Journal of Algebraic Systems
Vol. 7, No. 2, (2020), pp 155-165

P-CLOSURE IN PSEUDO BCI-ALGEBRAS

H. HARIZAVI*

Abstract. In this paper, for any non-empty subset C of a pseudo BCI-algebra $\mathfrak{X}$, the concept of p-closure of C, denoted by $C^{\mathfrak{pc}}$, is introduced and some related properties are investigated. Applying this concept, a characterization of the minimal elements of $\mathfrak{X}$ is given. It is proved that $C^{\mathfrak{pc}}$ is the least closed pseudo BCI-ideal of $\mathfrak{X}$ containing C and $K(\mathfrak{X})$ for any ideal C of $\mathfrak{X}$. Finally, by using the concept of p-closure, a closure operator is introduced.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras as a generalization of set-theoretic difference and propositional calculi [5, 6]. We refer useful textbooks for BCK/BCI-algebra to [9, 10]. The notion of pseudo BCI-algebras was introduced by W.A. Dudek and Y.B. Jun [4] in 2008 as an extention of BCI-algebras, and investigated some related properties. Y.B. Jun, et, al. introduced the notion of pseudo BCI-ideals and pseudo BCI-homomorphism, and showed that the pseudo BCK-part of pseudo BCI-algebras is a pseudo BCI-ideal. In [2], G. Dymek introduced the notion of p-semisimple pseudo BCI-algebras, and established some necessary and sufficient condition for a pseudo BCI-algebra to be p-semisimple pseudo BCI-algebra. Also, he proved that there is a one to one relationship between p-semisimple pseudo BCI-algebra and groups. In [8], Y.H. Kim and K.S. So defined the minimal elements of pseudo BCI-algebras, and showed that the set of all minimal elements of a

MSC(2010): Primary: 06F35; Secondary: 03G25.
Keywords: Pseudo BCI-algebra, pseudo BCI-ideal, p-closure, closure operator.
Received: 20 October 2017, Accepted: 27 January 2019.
∗Corresponding author.

pseudo BCI-algebra X forms a subalgebra of X. Recently, G. Dymek [1] introduced the notion of period of elements of pseudo BCI-algebras and investigated their properties. It is known that for any non-empty subset C of a BCI-algebra $\mathfrak{X}$, the generated ideal $\langle C \cup C^{\circ} \rangle$ is the least closed ideal of $\mathfrak{X}$ containing C, where $C^{\circ} = \{0 * x \mid x \in C\}$ [10]. According to this fact, for any non-empty subset C of a pseudo BCI-algebra $\mathfrak{X}$, the concept of p-closure of C, denoted by $C^{\mathfrak{pc}}$, is defined as $C^{\mathfrak{pc}} := \{x \in X \mid a * x \in C \text{ and } a \diamond x \in C \text{ for some } a \in C\}$, and some related properties are investigated. Applying this concept, a characterization of the minimal elements of X is given. A necessary and sufficient condition for a pseudo BCI-algebra to be a p-semisimple BCI-algebra is given. It is proved that $C^{\mathfrak{pc}}$ is the least closed pseudo BCI-ideal containing C and $K(\mathfrak{X})$ for any ideal C of $\mathfrak{X}$. Finally, by using the concept of p-closure, a closure operator is introduced.

2. Preliminary

In this section, we review some definitions and properties that will be used in this paper. For more details, we refer the reader to [9, 4].

An algebra $(X, *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions: for any $x, y, z \in X$,

BCI-1: $((x * y) * (x * z)) * (z * y) = 0$,

BCI-2: $x * 0 = 0$,

BCI-3: $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra $(X, *, 0)$ satisfying $0 * x = 0$ for all $x \in X$ is called a BCK-algebra.

In any BCI-algebra (and BCK-algebra) X, one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

Definition 2.1. A pseudo BCI-algebra is a structure $\mathfrak{X} = (X, \preceq, *, \diamond, 0)$, where $\preceq$ is a binary relation on set X, $*$ and $\diamond$ are binary operations on X and 0 is an elements of X satisfying the following axioms: for all $x, y, z \in X$,

- (a_1) $(x * y) \diamond (x * z) \preceq z * y$, $(x \diamond y) * (x \diamond z)) \preceq z \diamond y$,
- (a_2) $x * (x \diamond y) \preceq y$, $x \diamond (x * y) \preceq y$,
- (a_3) $x \preceq x$,
- (a_4) $x \preceq y$, $y \preceq x \Longrightarrow x = y$,
- (a_5) $x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo BCI-algebra $\mathfrak{X} = (X, \preceq, *, \diamond, 0)$ satisfying $0 \preceq x$ for all $x \in X$ is called a pseudo BCK-algebra.

It is obvious that every pseudo BCI-algebra (resp: pseudo BCK-algebra) satisfying $x * y = x \diamond y$ for any $x, y \in X$ is a BCI-algebra (resp: BCK-algebra).

