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A KIND OF F-INVERSE SPLIT MODULES

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Abstract. Let M be a right module over a ring R. In this manuscript, we shall study on a special case of F-inverse split modules where F is a fully invariant submodule of M introduced in [12]. We say M is $\overline{Z}^2(M)$-inverse split provided $f^{-1}(\overline{Z}^2(M))$ is a direct summand of M for each endomorphism f of M. We prove that M is $\overline{Z}^2(M)$-inverse split if and only if M is a direct sum of $\overline{Z}^2(M)$ and a $\overline{Z}^2$-torsionfree Rickart submodule. It is shown under some assumptions that the class of right perfect rings R for which every right R-module M is $\overline{Z}^2(M)$-inverse split ($\overline{Z}(M)$-inverse split) is precisely that of right GV-rings.

1. Introduction

Throughout this paper R denotes a ring with identity, modules are unital right R-modules and $S = End_R(M)$ denotes the ring of all right R-module endomorphisms of a module M unless otherwise stated. Also $N \leq M$ states that N is a submodule of a module M.

The notions of Rickart and Baer rings have their roots in functional analysis with close links to C^*-algebras and von Neumann algebras. Kaplansky introduced the notion of Baer rings in 1955 [4] which was extended to quasi-Baer rings in 1967 [1]. A ring R is called *(quasi-)Baer* if the right annihilator of any nonempty subset (two-sided ideal) of R is generated by an idempotent as a right ideal. Closely related to the concept of Baer rings is the more general notion of right Rickart

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rings. The concept of right (left) Rickart rings has been comprehensively studied in the literature. A ring R is called *right Rickart* if the right annihilator of any single element of R is generated by an idempotent of R as a right ideal. Let R be a ring, M be a right R-module and $S = End_R(M)$. Following [5], M is a *Rickart* module if the right annihilator in M of any single element of S is generated by an idempotent of S, equivalently, $r_M(f) = Kerf$ is a direct summand of M for every $f \in S$. It is easy to see that for $M = R_R$, the notion of a Rickart module coincides with that of a right Rickart ring. Hence R_R is a Rickart module if R is a Baer ring, a von Neumann regular ring or a right hereditary ring (see [5]). In [5], the authors investigated Rickart modules and their properties and study the connections between a Rickart module and its endomorphism ring.

A submodule N of a module M is said to be *small* in M if $N+K \neq M$ for any proper submodule K of M. Also a module L is said to be a *small module*, in case L is a small submodule of a module T. Following [2], a module M is called *lifting* if every submodule $N \leq M$ there exists a direct summand D of M such that $N/D \ll M/D$. A submodule N of M is called a *supplement* in M if there is a submodule K of M such that $M = N+K$ and $N \cap K \ll N$. A module M is called *supplemented* if every submodule of M has a supplement in M. A module M is *amply supplemented* if $M = A + B$, then A contains a supplement of B in M. A lifting module is amply supplemented and hence supplemented. Let R be a ring and M a right R-module. In [10], Talebi and Vanaja defined $\overline{Z}(M)$ as a dual of singular submodule as follows: $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U, U \in \mathcal{S}\}$ (here $\mathcal{S}$ denotes the class of all small right R-modules). They called M a *cosingular (noncosingular)* module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Clearly every small module is cosingular. In [10], $\overline{Z}^{\alpha}(M)$ is defined by $\overline{Z}^{0}(M) = M$, $\overline{Z}^{\alpha+1}(M) = \overline{Z}(\overline{Z}^{\alpha}(M))$ and $\overline{Z}^{\alpha}(M) = \bigcap_{\beta<\alpha} \overline{Z}^{\beta}(M)$ if α is a limit ordinal. Hence there is a descending chain $M = \overline{Z}^{0}(M) \supseteq \overline{Z}(M) \supseteq \overline{Z}^{2}(M) \supseteq \dots$ of submodules of M.

