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COMPUTING THE EIGENVALUES OF CAYLEY GRAPHS OF ORDER p^2q

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ABSTRACT. A graph is called symmetric if its full automorphism group acts transitively on the set of arcs. The Cayley graph $\Gamma = Cay(G, S)$ on group G is said to be normal symmetric if $N_A(R(G)) = R(G) \rtimes Aut(G, S)$ and $N_A(R(G))$ acts transitively on the set of arcs of Γ. In this paper, we determine the spectra of all connected minimal normal symmetric Cayley graphs of order p^2q, where p, q are prime numbers.

1. INTRODUCTION

Throughout this paper all groups are assumed to be finite. An important development of graph spectra is the interaction between algebraic graph theory and finite group theory. The concepts and methods of algebraic spectral methods bring useful tools to study the spectrum of Cayley graphs.

The aim of this paper is to investigate the spectrum of Cayley graphs of order p^2q via their character table, where $p, q > 2$ are distinct prime numbers. The most important works on the problem of computing the eigenvalues of Cayley graphs was done by Babai in 1979, see [1]. He used the methods based on the results of algebraic graph theory to obtain a relation between powers of eigenvalues and then by solving a system linear equation, the spectrum of the graph can be determined. In [10] the authors proposed a formula for computing the spectrum of Cayley graph $\Gamma = Cay(G, S)$ in terms of character table of G, where

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S is a symmetric normal subset of G, see also [5, 6, 7]. The main results of this paper are related to this formula. In Section 2, we give the necessary definitions and some preliminary results. In Section 3, we introduce all groups of order p^2q, where p, q are primes and finally in Section 4, the Cayley graphs spectra of order p^2q in terms of their character tables were given. Here, our notation is standard and mainly taken from the standard books of algebraic graph theory and representation theory of finite groups such as [2, 3, 9].

2. Definitions and Preliminaries

Let q be a power of a prime number p. A representation of degree n of group G is a homomorphism $\alpha : G \to GL(n, q)$, where $\alpha(g) = [g]_\beta$ for some basis β. The homomorphism $\alpha : G \to \mathbb{C}^*$ with $\alpha(g) = 1$, for all $g \in G$, is called a trivial representation. The character $\chi_\varphi : G \to \mathbb{C}$ afforded by representation φ is defined as $\chi_\varphi(g) = tr([g]_\beta)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1) = 1$. We denote the set of all irreducible characters of G by $Irr(G)$.

A character table is a matrix whose rows and columns are corresponding to the irreducible characters and the conjugacy classes of G, respectively.

Let G be a group, for every element $g \in G$, we denote the conjugacy class of g by g^G. Assume that N is a normal subgroup of G and $\widetilde{\chi}$ is a character of G/N, then the character χ of G which is given by

$$\chi(g) = \widetilde{\chi}(Ng), \ \forall g \in G,$$

is called the lift of $\widetilde{\chi}$ to G.

Let G and H be two finite groups, then the direct product group $G \times H$ is a group whose elements are the cartesian product of sets G, H and for $(g_1, h_1), (g_2, h_2) \in G \times H$ the related binary operation is defined as $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$.

Theorem 2.1. [9] *Let G and H be two finite groups with $Irr(G) = \{\varphi_1, \ldots, \varphi_r\}$ and $Irr(H) = \{\eta_1, \ldots, \eta_s\}$. Let $M(G)$ and $M(H)$ be the character tables of G and H, respectively. Then the direct product $G \times H$ has exactly rs irreducible characters $\varphi_i\eta_j$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. In particular, the character table of $G \times H$ is*

$$M(G \times H) = M(G) \otimes M(H),$$

where $\otimes$ denotes the Kronecker product.

Let p be a prime number and G a group of order $p^\alpha.m$, where $p \nmid m$. Every subgroup of order p^α is called a Sylow subgroup of G and the set of all Sylow subgroups of G is denoted by $Syl_p(G)$.