Any pseudo BCI-algebra $\mathfrak{X}$ satisfies the following conditions: for any $x, y, z \in X$,
(p_1) $x \preceq 0 \Rightarrow x = 0$,
(p_2) $x \preceq y \Rightarrow x * z \preceq y * z,\ x \diamond z \preceq y \diamond z$,
(p_3) $x \preceq y \Rightarrow z * y \preceq z * x,\ z \diamond y \preceq z \diamond x$
(p_4) $x \preceq y,\ y \preceq z \Rightarrow x \preceq z$,
(p_5) $(x * y) \diamond z = (x \diamond z) * y$,
(p_6) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y$,
(p_7) $(x * y) * (z * y) \preceq x * z,\ (x \diamond y) \diamond (z \diamond y) \preceq x \diamond z$,
(p_8) $x * (x \diamond (x * y)) = x * y$ *and* $x \diamond (x * (x \diamond y)) = x \diamond y$,
(p_9) $x * 0 = x = x \diamond 0$,
(p_{10}) $x * x = 0 = x \diamond x$,
(p_{11}) $0 * (x \diamond y) \preceq y \diamond x$,
(p_{12}) $0 \diamond (x * y) \preceq y * x$,
(p_{13}) $0 * x = 0 \diamond x$,
(p_{14}) $0 * (x * y) = (0 * x) \diamond (0 * y)$,
(p_{15}) $0 \diamond (x \diamond y) = (0 \diamond x) * (0 \diamond y)$.

For any BCI-algebra (and BCK-algebra) X, using axioms (a_3), (a_4) and property (p_3), the relation order $\preceq$ defined by axiom (a_5), that is,

$$(\forall x, y \in X)\ x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0,$$

is a partial order.

A non-empty subset S of a pseudo BCI-algebra $\mathfrak{X}$ is called a subalgebra of $\mathfrak{X}$ if $x * y \in S$ and $x \diamond y \in S$ for all $x, y \in S$. It is easily seen that the set $K(\mathfrak{X}) = \{x \in X \mid 0 \preceq x\}$ is a subalgebra of $\mathfrak{X}$ (called the maximal pseudo BCK-algebra of $\mathfrak{X}$). Then $(K(\mathfrak{X}), \preceq, *, \diamond, 0)$ is a pseudo BCK-algebra and so a pseudo BCI-algebra $\mathfrak{X}$ is a pseudo BCK-algebra if and only if $X = K(\mathfrak{X})$.

An element a of a pseudo BCI-algebra X is called minimal if for any $x \in X$ the following holds:

$$a \preceq x \Longrightarrow a = x.$$

We will denote by $M(\mathfrak{X})$ the set of all minimal elements of X. Obviously, $0 \in M(\mathfrak{X})$. In [6], it has proved that $a \in X$ is minimal if and only if $a = 0 * (0 \diamond a)$ if and only if $a = 0 * x$ for some $x \in X$. Therefore $M(\mathfrak{X}) = \{x \in X \mid x = 0 \diamond (0 * x)\} = \{0 * x \mid x \in X\}$. A pseudo BCI-algebra $\mathfrak{X}$ is called p-semisimple if any element of X is minimal. It is easily to seen that $K(\mathfrak{X}) \cap M(\mathfrak{X}) = \{0\}$.

Proposition 2.2. [2] *Let $\mathfrak{X}$ be a pseudo BCI-algebra. Then for any $x, y \in X$ the following are equivalent:*

(i) $\mathfrak{X}$ *is a p-semisimple,*

(ii) $x * (x \diamond y) = y = x \diamond (x * y)$,
(iii) $0 * (0 \diamond x) = x = 0 \diamond (0 * x)$.

For any minimal element $a \in X$, the branch of a is defined by $V(a) := \{x \in X \mid x \succeq a\}$. Obviously, $a \in V(a)$ and hence $V(a) \neq \emptyset$.

Let X be a pseudo-BCI-algebra. For any none-empty subset J of X and any element $y \in X$ we denote

$$*(y, J) := \{x \in X \mid x * y \in J\} \text{ and } \diamond(y, J) := \{x \in X \mid x \diamond y \in J\}.$$

Definition 2.3. [7] A subset J of a pseudo BCI-algebra X is called a pseudo BCI-ideal of $\mathfrak{X}$ if

(I1) $0 \in J$,
(I2) $(\forall y \in J)$ $(*(y, J) \subseteq J$ *and* $\diamond(y, J) \subseteq J.)$

Theorem 2.4. [7] *If J is a pseudo BCI-ideal of a pseudo BCI-algebra $\mathfrak{X}$, then the following hold: for any $x, y, z \in X$,*

(i) *$x \in J$ and $y \preceq x \Longrightarrow y \in J$,*
(ii) *$y \in J$ and $z * y \in J \Longrightarrow z \in J$,*
(iii) *$y \in J$ and $z \diamond y \in J \Longrightarrow z \in J$.*

A pseudo BCI-ideal J of a pseudo BCI-algebra $\mathfrak{X}$ is called closed if J is closed under operations $*$ and $\diamond$. A pseudo BCI-ideal J of a pseudo BCI-algebra X is closed if and only if $0 * x = 0 \diamond x \in J$ for any $x \in J$ (see [7]).