Recall that a submodule F of a module M is called *fully invariant* if $h(F) \subseteq F$ for every $h \in End_R(M)$. Let F be a fully invariant submodule of a module M. Then M is said to be *F-inverse split* [12], if $h^{-1}(F)$ is a direct summand of M for every $h \in End_R(M)$. In [13], the authors defined $\overline{Z}(M)$-inverse split modules and investigated their properties. A module M is *$\overline{Z}(M)$-inverse split* provided $h^{-1}(\overline{Z}(M))$ is a direct summand of M for each endomorphism h of M. They proved that a module M is $\overline{Z}(M)$-inverse split if and only if M is decomposed to a noncosingular submodule and a cosingular Rickart submodule if and only if $M = \overline{Z}(M) \oplus N$ where N is cosingular Rickart. In this article, we define $\overline{Z}^2(M)$-inverse split modules and try to investigate their general properties. We say M is $\overline{Z}^2(M)$-inverse split provided $h^{-1}(\overline{Z}^2(M))$ is a direct summand of M for each endomorphism h of M. We prove that a module M is $\overline{Z}^2(M)$-inverse split if and only if M is decomposed to $\overline{Z}^2(M)$ and a $\overline{Z}^2$-torsion free Rickart submodule N. We also present, under some assumptions, new characterizations of right perfect right GV-rings in terms of $\overline{Z}^2$-inverse split modules.

2. $\overline{Z}^2$-INVERSE SPLIT MODULES

In this section, we are interested in studying on a special case of F-inverse split modules. There are many important fully invariant submodules of a module. Among all of them, the second cosingular submodule of a module M, namely $\overline{Z}^2(M)$, has a key role in determining of some important modules such as lifting modules and amply supplemented modules.

We start with introducing $\overline{Z}^2$-inverse split modules.

Definition 2.1. A module M is called $\overline{Z}^2(M)$-inverse split whenever $f^{-1}(\overline{Z}^2(M))$ is a direct summand of M for every $f \in S$.

Here are examples of some known $\overline{Z}^2$-inverse split modules.

Example 2.2. (1) The class of $\overline{Z}^2$-inverse split modules contains a large class of modules namely semisimple modules. In particular, every Artinian (Noetherian) module M over a Boolean ring R is $\overline{Z}^2(M)$-inverse split.

(2) It is clear that, every noncosingular module M is $\overline{Z}^2(M)$-inverse split. So that, every right R-module M over a right V-ring R, is $\overline{Z}^2(M)$-inverse split.

(3) Every injective right R-module M over a right hereditary ring is noncosingular (see [10, Proposition 2.7]) and hence is $\overline{Z}^2(M)$-inverse split.

(4) For a cosingular module M, two concepts "Rickart" and "$\overline{Z}^2(M)$-inverse split" coincide. Since a projective $\mathbb{Z}$-module has the form $M = \mathbb{Z}^{(I)}$ where I is an arbitrary nonempty index set, M is cosingular. From [5, Theorem 2.26], M is Rickart. So that M is $\overline{Z}^2(M)$-inverse split.

We exhibit a characterization of $\overline{Z}^2(M)$-inverse split modules which will be applied excessively throughout the paper.

Theorem 2.3. *Let M be a module. Then the following statements are equivalent:*

(1) *M is $\overline{Z}^2(M)$-inverse split;*

(2) *$M = \overline{Z}^2(M) \oplus N$ where N is a Rickart module with $\overline{Z}^2(N) = 0$;*