3. Main Results

Let $p > q$ be prime numbers where $q|p-1$. A Frobenius group of order pq has the following presentation:

$$F_{p,q} = \langle a, b : \; a^p = b^q = 1, b^{-1}ab = a^u \rangle, \tag{3.1}$$

where u is an element of order q in the multiplicative group $\mathbb{Z}_p^*$.

According to [4, 8] the structures of groups of order p^2q, where $p < q$ are as follows:

- $G_1 = \mathbb{Z}_{p^2q}$,
- $G_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $G_3 = \mathbb{Z}_p \times F_{q,p}$ $(p|q-1)$,
- $G_4 = F_{q,p^2}$ $(p^2|q-1)$,
- $G_5 = \langle a, b : a^{p^2} = b^q = e, a^{-1}ba = b^u, \; u^p \equiv 1 \; (mod \; q)\rangle$ $(p^2|q-1)$.

The structures of groups of order p^2q, where $p > q$ are as follows:

- $H_1 = \mathbb{Z}_{p^2q}$,
- $H_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $H_3 = \mathbb{Z}_p \times F_{p,q}$ $(q|p-1)$,
- $H_4 = \langle a, b : a^q = b^{p^2} = 1, a^{-1}ba = b^\alpha, \; \alpha^q \equiv 1 \; (mod \; p^2)\rangle$ $(q|p-1)$,
- $H_5 = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = c, a^{-1}ca = b^{-1}c^{2\alpha}, bc = cb, (\alpha + \sqrt{\alpha^2-1})^q = 1 \; (mod \; p) \; \rangle$, where $q|p+1$ and $\alpha^2 - 1$ is not a perfect square, namely an integer that is not the square of an integer.
- $H_{5+i} = \langle a, b, c \mid a^q = b^p = c^p = 1, a^{-1}ba = b^\beta, a^{-1}ca = c^{\beta^i}, bc = cb, \; \beta^q \equiv 1 \; (mod \; p)\rangle$, where $\mathcal{B} = \{1, 2, 3, \ldots, \frac{q-1}{2}, q-1\}, i \in \mathcal{B}$ and $q|p-1$.

The aim of this section is to investigate the character table of groups of order p^2q. The character table of cyclic and Frobenius groups can be found in [9]. On the other hand, the character table of product groups can be computed directly by using Theorem 2.1. Hence, it remains to compute the character table of group G, where $G \in \{G_5, H_4, H_5, H_{5+i}\}$. It is not difficult to prove that $G_5' = \langle b \rangle$ and $Z(G_5) = \langle a^p \rangle$, where $Z(G_5)$ denotes the center of G_5.

Lemma 3.1. [4] *The conjugacy classes of $G = G_5$ are*

$$\begin{aligned}
&\{1\}, \{a^p\}, \ldots, \{a^{p(p-1)}\},\\
&(b^{v_i})^G = \{b^{v_i u} \mid u \in U\},\ 1 \leq i \leq (q-1)/p,\\
&(a^n)^G = \{a^n b^i \mid 1 \leq i \leq q-1\},\ 1 \leq n \leq p^2-1, p \nmid n,\\
&(a^{kp} b^{v_i})^G = \{a^{kp} b^{v_i u} \mid u \in U\},\ 1 \leq i \leq (q-1)/p, 1 \leq k \leq p-1,
\end{aligned}$$

where $U = \langle z \rangle$ is a subgroup of order p in $\mathbb{Z}_q^*$ and v_i's are distinct coset representative of U in $\mathbb{Z}_q^*$.

Theorem 3.2. [5] *Let $p, q (p < q)$ be distinct prime numbers, $0 \leq m, n \leq p^2-1, p \nmid n, 1 \leq i, j \leq (q-1)/p, 1 \leq r, k \leq p-1, 1 \leq t \leq q-1$, $\tau = e^{\frac{2\pi i}{p}}$ and $\gamma = e^{\frac{2\pi i}{q}}$. Then all irreducible characters of G_5 are as reported in Table 3.*

g	a^{kp}	b^{v_i}	a^n	$a^{kp}b^{v_i}$
χ_m	ϵ^{kpm}	1	ϵ^{mn}	ϵ^{kpm}
φ_j	p	$\sum_{u\in U}\gamma^{v_i v_j u}$	0	$\sum_{u\in U}\gamma^{v_i v_j u}$
$\eta_{r,t}$	$p\tau^{rk}$	$\sum_{j=0}^{p-1}\gamma^{v_i t z^j}$	0	$\tau^{rk}\sum_{u\in U}\gamma^{v_i u t}$

Table 3. The character table of group G_5.