3. Main results

In this section, we start by introducing the concept of p-closure for a non-empty subset C of a pseudo BCI-algebra X, and then investigate some related properties.

In what follows, let $\mathfrak{X}$ denote a pseudo BCI-algebra unless otherwise specified.

Definition 3.1. For any non-empty subset C of X, we define the p-closure of C by the set

$$C^{\mathfrak{pc}} := \{x \in X \mid a * x \in C \text{ and } a \diamond x \in C \text{ for some } a \in C\}.$$

Obviously, $0 \in C^{\mathfrak{pc}}$.

The following lemma is an immediate consequence from Definition 3.1 and (p_9).

Lemma 3.2. *For any non-empty subsets C and D of X, the following holds:*

(i) *if $C \subseteq D$, then $C^{\mathfrak{pc}} \subseteq D^{\mathfrak{pc}}$,*
(ii) *if $0 \in C$, then $C \subseteq C^{\mathfrak{pc}}$.*

In the following theorem, we give a characterization of the minimal elements of X.

Theorem 3.3. *An element a of X is minimal if and only if $\{a\}^{\mathfrak{pc}} = K(\mathfrak{X})$.*

Proof. ($\Rightarrow$) Let a be a minimal element of X. Assume that $x \in \{a\}^{\mathfrak{pc}}$. Then $a * x = a = a \diamond x$ and so, using (p_5), we have $0 = (a * x) \diamond a = (a \diamond a) * x = 0 * x$. It follows that $x \in K(\mathfrak{X})$. Hence $\{a\}^{\mathfrak{pc}} \subseteq K(\mathfrak{X})$. To prove the reverse inclusion, let $x \in K(\mathfrak{X})$. Then $0 * x = 0$, and so we have

$$\begin{aligned} a * x &= (0 \diamond (0 * a)) * x && \text{by the minimality of a} \\ &= (0 * x) \diamond (0 * a) && \text{by } (p_5) \\ &= 0 \diamond (0 * a) && \text{by } (p_{13}) \\ &= a, && \text{by the minimality of a} \end{aligned}$$

that is, $a*x = a$, which implies that $x \in \{a\}^{\mathfrak{pc}}$. Therefore $K(\mathfrak{X}) \subseteq \{a\}^{\mathfrak{pc}}$ and so $\{a\}^{\mathfrak{pc}} = K(\mathfrak{X})$.

($\Leftarrow$) Assume that $\{a\}^{\mathfrak{pc}} = K(\mathfrak{X})$. Let $b \in X$ with $b \preceq a$. Then $0 \preceq a*b$ and so $a*b \in K(\mathfrak{X})$. Thus $a*b \in \{a\}^{\mathfrak{pc}}$ and hence $a \diamond (a*b) = a$. It follows from (p_5) that $a*b = (a \diamond (a*b))*b = (a*b) \diamond (a*b) = 0$, that is, $a \preceq b$. Hence $a = b$. Therefore a is a minimal element of X. □

In the following theorem, we give a necessary and sufficient condition for a pseudo BCI-algebra to be a pseudo BCK- algebra.

Theorem 3.4. *$\mathfrak{X}$ is a pseudo BCK-algebra if and only if $\{0\}^{\mathfrak{pc}} = X$.*

Proof. ($\Rightarrow$) Let $\mathfrak{X}$ be a pseudo BCK-algebra. Then for any $x \in X$, $0 * x = 0 = 0 \diamond x$. It follows that $x \in \{0\}^{\mathfrak{pc}}$ for any $x \in X$. Therefore $\{0\}^{\mathfrak{pc}} = X$.

($\Leftarrow$) Assume that $\{0\}^{\mathfrak{pc}} = X$. Then using Theorem 3.3, we get $X = K(\mathfrak{X})$. This implies that $\mathfrak{X}$ is a pseudo BCK-algebra. □

Corollary 3.5. *$\mathfrak{X}$ is a pseudo BCK-algebra if and only if $C^{\mathfrak{pc}} = X$ for any subset C of X containing 0.*

Proof. Using Lemma 3.2(i) and Theorem 3.4, the proof is straightforward. □

In the following, we introduce some subsets of X whose p-closure are maximal pseudo BCK-algebra of $\mathfrak{X}$.

Theorem 3.6. *For any $\mathfrak{X}$, the following hold:*

(i) *if C is a subset of $K(\mathfrak{X})$ and $0 \in C$, then $C^{\mathfrak{pc}} = K(\mathfrak{X})$,*

(ii) *$K(\mathfrak{X})^{\mathfrak{pc}} = K(\mathfrak{X})$,*

(iii) *for any element* c *of* X, $\{A(c)\}^{\mathfrak{pc}} = K(\mathfrak{X})$, *where* $A(c) = \{x \in X \mid x \preceq c\}$.