(3) *$M = K \oplus L$ where K is noncosingular and L is a Rickart module with $\overline{Z}^2(L) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let M be $\overline{Z}^2(M)$-inverse split. So, $id_M^{-1}(\overline{Z}^2(M)) = \overline{Z}^2(M)$ is a direct summand of M. Set $M = \overline{Z}^2(M) \oplus N$ for $N \leq M$. We shall prove N is Rickart. To verify this assertion, suppose $g \in End(N)$. Then $f = jogo\pi_N$ is an endomorphism of M where $\pi_N : M \to N$ is the projection of M on N and $j : N \to M$ is the inclusion. Now, being M a $\overline{Z}^2(M)$-inverse split module leads us that $f^{-1}(\overline{Z}^2(M))$ is a direct summand of M. By a normal verification, we conclude that $f^{-1}(\overline{Z}^2(M)) = Kerg \oplus \overline{Z}^2(M)$. Hence $Kerg$ is a direct summand of M. As $Kerg$ is contained in N, we have $Kerg$ is a direct summand of N, showing N is Rickart.

(2) $\Rightarrow$ (1) Let $M = \overline{Z}^2(M) \oplus N$ where N is Rickart. Let $f \in End(M)$. Consider $h = \pi_N ofoj : N \to N$ which is an endomorphism of N such that $\pi_N : M \to N$ is the projection of M on N and $j : N \to M$ is the inclusion. Being N a Rickart module implies $Kerh$ is a direct summand of N. Set $Kerh \oplus L = N$. It is not hard to check that $f^{-1}(\overline{Z}^2(M)) = Kerh \oplus \overline{Z}^2(M)$. By the decomposition $M = (Kerh \oplus L) \oplus \overline{Z}^2(M)$ we come to a conclusion that $f^{-1}(\overline{Z}^2(M))$ is a direct summand.

(2) $\Rightarrow$ (3) Suppose that $M = \overline{Z}^2(M) \oplus N$. Then $\overline{Z}^4(M) = \overline{Z}^3(M) = \overline{Z}^2(M)$ showing that $\overline{Z}^2(M)$ is noncosingular.

(3) $\Rightarrow$ (2) If $M = K \oplus L$ where K is noncosingular and L is Rickart with $\overline{Z}^2(L) = 0$ then it is obvious that $K = \overline{Z}^2(M)$. □

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Every $\overline{Z}(M)$-inverse split module M is $\overline{Z}^2(M)$-inverse split.*

Example 2.5. Every Rickart module M with $\overline{Z}^2(M)$ a direct summand of M, is a $\overline{Z}^2(M)$-inverse split module. Let now M be a lifting Rickart module. Then by [10, Theorem 4.1], there is a decomposition $M = \overline{Z}^2(M) \oplus N$. It follows that M is $\overline{Z}^2(M)$-inverse split. In particular, every nonsingular injective (extending) lifting module M is $\overline{Z}^2(M)$-inverse split by [5, Example 2.3].

Definition 2.6. Let M be a module. We call M a $\overline{Z}^2$-*torsionfree* module provided $\overline{Z}^2(M) = 0$.

It is easy to see that every cosingular module is $\overline{Z}^2$-torsionfree. The class of $\overline{Z}^2$-torsionfree modules is closed under submodules, direct sums and direct products (see [10, Proposition 2.1]). It is also followed by [6, Theorem 4.41] and [10, Proposition 2.1 and Theorem 3.5] that for a perfect ring R, the class of $\overline{Z}^2$-torsionfree R-modules is also closed under factor modules.

Recall that a module M satisfies (D_0) in case $M = M_1 \oplus M_2$ implies M_1 and M_2 are relatively projective. We present a new characterization of right GV-rings in terms of $\overline{Z}^2$-inverse split modules.

We should note that the proofs for equivalences of (1), (2) and (3) of the following can be found distinctly in [8]. We state them here to make useful connections with $\overline{Z}^2$-inverse split modules.

Before presenting next result, it is worth to recall that a ring R is a right GV-ring (generalized V-ring) provided every simple singular right R-module is injective. In [8], some characterizations of right GV-rings are given. Among them, it is proved that a ring R is right GV if and only if every simple cosingular right R-module is projective if and only if every small right R-module is projective ([8, Theorem 3.1 and Corollary 3.3]).