Lemma 3.3. *The conjugacy classes of $G = H_4$ are*

$$\begin{aligned}
&\{1\},\\
&(a^n)^G = \{(a^n)^{b^m} \mid 1 \leq m \leq p^2\},\ 1 \leq n < q,\\
&(b^{v_{i1}})^G = \{(b)^{v_{ij}} | \ j = 1, 2, 3, \cdots, q\},\ 1 \leq i \leq (p^2-1)/q,
\end{aligned}$$

where v_{ij}'s are distinct coset representatiive of $\{1, \alpha, \alpha^2, \ldots, \alpha^{q-1}\}$ in $\mathbb{Z}_{p^2}^*$.

Proof. Suppose $G = H_4$, since $p > q$ and $|G| = p^2 q$, it is clear that $\langle b \rangle \lhd G$ and so $(b^i)^G \subset \langle b \rangle$, $|(b^i)^G| = \frac{p^2 q}{p^2} = q$. In other words,

$$(b^i)^G = \{(b^i)^{a^j} | j = 1, 2, 3, \ldots, q\} = \{b^i, b^{i\alpha}, b^{i\alpha^2}, \ldots, b^{i\alpha^{q-1}}\}.$$

Suppose v_i's are distinct coset representative of $\{1, \alpha, \alpha^2, \ldots, \alpha^{q-1}\}$ in $\mathbb{Z}_{p^2}^*$. It is clear that $(b^i)^G$'s can be displayed by $(b^{v_i})^G$'s. On the other hand, according to the presentation of H_4 and by counting members of H_4, it turns out that the Seylow p-subgroup is of order p^2 and thus $|(a^n)^G| = p^2$. Hence, $(a^n)^G = \{(a^n)^{b^m} \mid 1 \leq m \leq p^2\}$ for $1 \leq n < q$ and the proof is complete. $\square$

Theorem 3.4. *Let p, q $(p > q,\ q \mid p-1)$ be distinct prime numbers, $1 \leq m, n \leq q-1, 1 \leq i, k \leq (p^2-1)/q, 1 \leq j \leq q$, where $\gamma = e^{\frac{2\pi i}{q}}$ and $\tau = e^{\frac{2\pi i}{p^2}}$. Then all irreducible characters of H_4 are as reported in Table 4.*

g	1	a^n	$b^{v_{i1}}$
χ_0	1	1	1
χ_m	1	γ^{mn}	1
$\varphi_{v_{k1}}$	q	0	$\displaystyle\sum_{j=1}^{q} \tau^{v_{k1}v_{ij}}$

Table 4. The character table of group H_4.

Proof. It follows from Lemma 3.1 that $G = H_4$ has $q + (p^2-1)/q$ irreducible characters. Among them q characters are linear, since $|G/G^{'}| = q$ $(G^{'} = \mathbb{Z}_{p^2})$. On the other hand, $G/G^{'}$ is a cyclic group of order q and thus all its irreducible characters are of the form $\tilde{\chi}_m : G/G^{'} \to \mathbb{C}^*$ with $\tilde{\chi}_m((a)^n G^{'}) = \gamma^{mn}$, where $\gamma = e^{\frac{2\pi i}{q}}$ and $1 \leq n, m \leq q$. By lifting these characters, we get q linear characters $\chi_m (1 \leq m \leq q)$ such that

$$\begin{aligned} \chi_m(a^n) &= \tilde{\chi}_m(a^n G^{'}) = \gamma^{nm},\ 1 \leq n \leq q \textit{ and} \\ \chi_m(b^{v_{i1}}) &= \tilde{\chi}_m(b^{v_{i1}} G^{'}) = \tilde{\chi}_m(G^{'}) = 1,\ 1 \leq i \leq \frac{p^2-1}{q}. \end{aligned}$$

