimple ring. Assume that

$$S \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{t \text{ times}} \times E_1 \times E_2 \times \cdots \times E_m,$$

where E_i is a field for all $1 \leq i \leq m$ and $3 \leq e_i = |E_i| \leq e_{i+1} = |E_{i+1}|$ for all $1 \leq i \leq m-1$. Since $\Gamma(R) \cong \Gamma(S)$, we have

$$2^t f_1 f_2 \ldots f_n = 2^t e_1 e_2 \ldots e_m. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} f_1 f_2 \ldots f_{n-1}(f_n - 1) &= \Delta_2(\Gamma(R)) = \Delta_2(\Gamma(S)) \\ &= e_1 e_2 \ldots e_{m-1}(e_m - 1). \end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2) we deduce that $f_n = e_m$ and hence $F_n \cong E_m$. By the Cancelation Theorem ([9, Proposition 9.6]) and Lemma 3.8, we have

$$\begin{aligned} \overline{\Gamma}(\mathbb{Z}_2^t \times F_1 \times \cdots \times F_{n-1}) &\cong \overline{\Gamma}(\mathbb{Z}_2^t) \otimes \overline{\Gamma}(F_1) \otimes \cdots \otimes \overline{\Gamma}(F_{n-1}) \\ &\cong \overline{\Gamma}(\mathbb{Z}_2^t) \otimes \overline{\Gamma}(E_1) \otimes \cdots \otimes \overline{\Gamma}(E_{m-1}) \\ &\cong \overline{\Gamma}(\mathbb{Z}_2^t \times E_1 \times \cdots \times E_{m-1}) \end{aligned}$$

By repeating this argument, we conclude that $n = m$ and $F_i \cong E_i$ for every $1 \leq i \leq n$. Hence $R \cong S$. $\square$

We end this section by the following conjecture.

Conjecture 1. Let R and S be two finite rings such that $\Gamma(R) \cong \Gamma(S)$. Then $\frac{R}{J(R)} \cong \frac{S}{J(S)}$.

4. The spectrum of $\Gamma(R)$

The *eigenvalues* of a graph are eigenvalues of its adjacency matrix, and the spectrum of a graph is the collection of its eigenvalues together with multiplicities. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a graph G and $m_1, m_2, \ldots, m_k$ the corresponding multiplicities, then we denote the spectrum of G by

$$\mathrm{Spec}(\Gamma(R)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}.$$

Let R be a finite commutative ring and n, m be two natural integer numbers. Let $I_n \in M_n(R)$ denote the identity matrix and let $J_{n,m}$ be the $n \times m$ matrix with all entries equal to 1. If $a \in R$, then it is easy to see that the characteristic polynomial of aJ_n is equal to $\lambda^{n-1}(\lambda - na)$.

Theorem 4.1. *Let R be a finite commutative local ring with maximal ideal $\mathfrak{m}$ such that $|\frac{R}{\mathfrak{m}}| > 2$. Then*

$$\mathrm{Spec}(\Gamma(R)) = \begin{pmatrix} 0 & a & b \\ |R|-2 & 1 & 1 \end{pmatrix},$$

where a, b are roots of the equation $\lambda^2 - |U(R)|\lambda - |\mathfrak{m}||U(R)|$.

Proof. Let $n = |R|$, $m = |\mathfrak{m}|$ and M be the adjacency matrix of $\Gamma(R)$ in such away that, the elements of $U(R)$ labeled by $1, \ldots, n-m$ and the elements of $\mathfrak{m}$ labeled by $(n-m)+1, \ldots, n$.