Theorem 2.7. *Consider the following statements for a right perfect ring R:*

(1) *R is a right GV-ring;*

(2) *Every $\overline{Z}^2$-torsionfree right R-module is projective;*

(3) *Every right R-module is a direct sum of a noncosingular right R-module and a semisimple right R-module;*

(4) *Every right R-module M is $\overline{Z}^2(M)$-inverse split.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). *They are equivalent in case every $\overline{Z}^2$-torsionfree module satisfies (D_0).*

Proof. (1) $\Rightarrow$ (2) Assume that R is right GV. Let $0 \neq M$ be a $\overline{Z}^2$-torsionfree R-module, $0 \neq x \in M$ and K a maximal submodule of xR. Now the simple R-module xR/K is either singular or projective (but not both). If xR/K is singular, then it will be noncosingular by [7, Theorem 4.1]. Consider the natural epimorphism $\pi\colon xR \to xR/K$. Since R is a right perfect ring, by [6, Theorem 4.41] xR is amply supplemented. Therefore by [10, Theorem 3.5] we conclude that $0 = \pi(\overline{Z}^2(xR)) = \overline{Z}^2(xR/K) = \overline{Z}(xR/K) = xR/K$, which is a contradiction. Then xR/K is projective and so K is a direct summand of xR. Hence xR and as well as xR, the module M is semisimple. Let $M = \oplus_{i\in I} M_i$ where each M_i is simple. By mentioned argument in above, each M_i is projective. Hence M is projective.

(2) $\Rightarrow$ (3) Let M be a right R-module. Since R is a right perfect ring, M is amply supplemented by [6, Theorem 4.41]. Now consider natural epimorphism $\pi : M \to M/\overline{Z}^2(M)$. Hence by [10, Theorem 3.5], we have $0 = \pi(\overline{Z}^2(M)) = \overline{Z}^2(M/\overline{Z}^2(M))$. Now, we conclude by assumption that $\overline{Z}^2(M)$ is a direct summand of M. Suppose $M = \overline{Z}^2(M) \oplus N$ for a submodule N of M. It is clear that $\overline{Z}^2(M)$ is noncosingular. We shall show that N is semisimple. To verify this assertion, let $0 \neq x \in N$. As xR is finitely generated, it contains a maximal submodule say K. Consider the simple module xR/K which must be small or injective, but not both. Assume that xR/K is injective. Then xR/K is noncosingular. Now, designate the natural epimorphism $\pi : xR \to xR/K$. Being R a right perfect ring combining with [10, Theorem 3.5] implies that $0 = \pi(\overline{Z}^2(xR)) = \overline{Z}^2(xR/K) = \overline{Z}(xR/K) = xR/K$ which causes a contradiction. By the way, xR/K is small and therefore by assumption xR/K is projective concluding that K is a direct summand of xR. Hence xR is semisimple which implies N is semisimple.

(3) $\Rightarrow$ (1) Let M be a simple singular right R-module. Then M is either small or injective. If M is small, then it is projective which is a contradiction. It follows that M is injective.

(3) $\Rightarrow$ (4) Let M be a right R-module. By (3), there is a decomposition $M = N \oplus S$ where N is noncosingular and S is semisimple. Let us consider S as $(\oplus_{\alpha\in A}(S_\alpha)) \oplus (\oplus_{\beta\in B}(S'_\beta))$ while for each $\alpha \in A$, S_α is noncosingular and S'_β for each $\beta \in B$ is small (note that a simple module is either injective (noncosingular) or small). Now, $M = [N \oplus (\oplus_{\alpha\in A}(S_\alpha))] \oplus [(\oplus_{\beta\in B}(S'_\beta))]$ is a direct sum of $\overline{Z}^2(M)$ and a semisimple (Rickart) module. Hence, M is $\overline{Z}^2(M)$-inverse split by Theorem 2.3.