Let $H = \mathbb{Z}_{p^2} = \langle b \rangle$, it is not difficult to see that H has p^2 linear characters such that

$$\tilde{\varphi}_k(b^i) = \tau^{ki}, 1 \leq k \leq p^2.$$

For given $k \in \{1, 2, 3, \ldots, p^2 - 1/q\}$, the degree of induced character $\varphi_{v_{k1}} := \tilde{\varphi}_{v_{k1}} \uparrow G$ is

$$\varphi_{v_k 1}(1) = \tilde{\varphi}_{v_{k1}} \uparrow G(1) = \frac{|G|}{|\langle b \rangle|}(\tilde{\varphi}_{v_{k1}})(1) = \frac{p^2 q}{p^2} = q.$$

On the other hand, we have

$$\begin{aligned} |C_G(b^i)| = |C_H(b^i)| \ &= \ p^2,\ 1 \leq i, j \leq p, \\ (a^n)^G \cap H = \phi,&\ 1 \leq n \leq q-1, \\ ((b)^{v_{i1}})^G \cup H = ((b)^{v_{i1}})^G,&\ 1 \leq i \leq (p^2-1)/q. \end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\varphi}_{v_{k1}} \uparrow G(1) &= q, \\
\tilde{\varphi}_{v_{k1}} \uparrow G((a^n)^G) &= 0,\ 1 \le n \le q-1 \\
\tilde{\varphi}_{v_{k1}} \uparrow G((b^{v_{i1}})^G) &= \sum_{j=1}^{q} \tilde{\varphi}_{v_{k1}}(b^{v_{ij}}) \\
&= \sum_{j=1}^{q} \tau^{v_{k1}v_{ij}},\ 1 \le i,k \le (p^2-1)/q.
\end{aligned}$$

So, we get $(p^2-1)/q$ characters of G with degree q. It remains to show these are all distinct irreducible characters of G. Thus

$$\sum_{\chi_i \in Irr(G)} \chi_i^2(1) = |G| = p^2q \ and \sum_{\chi_i \in Irr(G), \chi_i(1) \neq 1} \chi_i^2(1) = |G| = (p^2-1)q.$$

Now by regarding to $(p^2-1)/q$ non-linear irreduicible characters of G, it is not difficult to see that $\chi_i(1) = q$, for all $\chi_i \in Irr(G)$ and $\chi_i(1) \neq 1$. Hence, $\varphi_{v_{k1}}$ is an irreduicible character or a linear combination of linear irreduicible characters. The second one is impossible, because otherwise, we must have $G' = \mathbb{Z}_{p^2} \subset Ker_{\varphi_{v_{k1}}}$, a contradiction. It follows that $\varphi_{v_{k1}}$ is irreducible. It is not difficult to verify that all $\varphi_{v_{kj}}$'s are distinct and the proof is complete. $\square$

Remark. Suppose that the group H is matrix representation of $GF^*(p^2)$ in $GL(2,p)$ and

$$K = \langle \begin{pmatrix} 0 & -1 \\ 1 & 2\alpha \end{pmatrix} \rangle, \tag{3.2}$$

is a subgroup of order q of H. In the following, suppose $\{K_i\}_{i=1,2,3,\ldots,\frac{p^2-1}{q}}$ are distinct cosets of K in H and $v_{ij} = \begin{pmatrix} \beta_{ij} \\ \lambda_{ij} \end{pmatrix}$, $(j = 1,2,3,\ldots,q)$ is the first column of matrices in K_i.