$$\det(\lambda I_n - M) = \left[\begin{array}{ccccc|cccccc} \lambda-1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 & -1 & \ldots & -1 \\ -1 & \lambda-1 & -1 & \ldots & -1 & -1 & -1 & -1 & -1 & \ldots & -1 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \ldots & & \lambda-1 & -1 & -1 & -1 & -1 & \ldots & -1 \\ \hline -1 & -1 & -1 & \ldots & -1 & \lambda & 0 & 0 & 0 & \ldots & 0 \\ -1 & -1 & -1 & \ldots & -1 & 0 & \lambda & 0 & 0 & \ldots & 0 \\ -1 & -1 & -1 & \ldots & -1 & 0 & 0 & \lambda & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \\ -1 & -1 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & & \lambda \end{array}\right].$$

Let $A = \lambda I_{n-m} - J_{n-m}$, $B = -J_{n-m,m}$, $C = -J_{m,n-m}$ and $D = \lambda I_m$. If $\lambda \neq 0$, then by Schur complement formula (see for example, [27, Exercise 2.10(4)]), we have $\det(\lambda I_n - M) = \det(D)\det(A - BD^{-1}C)$. It is easy to see that

$$A - BD^{-1}C = \begin{bmatrix} \lambda - a & -a & -a & -a & \cdots & -a \\ -a & \lambda - a & -a & -a & \cdots & -a \\ -a & -a & \lambda - a & -a & \cdots & -a \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -a & -a & -a & -a & \cdots & \lambda - a \end{bmatrix},$$

where $a = 1 + \frac{m}{\lambda}$. It follows that $\det(\lambda I_n - M) = \lambda^{n-2}(\lambda^2 - |U(R)|\lambda - |\mathfrak{m}||U(R)|)$. This completes the proof. $\square$

Let R be a commutative local ring with maximal ideal $\mathfrak{m}$ such that $|\frac{R}{\mathfrak{m}}| = 2$. Then by Theorem 2.3, $\Gamma(R) \cong K_{n,n}$, where $n = \frac{|R|}{2}$. Therefore by [6, Theorem 3.4(ii)], we have

$$\operatorname{Spec}(\overline{\Gamma}(R)) = \operatorname{Spec}(\Gamma(R)) = \begin{pmatrix} 0 & n & -n \\ 2n-2 & 1 & 1 \end{pmatrix}.$$

So by Lemma 3.6 and [6, Lemma 3.25], we can calculate the spectrum of $\overline{\Gamma}(R)$.

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ON THE REFINEMENT OF THE UNIT AND UNITARY CAYLEY GRAPHS OF RINGS

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تظریف گراف‌های یکه و کیلی یکانی حلقه‌ها

میثم رضاقلی‌بیگی و علیرضا نقی‌پور
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فرض کنیم R یک حلقه (نه لزوماً تعویض‌پذیر) با عنصر همانی ناصفر باشد. $\Gamma(R)$ را گرافی تعریف می‌کنیم که مجموعه راسی آن R بوده و راس‌های x و y توسط یک یال متصل‌اند اگر و تنها اگر عنصرهای یکه u و v از R موجود باشند به طوری که $x + uyv$ یک عنصر یکه از R باشد. در این مقاله، ضمن مطالعه خواص پایه‌ای $\Gamma(R)$، همبندی و کمر گراف $\Gamma(R)$ برای حلقه آرتینی R بررسی می‌شود. همچنین تعیین می‌کنیم که چه‌زمانی گراف $\Gamma(R)$ یک دور است. ثابت خواهیم کرد که اگر $\Gamma(R) \cong \Gamma(M_n(F))$، آن‌گاه $R \cong M_n(F)$، جایی که R یک حلقه و F یک میدان متناهی است. نشان می‌دهیم که اگر R یک حلقه متناهی و تعویض‌پذیر نیم‌ساده و S یک حلقه تعویض‌پذیر باشد که $\Gamma(R) \cong \Gamma(S)$، آن‌گاه $R \cong S$. در پایان، طیف گراف $\Gamma(R)$ را برای حلقه تعویض‌پذیر و متناهی R به‌دست می‌آوریم.

کلمات کلیدی: حلقه، حلقه‌های ماتریسی، رادیکال جیکوبسن، گراف‌های یکه، گراف‌های کیلی یکانی، طیف.