Now let every $\overline{Z}^2$-torsionfree module satisfy (D_0).

(4) $\Rightarrow$ (3) Let M be a module. Then by assumption, $M = \overline{Z}^2(M) \oplus N$ where N is a Rickart module. We shall prove that N is semisimple. To verify this assertion, take an arbitrary nonzero x in N. Being xR finitely generated implies xR has at least a maximal submodule. Suppose K is a maximal submodule of xR. As $\overline{Z}^2(xR \oplus xR/K) = 0$, it satisfies (D_0) by assumption (note that $\overline{Z}^2(xR/K) = (\overline{Z}^2(xR) + K)/K = 0$ as R is right perfect). It follows that xR/K is xR-projective which

implies that K is a direct summand of xR. Therefore, xR and hence N are semisimple.

□

Theorem 2.7 combining characterizations of GV-rings in [8], gives us the following:

Corollary 2.8. *Let R be a right perfect ring. Consider the following:*

(1) *R is a right GV-ring;*

(2) *Every (simple) cosingular right R-module is projective;*

(3) *Every right R-module is a direct sum of a noncosingular right R-module and a semisimple right R-module;*

(4) *Every right R-module M is $\overline{Z}^2(M)$-inverse split.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). *They are equivalent in case every $\overline{Z}^2$-torsionfree module satisfies (D_0).*

Proof. It follows from Theorem 2.7, [8, Theorems 3.1 and 3.18]. □

We shall prove under some assumptions that the class of right perfect rings R for which every right R-module M is $\overline{Z}(M)$-inverse split is precisely that of right GV-rings.

Corollary 2.9. *Let R be a right perfect ring such that every $\overline{Z}^2$-torsionfree module satisfies (D_0). Then the following statements are equivalent:*

(1) *R is a right GV-ring;*

(2) *Every right R-module M is $\overline{Z}(M)$-inverse split;*

(3) *Every right R-module M is $\overline{Z}^2(M)$-inverse split.*

Proof. (1) $\Rightarrow$ (2) Let M be a right R-module. It follows from Corollary 2.8 that $M = N \oplus S$ where N is noncosingular and S is semisimple. Similar to argument mentioned in (3) $\Rightarrow$ (4) of the proof of Theorem 2.7, we can conclude that $M = \overline{Z}(M) \oplus H$ where H is semisimple. Hence by [13, Theorem 3.3], M is $\overline{Z}(M)$-inverse split.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) It follows from Theorem 2.7. □

Corollary 2.10. *Let R be a right perfect ring such that every noncosingular submodule of a module is a direct summand of that module. If every $\overline{Z}^2$-torsionfree right R-module satisfies (D_0), then every right R-module M is $\overline{Z}^2(M)$-inverse split.*

Proof. Let M be an arbitrary right R-module. By assumption $\overline{Z}^2(M)$ is a direct summand of M. Set $M = \overline{Z}^2(M) \oplus L$ for some submodule L of

M. It is clear that $\overline{Z}^2(L) = 0$, so that by assumption L satisfies (D_0). By a similar argument stated in $(4) \Rightarrow (3)$ of the proof of Theorem 2.7, L is semisimple. Therefore, M is $\overline{Z}^2(M)$-inverse split by Theorem 2.3. □

Corollary 2.11. *Let R be a left and right Artinian serial ring with $J(R)^2 = 0$. If every injective module is noncosingular, then every left and right R-module M is $\overline{Z}^2(M)$-inverse split. In particular, over a hereditary left and right Artinian serial ring R with $J(R)^2 = 0$, every R-module M is $\overline{Z}^2(M)$-inverse split.*

Proof. By [2, 29.10], every R-module M has a decomposition $M = E \oplus S$ where E is an injective R-module and S is a semisimple R-module. Now, by assumption E is noncosingular. The result follows from Theorem 2.7. The last assertion follows from first part and [10, Proposition 2.7]. □

The following introduces a ring over which every module M is $\overline{Z}^2(M)$-inverse split.