Lemma 3.5. *The conjugacy classes of $G = H_5$ are*

$\{1\}$,
$(a^n)^G = \{(a^n)^{b^m c^k} \mid 1 \le m \le p, 1 \le k \le p\},\ 1 \le n < q,$
$((bc)^{v_{i1}})^G = \{(bc)^{v_{ij}} = (bc)^{v_{ij}} = b^{\beta_{ij}} c^{\lambda_{ij}} \mid j = 1,2,3,\cdots,q\}$
$1 \le i \le (p^2-1)/q.$

Proof. Since $q \mid p+1$ and $q \nmid p-1$, we have $|Syl_q(G)| = p^2$. On the other hand, all Sylow q-subgroups of G are conjugate, hence $o((a^n)^G) = p^2$ and $(a^n)^G = \{(a^n)^{b^m c^k} \mid 1 \le m \le p, 1 \le k \le p\}$, where $1 \le n <$

q. We know that the unique Sylow p-subgroup G is normal, hence $\langle b,c\rangle = \mathbb{Z}_p\times\mathbb{Z}_p$ is the union of conjugacy classes of G and $o((b^ic^j)^G) = |G|/p^2 = p^2q/p^2 = q$ $((i,j)\neq(0,0))$. Hence, $\langle b,c\rangle - \{1\}$ contains $(p^2-1)/q$ conjugacy classes of order q. Finally, $\begin{pmatrix} 0 & -1 \\ 1 & 2\alpha \end{pmatrix}$ is the matrix presentation of $\varphi \in Aut(\mathbb{Z}_p\times\mathbb{Z}_p) \cong GL(2,p)$ in semi-direct product $(\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes_\varphi \mathbb{Z}_q$. Hence

$$\{(b^ic^j)^G = \{(b^ic^j)^{a^k} = a^{-k}(b^ic^j)a^k = \varphi^k(b^ic^j) \mid k = 1,2,3,\ldots,q\}.$$

This completes the proof. □

Theorem 3.6. *Let p,q be distinct prime numbers, where $p>q$ and $q \mid (p+1)$. Suppose $1\le m,n\le q-1, 1\le i,\le (p^2-1)/q, 1\le k,l\le p, 1\le j\le q, \gamma = e^{\frac{2\pi i}{q}}$ and $\tau = e^{\frac{2\pi i}{p}}$. Then all irreducible characters of H_5 are as reported in Table 5.*

g	1	a^n	$b^{\alpha_{i1}}c^{\beta_{i1}}$
χ_0	1	1	1
χ_m	1	γ^{mn}	1
$\varphi_{k,l}$	q	0	$\sum_{j=1}^{q}\tau^{k\alpha_{ij}+l\beta_{ij}}$

Table 5. The character table of group H_5.

Proof. It follows from Lemma 3.1 that $G = H_5$ has $q+(p^2-1)/q$ irreducible characters and among them q characters are linear, since $|G/G'| = q$ $(G' = \mathbb{Z}_p\times\mathbb{Z}_p)$. On the other hand, G/G' is a cyclic group of order q and then all its irreducible characters are of the form $\tilde{\chi}_m : G/G' \to \mathbb{C}$ with $\tilde{\chi}_m((a)^nG') = \gamma^{mn}$, where $\gamma = e^{\frac{2\pi i}{q}}$ and $1\le n,m\le q$. By lifting these characters, we get q linear characters $\chi_m(1\le m\le q)$ such that

$$\begin{aligned}\chi_m(a^n) &= \tilde{\chi}_m(a^nG') = \gamma^{nm},\ 1\le n\le q,\\ \chi_m(b^ic^j) &= \tilde{\chi}_m(b^ic^jG') = \tilde{\chi}_m(G') = 1,\ 1\le i,j\le p.\end{aligned}$$

Let $H = \mathbb{Z}_p\times\mathbb{Z}_p = \langle b,c\rangle$, it is not difficult to see that H has p^2 linear characters such that

$$\tilde{\varphi}_{kl}(b^ic^j) = \tau^{ki+lj},\ 1\le k,l\le p.$$

For given elements $(k,l) \in \mathbb{Z}_p \times \mathbb{Z}_p - \{(p,p)\}$, the degree of induced character $\varphi_{kl} := \tilde{\varphi}_{kl} \uparrow G$ is

$$\varphi_{kl}(1) = \tilde{\varphi}_{kl} \uparrow G(1) = \frac{|G|}{|\langle b, c\rangle|}(\tilde{\varphi}_{kl})(1) = \frac{p^2 q}{p^2} = q.$$