Proof. (i) Since $\{0\} \subseteq C \subseteq K(\mathfrak{X})$, it follows from Lemma 3.2(i) that $\{0\}^{\mathfrak{pc}} \subseteq C^{\mathfrak{pc}} \subseteq K(\mathfrak{X})^{\mathfrak{pc}}$. Thus by Theorem 3.3, we obtain $K(\mathfrak{X}) \subseteq C^{\mathfrak{pc}} \subseteq K(\mathfrak{X})$, which implies that $C^{\mathfrak{pc}} = K(\mathfrak{X})$.

(ii) It is an immediate consequence of (i).

(iii) Let $x \in K(\mathfrak{X})$. Then $0 * x = 0 = 0 \diamond x$ and so $(c * x) \diamond c = (c \diamond c) * x = 0 * x = 0$. This implies that $c * x \preceq c$ and so $c * x \in A(c)$. Moreover, $c \in A(c)$. Hence, $x \in A(c)^{\mathfrak{pc}}$ and so $K(\mathfrak{X}) \subseteq A(c)^{\mathfrak{pc}}$. Now let $x \in A(c)^{\mathfrak{pc}}$. Then there exists $t \in A(c)$ such that $t * x \preceq c$, that is, $(t * c) * x = 0$. On the other hand, from $t \in A(c)$ we have $t * c = 0$. Thus, $0 * x = 0$ and so $x \in K(\mathfrak{X})$. Therefore $A(c)^{\mathfrak{pc}} = K(\mathfrak{X})$. □

Proposition 3.7. *For any subset* C *of* X *containing* $M(\mathfrak{X})$, $C^{\mathfrak{pc}} = X$.

Proof. (i) Let $x \in X$. We know that $0 * (0 * x)$ is a minimal element of $\mathfrak{X}$, and so $0 * (0 * x) \in M(\mathfrak{X})$. Thus, $0 * (0 * x) \in C$. Now, using (p_5), we get $(0 * (0 * x)) * x = 0 \in C$ and $(0 * (0 * x)) \diamond x = 0 \in C$, which implies $x \in C^{\mathfrak{pc}}$. Therefore $C^{\mathfrak{pc}} = X$. □

Lemma 3.8. *Let* C *be a subalgebra of* X. *Then the following statement are equivalent: for any* $x \in X$,

(i) $x \in C^{\mathfrak{pc}}$.
(ii) $0 * x \in C$.
(iii) $0 * x \in C^{\mathfrak{pc}}$.

Proof. (i)⇒(ii) Let $x \in C^{\mathfrak{pc}}$. Then $a * x \in C$ and $a \diamond x \in C$ for some $a \in C$, and so, since C is closed, we get $(a * x) \diamond a \in C$. On the other hand, we have $(a * x) \diamond a = (a \diamond a) * x = 0 * x$. Therefore $0 * x \in C$.

(ii)⇒(iii) This is obvious by Lemma 3.2(ii).

(iii)⇒(i) Let $0*x \in C^{\mathfrak{pc}}$. Then there exists $a \in C$ such that $a*(0*x) \in C$ and $a \diamond (0 * x) \in C$. Since C is closed, we obtain $(a * (0 * x)) \diamond a \in C$. But using (p_5), we have $(a*(0*x))\diamond a = 0*(0*x)$. Hence $0*(0*x) \in C$. Now, by (p_5), we get $(0 * (0 * x)) * x = 0 = (0 * (0 * x)) \diamond x$. Therefore, it follows from $0 \in C$ that $x \in C^{\mathfrak{pc}}$. □

The following follows from Lemma 3.8.

Corollary 3.9. *If* C *is a subalgebra of* X, *then so is* $C^{\mathfrak{pc}}$.

In the following theorem, for any subalgebra C of X, we give a characterization of $C^{\mathfrak{pc}}$ by some branches of C.

Theorem 3.10. *If* C *is a subalgebra of* X, *then* $C^{\mathfrak{pc}} = \bigcup_{c \in C} V(0 * c)$.

Proof. Let $x \in C^{\mathfrak{pc}}$. Then by Lemma 3.8, $0*x \in C$. Since $0*(0*x) \preceq x$, by putting $c = 0*x$, we get $x \in V(0*c)$. This implies that $C^{\mathfrak{pc}} \subseteq \bigcup_{c\in C} V(0*c)$. In order to show the reverse inclusion, let $x \in \bigcup_{c\in C} V(0*c)$. Then there exists $c \in C$ such that $x \in V(0*c)$). Thus, $0*c \preceq x$ and so $(0*c)*x = 0$ and $(0*c)\diamond x = 0$. Moreover, since C is a subalgebra, we have $0*c \in C$ and hence $x \in C^{\mathfrak{pc}}$. Therefore $\bigcup_{c\in C} V(0*c) \subseteq C^{\mathfrak{pc}}$, and so the proof is completed. □

In the following, we establish an important property of the p-closure.