Example 2.12. ([8, Example 3.15]) Let F be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ the ring of 2×2 upper triangular matrices over F. By [3, Example 13.6], every singular (left and right) R-module is injective. Hence R is a left and right GV-ring. Since $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, R can not be a (left and right) V-ring. Also R is (left and right) hereditary Artinian serial from [3, Example 13.6]. It is easy to check that $J(R)^2 = 0$. Hence by Corollary 2.11, every right R-module M is $\overline{Z}^2(M)$-inverse split.

There are $\overline{Z}^2(M)$-inverse split modules which are not Rickart. Now, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}^{(I)}$ for an arbitrary non-empty indexed set I. Then M is not Rickart, since $\mathbb{Z}_{p^\infty}$ is not a Rickart $\mathbb{Z}$-module by [5, Example 2.17]. On the other hand, M is noncosingular and so it is $\overline{Z}^2(M)$-inverse split. Generally, every non-Rickart noncosingular module provides an example of a $\overline{Z}^2(M)$-inverse split module that is not Rickart.

Proposition 2.13. *Let M be an indecomposable module. Then the following are equivalent:*

(1) *M is $\overline{Z}^2(M)$-inverse split;*

(2) *M is noncosingular or M is Rickart with $\overline{Z}^2(M) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let M be $\overline{Z}^2(M)$-inverse split. Then by Theorem 2.3, $M = \overline{Z}^2(M) \oplus N$ where N is Rickart. Being M indecomposable implies $M = \overline{Z}^2(M)$ or $M = N$. First case yields M is noncosingular and the second one implies M is Rickart with $\overline{Z}^2(M) = 0$.

(2) $\Rightarrow$ (1) It is straightforward. □

Example 2.14. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^n}$ where p is a prime and $n \in \mathbb{N}$. Then M is an indecomposable cosingular $\mathbb{Z}$-module. Now, earmark the endomorphism $h : M \to M$ by $h(x) = px$. It is clear that $0 \neq Kerh < M$. Therefore, M is not a Rickart $\mathbb{Z}$-module. As M is cosingular indecomposable, M is not $\overline{Z}^2(M)$-inverse split by Proposition 2.13.

Recall from [9] that a module M has C^**-property* provided that every submodule N of M contains a direct summand D of M such that N/D is cosingular. Let R be a ring. Then every right R-module satisfies C^* if and only if every right R-module is a direct sum of an injective module and a cosingular module (see [9, Theorem 2.9]). Recall also from [2] that a ring R is right Harada in case every injective right R-module is lifting. It follows from [2, 28.10] that R is right Harada if and only if every right R-module is decomposed to an injective right R-module and a small right R-module. So, over a right Harada ring every right R-module satisfies C^*.

Proposition 2.15. *Let R be a right perfect ring such that every right R-module has C^*-property. Then every Rickart R-module M is $\overline{Z}^2(M)$-inverse split. In particular, every Rickart module M over a Harada ring (quasi-Frobenius ring) is $\overline{Z}^2(M)$-inverse split.*

Proof. Let M be a Rickart module. As R is right perfect, $\overline{Z}^2(M)$ is a noncosingular submodule of M. Now, from [9, Theorem 2.9], there is a decomposition $\overline{Z}^2(M) = E \oplus C$ such that E is injective and C is cosingular. It follows that $\overline{Z}^2(M)$ is injective and hence a direct summand of M. Being M a Rickart module implies that M is $\overline{Z}^2(M)$-inverse split. □

The following contains an example of a $\overline{Z}^2(M)$-inverse split module which is not $\overline{Z}(M)$-inverse split showing that the concept of $\overline{Z}^2(M)$-inverse split modules is a proper generalization of the $\overline{Z}(M)$-inverse split modules.

Example 2.16. Let K be a field and

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ d & 0 & 0 & c \end{pmatrix} \mid a, b, c, d \in K \right\}.$$

Then R is a subring of $M_{4\times 4}(K)$. Now consider $e = e_{11} + e_{22}$ and $f = e_{33} + e_{44}$ where e_{ij} is an element of R such that (i, j)-component is 1 and elsewhere is 0. Then e and f are two idempotents in R and $R = eR \oplus fR$. The ring R is a (4-dimensional) Frobenius algebra and eR is an indecomposable projective module where $Soc(eR) = e_{23}R$ is the only non-trivial proper submodule of eR (it can be easily checked). Therefore, eR is a local right R-module with $Soc(eR) = Rad(eR) \ll eR$. Now by [11, Corollary 2.8], we have $\overline{Z}(eR) = Soc(eR) \ll eR$, so that $\overline{Z}^2(eR) = 0$. Note that the only proper submodule of eR is $Soc(eR)$. Now, suppose that $\varphi : eR \to eR$ is an arbitrary nonzero endomorphism of eR. Then $Ker\varphi = 0$ or $Ker\varphi = Soc(eR)$. Since $eR/Soc(eR)$ is not isomorphic to a submodule of eR, we conclude that $Ker\varphi = 0$. In fact, $eR/Soc(eR)$ is isomorphic to a submodule of fR. Hence, $Ker\varphi$ is a direct summand of eR implying that eR is a Rickart right R-module. Now by Theorem 2.3, eR is $\overline{Z}^2(eR)$-inverse split while eR is not $\overline{Z}(eR)$-inverse split as $\overline{Z}(eR) \ll eR$.

A submodule N of a $\overline{Z}^2(M)$-inverse split module M need not be $\overline{Z}^2(N)$-inverse split. Now, consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_2$. By Theorem 2.3 and [5, Example 2.5], M is $\overline{Z}^2(M)$-inverse split. Set $N = \mathbb{Z} \oplus \mathbb{Z}_2$ which is a cosingular submodule of M. By [5, Example 2.5], N is a Rickart $\mathbb{Z}$-module. Hence N is not a $\overline{Z}^2(N)$-inverse split module. We next show that a direct summand of a $\overline{Z}^2$-inverse split module inherits the property.

Proposition 2.17. *Let M be a $\overline{Z}^2(M)$-inverse split module and N a direct summand of M. Then N is $\overline{Z}^2(N)$-inverse split.*

Proof. Let $M = N \oplus K$ be a $\overline{Z}^2(M)$-inverse split module with N and K submodules of M. By Theorem 2.3, there is a decomposition $M = \overline{Z}^2(M) \oplus L$ where L is Rickart. Since $\overline{Z}^2(M) = \overline{Z}^2(N) \oplus \overline{Z}^2(K)$, we conclude that $M = \overline{Z}^2(N) \oplus \overline{Z}^2(K) \oplus L$. Modular law implies $N = \overline{Z}^2(N) \oplus [(\overline{Z}^2(K) \oplus L) \cap N]$. Let $Y = (\overline{Z}^2(K) \oplus L) \cap N$ and $f \in End_R(Y)$. It just remains to prove that Y is a Rickart module. It is easy to check that $h = jof o\pi_Y$ is an endomorphism of M where

$j : Y \to M$ is the inclusion map and $\pi_Y : M \to Y$ is the projection of M on Y. Being M a $\overline{Z}^2(M)$-inverse split module conduces that $h^{-1}(\overline{Z}^2(M)) = \overline{Z}^2(N) \oplus \overline{Z}^2(K) \oplus Kerf$ is a direct summand of M. By modular law, $Kerf$ is a direct summand of Y resulting that Y is a Rickart module. □

Theorem 2.18. *The following are equivalent for a module M:*

(1) *M is $\overline{Z}^2(M)$-inverse split and $Kerf$ is a direct summand of $f^{-1}(\overline{Z}^2(M))$ for any $f \in S$;*