Theorem 3.11. *If C is a pseudo BCI-ideal of $\mathfrak{X}$, then $C^{\mathfrak{pc}}$ is a pseudo BCI-ideal of $\mathfrak{X}$, too.*

Proof. We first prove that $C^{\mathfrak{pc}}$ is a pseudo BCI-ideal of $\mathfrak{X}$. Clearly, $0 \in C^{\mathfrak{pc}}$. Now, we show that $*(y, C^{\mathfrak{pc}}) \subseteq C^{\mathfrak{pc}}$ and $\diamond(y, C^{\mathfrak{pc}}) \subseteq C^{\mathfrak{pc}}$ for any $y \in C^{\mathfrak{pc}}$. Let $x \in *(y, C^{\mathfrak{pc}})$. Then $x*y \in C^{\mathfrak{pc}}$, and so there exists $b \in C$ such that $b*(x*y) \in C$ and $b\diamond(x*y) \in C$. Also, from $y \in C^{\mathfrak{pc}}$, we have $a*y \in C$ and $a\diamond y \in C$ for some $a \in C$. We first show that $b\diamond(0*a) \in C$. It is easy to see that $(b\diamond(0*a))*b = (b*b)\diamond(0*a) = 0\diamond(0*a) \preceq a$. Thus, since $a, b \in C$, we conclude

$$b\diamond(0*a) \in C. \tag{3.1}$$

Now, we show that $x \in C^{\mathfrak{pc}}$. For this purpose, using (p_5) and axiom $(a1)$, we have

$$\begin{aligned} ((b\diamond(0*a))\diamond x)*(b\diamond(x*y)) &= ((b\diamond(0*a))*(b\diamond(x*y)))\diamond x \\ &\preceq ((x*y)\diamond(0*a))\diamond x \\ &= ((x\diamond(0*a))*y)\diamond x \\ &= ((x\diamond(0*a))\diamond x)*y \end{aligned}$$

Thus

$$((b\diamond(0*a))\diamond x)*(b\diamond(x*y)) \preceq ((x\diamond(0*a))\diamond x)*y. \tag{3.2}$$

On the other hand, using (p_5) and axiom $(a1)$ again, we have

$$\begin{aligned} (((x\diamond(0*a))\diamond x)*y)\diamond(a*y) &\preceq ((x\diamond(0*a))\diamond x)*a \\ &= ((x\diamond(0*a))*a)\diamond x \\ &= ((x*a)\diamond(0*a))\diamond x \\ &\preceq (x*0)\diamond x \\ &= 0. \end{aligned}$$

This implies that

$$((x\diamond(0*a))\diamond x)*y \preceq a*y \tag{3.3}$$

Combining (3.2) and (3.3), we obtain $((b \diamond (0 * a)) \diamond x) * (b \diamond (x * y)) \preceq a * y \in C$. Thus, since $b \diamond (x * y) \in C$, we get $(b \diamond (0 * a)) \diamond x \in C$. Similarly, applying $a \diamond (x * y) \in C$ and $a \diamond y \in C$, we can show that $(b \diamond (0 * a)) * x \in C$. Hence, by (3.1), we have $x \in C^{\mathfrak{pc}}$, and so $*(y, C^{\mathfrak{pc}}) \subseteq C^{\mathfrak{pc}}$. By the similar argument, we can show that $\diamond(y, C^{\mathfrak{pc}}) \subseteq C^{\mathfrak{pc}}$. Therefore $C^{\mathfrak{pc}}$ is a pseudo BCI-ideal of $\mathfrak{X}$. □

The following is another important property of the p-closure.

Theorem 3.12. *If C is a pseudo BCI-ideal of $\mathfrak{X}$, then $C^{\mathfrak{pc}}$ is a closed pseudo BCI-ideal of $\mathfrak{X}$ containing $K(\mathfrak{X})$.*

Proof. Let $x \in C^{\mathfrak{pc}}$. Then $a * x \in C$ and $a \diamond x \in C$ for some $a \in C$. Using (p_5), we get $(a * (0 * a)) \diamond a = 0 * (0 * a) \preceq a \in C$, and so $a * (0 * a) \in C$. Similarly, we have $a \diamond (0 * a) \in C$. Thus $0 * a \in C^{\mathfrak{pc}}$. Now, since $(0 * x) * (a * x) \preceq 0 * a \in C^{\mathfrak{pc}}$, it follows from $a * x \in C \subseteq C^{\mathfrak{pc}}$ that $0 * x \in C^{\mathfrak{pc}}$. Therefore $C^{\mathfrak{pc}}$ is closed. Also, using Theorem 3.3 and Lemma 3.2, we get $K(\mathfrak{X}) = \{0\}^{\mathfrak{pc}} \subseteq C^{\mathfrak{pc}}$, and so the proof is completed. □