(2) *M is Rickart and $\overline{Z}^2(M)$ is a direct summand of M.*

Proof. (1) $\Rightarrow$ (2) Let M be a $\overline{Z}^2(M)$-inverse split module and $f \in S$. Then $f^{-1}(\overline{Z}^2(M))$ is a direct summand of M and by hypothesis, $Kerf$ is a direct summand of $f^{-1}(\overline{Z}^2(M))$. It follows that M is Rickart. In addition, by Theorem 2.3, $\overline{Z}^2(M)$ is a direct summand of M.

(2) $\Rightarrow$ (1) Let M be a Rickart module and $M = \overline{Z}^2(M) \oplus N$ for some submodule N of M. Then N is Rickart and so M is $\overline{Z}^2(M)$-inverse split by Theorem 2.3. Being M Rickart leads us that $Kerf$ is a direct summand of M. The result follows from the fact that $Kerf$ is a submodule of $f^{-1}(\overline{Z}^2(M))$ for any $f \in S$. □

We shall state an analogue of [13, Theorem 3.12] for a $\overline{Z}^2(M)$-inverse split module.

Proposition 2.19. *Let $f : M \to M'$ be an epimorphism of modules where M is $\overline{Z}^2(M)$-inverse split. If $Kerf$ is noncosingular, then M' is $\overline{Z}^2(M')$-inverse split.*

Proof. Let M be $\overline{Z}^2(M)$-inverse split. Then by Theorem 2.3, $M = \overline{Z}^2(M) \oplus N$ where N is a Rickart module. It is easy to check that $\overline{Z}^2(N) = 0$. Taking image of M, we have $M' = f(\overline{Z}^2(M)) + f(N)$. As $Kerf$ is noncosingular, it is contained in $\overline{Z}^2(M)$. So by a similar argument given in the proof of [13, Theorem 3.12], we conclude that $M' = f(\overline{Z}^2(M)) \oplus f(N)$. Note that the condition $Kerf \subseteq \overline{Z}^2(M)$ implies there is an isomorphism between N and $f(N)$. Existing such isomorphism implies $f(N)$ is a Rickart module and $\overline{Z}^2(f(N)) = 0$, as well as $\overline{Z}^2(N) = 0$. Hence $M' = \overline{Z}^2(M') \oplus f(N)$. The result then follows from Theorem 2.3. □

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A KIND OF F-INVERSE SPLIT MODULES

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یک نوع از مدول‌های Fـ وارون شکافته‌شدنی

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فرض کنید M یک Rـ مدول راست روی حلقه R باشد. در این مقاله یک حالت خاص از مدول‌های Fـ وارون شکافته‌شدنی که در آن F یک زیرمدول کاملاً پایا از M است را مورد مطالعه قرار می‌دهیم. گوییم M یک مدول $\overline{Z}^{2}(M)$ـ وارون شکافته‌شدنی است هرگاه برای هر درونریختی f از مدول M، زیرمدول $f^{-1}(\overline{Z}^{2}(M))$ یک جمعوند مستقیم از M باشد. نشان دادیم M یک مدول $\overline{Z}^{2}(M)$ـ وارون شکافته‌شدنی است اگر و تنها اگر M مجموع مستقیم $\overline{Z}^{2}(M)$ و یک زیرمدول $\overline{Z}^{2}$ـ بدون تاب باشد. همچنین تحت شرایطی ثابت کردیم که حلقه‌ای که هر مدول مانند M روی آن $\overline{Z}^{2}(M)$ـ وارون شکافته‌شدنی باشد، دقیقاً یک Vـ حلقه‌ی تعمیم‌یافته است.

کلمات کلیدی: مدول ریکارت، مدول $\overline{Z}(M)$ـ وارون شکافته‌شدنی، مدول $\overline{Z}^{2}(M)$ـ وارون شکافته‌شدنی.