Lemma 3.13. *For any $\mathfrak{X}$,*

$$\begin{aligned} K(\mathfrak{X}) &= \{x \diamond (0 * (0 * x)) \mid \textit{for some } x \in X\} \\ &= \{x * (0 * (0 * x)) \mid \textit{for some } x \in X\}. \end{aligned}$$

Proof. (i) For any $x \in X$, we have

$$\begin{aligned} 0 * (x \diamond (0 * (0 * x))) &= (0 * x) \diamond (0 * (0 * (0 * x))) && \text{by } (p_{14}) \\ &= (0 * x) \diamond (0 * x) && \text{by } (p_8) \\ &= 0 \end{aligned}$$

Thus for any $x \in X$, $x \diamond (0 * (0 * x)) \in K(\mathfrak{X})$. Therefore $\{x \diamond (0 * (0 * x)) \mid \textit{for some } x \in X\} \subseteq K(\mathfrak{X})$. On the other hand, if $x \in K(\mathfrak{X})$, then $0 * x = 0$ and so $x = x \diamond (0 * (0 * x))$. This implies $K(\mathfrak{X}) \subseteq \{x \diamond (0 * (0 * x)) \mid \textit{for some } x \in X\}$. Therefore $K(\mathfrak{X}) = \{x \diamond (0 * (0 * x)) \mid \textit{for some } x \in X\}$. Similarly, we can show the second part of the lemma. □

In the following, we introduce an interesting property of the p-closure.

Theorem 3.14. *If C is a pseudo BCI-ideal of $\mathfrak{X}$, then $C^{\mathfrak{pc}} = (C^{\mathfrak{pc}})^{\mathfrak{pc}}$.*

Proof. Since $0 \in C^{\mathfrak{pc}}$, it follows from Lemma 3.2(ii) that $C^{\mathfrak{pc}} \subseteq (C^{\mathfrak{pc}})^{\mathfrak{pc}}$. To show the reverse inclusion, let $x \in (C^{\mathfrak{pc}})^{\mathfrak{pc}}$. By Theorem 3.12, $C^{\mathfrak{pc}}$

is a subalgebra of $\mathfrak{X}$ and so by Lemma 3.8, we get $0 * x \in C^{\mathfrak{pc}}$. Then, since $C^{\mathfrak{pc}}$ is closed, we have

$$0 * (0 * x) \in C^{\mathfrak{pc}}. \tag{3.4}$$

By Lemma 3.13, we have $x \diamond (0 * (0 * x)) \in K(\mathfrak{X})$. On the other hand, $K(\mathfrak{X}) \subseteq C^{\mathfrak{pc}}$. Hence $x \diamond (0 * (0 * x)) \in C^{\mathfrak{pc}}$, and so by (3.4), we get $x \in C^{\mathfrak{pc}}$. Therefore $(C^{\mathfrak{pc}})^{\mathfrak{pc}} \subseteq C^{\mathfrak{pc}}$, which completes the proof. □

Corollary 3.15. *For any $\mathfrak{X}$, the mapping $\mathfrak{pc} : \mathbb{I}(\mathfrak{X}) \to \mathbb{I}(\mathfrak{X})$ defined by $\mathfrak{pc}(C) = C^{\mathfrak{pc}}$ for any $C \in \mathbb{I}(\mathfrak{X})$ is a closure operator on $(\mathbb{I}(\mathfrak{X}), \subseteq)$, where $\mathbb{I}(\mathfrak{X})$ denotes the set of all pseudo BCI-ideals of $\mathfrak{X}$.*

Proof. It is an immediate consequence from Lemma 3.2 and Theorem 3.14. □

In the following theorem, we give a necessary and sufficient condition for a pseudo BCI- ideal to be closed.

Theorem 3.16. *Let C be a pseudo BCI-ideal of $\mathfrak{X}$. If we denote $C_\circ = \{x \in C \mid 0 * x \in C\}$, then the following are equivalent:*

(i) *C is closed,*
(ii) *$C = C_\circ$,*
(iii) *$C^{\mathfrak{pc}} = {C_\circ}^{\mathfrak{pc}}$.*

Proof. The proof of (i)⇒ (ii) and (ii)⇒ (iii) are easy.

(iii)⇒ (i) Assume that ${C_\circ}^{\mathfrak{pc}} = C^{\mathfrak{pc}}$ and $x \in C$. Then, by the closeness of $C^{\mathfrak{pc}}$, we have $0 * x \in C^{\mathfrak{pc}}$ and so by assumption, $0 * x \in {C_\circ}^{\mathfrak{pc}}$. Thus there exists $a \in C_\circ$ such that $a * (0 * x) \in C_\circ$ and $a \diamond (0 * x) \in C_\circ$. From this and definition of $C_\circ$ it follows that $0 * (a * (0 * x)) \in C$. Now we have

$$\begin{aligned}(0 * x) * a &= (0 \diamond (0 * (0 \diamond x))) * a && \text{by } (p_8)\\ &= (0 * a) \diamond (0 * (0 \diamond x)) && \text{by axiom (a2)}\\ &= 0 * (a * (0 * x)) && \text{by } (p_{14})\end{aligned}$$

Hence $(0 * x) * a \in C$ and so from $a \in C_0 \subseteq C$, we conclude $0 * x \in C$. Therefore C is closed. □

In the following, we consider the p-closure of intersection of a family of closed pseudo BCI-ideals of $\mathfrak{X}$.

Theorem 3.17. *For every family $\{C_\alpha\}_{\alpha \in I}$ of closed pseudo BCI-ideals of $\mathfrak{X}$, $(\bigcap_{\alpha \in I} C_\alpha)^{\mathfrak{pc}} = \bigcap_{\alpha \in I} {C_\alpha}^{\mathfrak{pc}}$.*

Proof. By Lemma 3.2(i), $(\bigcap_{\alpha \in I} C_\alpha)^{\mathfrak{pc}} \subseteq {C_\alpha}^{\mathfrak{pc}}$ for every $\alpha \in I$. Thus $(\bigcap_{\alpha \in I} C_\alpha)^{\mathfrak{pc}} \subseteq \bigcap_{\alpha \in I} {C_\alpha}^{\mathfrak{pc}}$. Now let $x \in \bigcap_{\alpha \in I} {C_\alpha}^{\mathfrak{pc}}$. Then for every $\alpha \in I$, there exists $c_\alpha \in C_\alpha$ such that $c_\alpha * x \in C_\alpha$. Using (p_7) and the fact that C_α is closed, we conclude $(0*x)*(c_\alpha*x) \preceq 0*c_\alpha \in C_\alpha$. Then, it follows from $c_\alpha*x \in C_\alpha$ that $0*x \in C_\alpha$ and so $0*x \in \bigcap_{\alpha\in I} C_\alpha$. Also, obviously, $0 \diamond x \in \bigcap_{\alpha\in I} C_\alpha$. Thus $x \in (\bigcap_{\alpha\in I} C_\alpha)^{\mathfrak{pc}}$, and consequently $\bigcap_{\alpha\in I}(C_\alpha)^{\mathfrak{pc}} \subseteq (\bigcap_{\alpha\in I} C_\alpha)^{\mathfrak{pc}}$. Therefore $(\bigcap_{\alpha\in I} C_\alpha)^{\mathfrak{pc}} = \bigcap_{\alpha\in I} {C_\alpha}^{\mathfrak{pc}}$. □

To give a characterization of the p-semisimple pseudo BCI-algebras, we recall the following notation [10].

For any non-empty subset C of $\mathfrak{X}$, we denote

$$C^\circ := \{0*x \mid x \in C\} = \{0 \diamond x \mid x \in C\}.$$

Lemma 3.18. *For any pseudo BCI-ideal C of $\mathfrak{X}$, the following hold:*

(i) $C^\circ \subseteq C^{\mathfrak{pc}}$,

(ii) $\langle C \cup C^\circ\rangle^{\mathfrak{pc}} = C^{\mathfrak{pc}}$.

Proof. (i) Let $0*x \in C^\circ$ for some $x \in C$. Then, from $0*(0*x) \preceq x$, we get $0*(0*x) \in C$. Also, obviously, $0 \diamond (0*x) \in C$. Theerfore $0*x \in C^{\mathfrak{pc}}$ and so $C^\circ \subseteq C^{\mathfrak{pc}}$.

(ii) By (i) and Lemma 3.2(ii), we have $C, C^\circ \subseteq C^{\mathfrak{pc}}$. Since $C^{\mathfrak{pc}}$ is a pseudo BCI-ideal, we obtain $C \subseteq \langle C \cup C^\circ\rangle \subseteq C^{\mathfrak{pc}}$, hence $C^{\mathfrak{pc}} \subseteq \langle C \cup C^\circ\rangle^{\mathfrak{pc}} \subseteq (C^{\mathfrak{pc}})^{\mathfrak{pc}}$. Thus by Theorem 3.14, we conclude $\langle C \cup C^\circ\rangle^{\mathfrak{pc}} = C^{\mathfrak{pc}}$. □

In the next theorem, we give a characterization of the p-semisimple pseudo BCI-algebras.

Theorem 3.19. *$\mathfrak{X}$ is p-semisimple $\Leftrightarrow$ $\langle C \cup C^\circ\rangle = C^{\mathfrak{pc}}$ for all pseudo BCI-ideal C of $\mathfrak{X}$.*

Proof. ($\Rightarrow$) This is obvious by Lemma 3.18(ii).

($\Leftarrow$) Assume that $\langle C \cup C^\circ\rangle = C^{\mathfrak{pc}}$ for any pseudo BCI-ideal C of $\mathfrak{X}$. Taking $C := \{0\}$, we get $C^\circ = \{0\}$ and so by Theorem 3.6(ii), we have $C^{\mathfrak{pc}} = K(\mathfrak{X})$. On the other hand, by assumption, we obtain $C^{\mathfrak{pc}} =< C \cup C^\circ >= \{0\}$. Therefore $K(\mathfrak{X}) = \{0\}$ and so by Lemma 3.13, we obtain $x \diamond (0*(0*x) = 0$ for any $x \in X$. On the other hand, $(0*(0*x) \diamond x = 0$. Therefore $0*(0*x) = x$ and so by Proposition 2.2, $\mathfrak{X}$ is a p-semisimple BCI-algebra. □

In the following theorem, we establish the main result of this paper.

Theorem 3.20. *For any pseudo BCI-ideal C of $\mathfrak{X}$, $C^{\mathfrak{pc}}$ is the least closed pseudo BCI-ideal of $\mathfrak{X}$ containing C and $K(\mathfrak{X})$.*

Proof. Combining Lemma 3.2(ii) and Theorems 3.11 and 3.12, we conclude $C^{\mathfrak{pc}}$ is a closed pseudo BCI-ideal of X containing C and $K(\mathfrak{X})$. To complete the proof, let D be another closed pseudo BCI-ideal of $\mathfrak{X}$ containing C and $K(\mathfrak{X})$, and let $x \in C^{\mathfrak{pc}}$. Then, since $C^{\mathfrak{pc}}$ is closed, we get $0 * x \in C^{\mathfrak{pc}}$. But from $C \subseteq D$, we have $C^{\mathfrak{pc}} \subseteq D^{\mathfrak{pc}}$. Thus $0 * x \in D^{\mathfrak{pc}}$ and so it follows from Lemma 3.8 that $0 * (0 * x) \in D$. We note that $x \diamond (0 * (0 * x)) \in K(\mathfrak{X})$ and so from $K(\mathfrak{X}) \subseteq D$, we obtain $x \diamond (0 * (0 * x)) \in D$. Hence, since $0 * (0 * x) \in D$, we conclude $x \in D$. Therefore $C^{\mathfrak{pc}} \subseteq D$, and so the proof is completed. □

Acknowledgement

The author would like to thank the referee for his careful reading and suggestions which helped to improve the paper. This work was supported by the research grant of the Shahid Chamran University of Ahvaz, Ahvaz, Iran.

References

1. G. Dymek, On a periodic part of pseudo BCI-algebras, *Discussiones Math., General Algebra Appl.*, **35** (2015), 139–157.
2. G. Dymek, On pseudo BCI-algebras, *Annales Universitatis Mariae Curie-Sklodowska Lublin Poloia*, **1** (2015), 59–71.
3. G. Dymek, On two classes of pseudo BCI-algebras, *Discussiones Math., General Algebra Appl.*, **31** (2011), 217–229.
4. W. A. Dudek and Y. B. Jun, Pseudo BCI-algebras, *East Asian Math. J.*, **24** (2008), 187–190.
5. Y. Imai and K. Iséki, On axiom system of propositional calculi, *Proc. Japan Acad.*, **42** (1966), 19–22.
6. K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad.*, **42** (1966), 26–29.
7. Y. B. Jun, H. S. Kim and J. Neggers, On pseudo BCI-ideals of pseudo BCI-algebras, *Matemat. Bech.*, **58** (2006), 39–46.
8. Y. H. Kim and K. S. So, On minimality in pseudo BCI-algebras, *Commun. Korean Math. Soc.*, **1** (2012), 7–13.
9. J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa Co., Seoul., 1994.
10. H. Yisheng, *BCI-Algebra*, Science Press, Beijing, 2006.

Habib Harizavi
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.
Email: harizavi@scu.ac.ir

Journal of Algebraic Systems

P-CLOSURE IN PSEUDO BCI-ALGEBRAS

H. HARIZAVI

p-بستار در شبه BCI-جبرها

حبیب حریزاوی
گروه ریاضی، دانشکده علوم ریاضی و کامپیوتر، دانشگاه شهید چمران اهواز، اهواز، ایران

چکیده مقاله : در این مقاله، برای هر زیرمجموعه‌ی ناتهی C از یک شبه BCI -جبر X، مفهوم p -بستار C با نمایش C^{pc}، معرفی شده است و برخی خواص مرتبط با آن مورد بررسی قرار گرفته است. با به‌کارگیری این مفهوم توصیفی از عناصر مینیمال X ارائه گردیده است. ثابت شده است که C^{pc} کوچکترین شبه ایدآل-BCI بسته‌ی X شامل C و $K(X)$ است. در نهایت، با به‌کارگیری مفهوم p -بستار، یک عملگر بستار بیان شده است.

کلمات کلیدی: p-بستار، شبه BCI-جبر، شبه BCI-ایدآل.