On the other hand, we have

$$\begin{aligned}
&|C_G(b^i c^j)| = |C_H(b^i c^j)| \ = \ p^2, \ 1 \le i, j \le p, \\
&(a^n)^G \cap H = \phi, \ 1 \le n \le q-1, \\
&((bc)^{v_{i1}})^G \cup H = ((bc)^{v_{i1}})^G, \ 1 \le i \le (p^2-1)/q.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\varphi}_{kl} \uparrow G(1) &= q, \\
\tilde{\varphi}_{kl} \uparrow G((a^n)^G) &= 0, \ (1 \le n \le q-1) \\
\tilde{\varphi}_{kl} \uparrow G(((bc)^{v_{i1}})^G) &= \sum_{j=1}^{q} (\varphi_{kl}((bc)^{v_{ij}}) = \sum_{j=1}^{q} (\varphi_{kl})((b^{\alpha_{ij}} c^{\beta_{ij}}) \\
&= \sum_{j=1}^{q} \tau^{k\alpha_{ij} + l\beta_{ij}}, \ 1 \le i \le (p^2-1)/q,
\end{aligned}$$

where $1 \le k, j \le p$ and $(k,l) \ne (p,p)$. This means that we have $p^2 - 1$ characters of q. It remains to show these are all irreducible characters of G. We have

$$\sum_{\chi_i \in Irr(G)} \chi_i^2(1) = |G| = p^2 q, \qquad \sum_{\chi_i \in Irr(G), \chi_i(1) \ne 1} \chi_i^2(1) = |G| = (p^2-1)q.$$

We can prove that $\chi_i(1) = q$ for all $\chi_i \in Irr(G), \chi_i(1) \ne 1$. Hence, φ_{kl} is an irreduicible character. This completes the proof. □

Lemma 3.7. *The conjugacy classes of $G = H_{5+i}$ are*

$$\begin{aligned}
&\{1\}, \\
&(a^n)^G = \{(a^n)^{b^m c^k} \mid 1 \le m \le p, 1 \le k \le p\}, \ 1 \le n < q, \\
&(b^{v_t} c^{u_t})^G = \{b^{v_t \beta^j} c^{u_t \beta^{ij}} | \ j = 1, 2, 3, \ldots, q\}, \ 1 \le t \le (p^2-1)/q,
\end{aligned}$$

where (v_t, u_t)'s are distinct coset representatives of $H = \{(\beta^k, \beta^{ik}) | \ k = 1, 2, 3, \ldots, q\}$ with multiplicity $H(n,m) = \{(n\beta^k, m\beta^{ik}) | \ k = 1, 2, \ldots, q\}$, where $\{(n,m) | \ 0 \le n, m \le p-1, (n,m) \ne (0,0)\}$ and $t = 1, 2, 3, \ldots, (p^2-1)/q$.

Proof. The proof follows by using Lemma 3.5. □

Theorem 3.8. *Let p, q $(p > q)$ be distinct prime numbers, where $q|p-1$, $1 \leq m, n \leq q-1, 1 \leq t, s \leq (p^2-1)/q, \gamma = e^{\frac{2\pi i}{q}}$ and $\tau = e^{\frac{2\pi i}{p}}$. Then all irreducible characters of H_{5+i} are as reported in Table 6.*

g	1	a^n	$b^{v_t}c^{u_t}$
χ_0	1	1	1
χ_m	1	γ^{mn}	1
$\varphi_{v_s u_s}$	q	0	$\sum_{j=1}^{q} \tau^{v_s v_t \beta^j + u_s u_t \beta^{ij}}$

Table 6. The character table of H_{5+i}.

Proof. The proof is similar to the proof of Lemma 3.3. □

A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. Let G be a finite group with symmetric subset S. We recall that S is a normal subset if and only if $g^{-1}Sg = S$, for all $g \in G$. The Cayley graph $\Gamma = Cay(G, S)$ with respect to S is a graph whose vertex set is $V(\Gamma) = G$ and the vertex x is adjacent with y if and only if $yx^{-1} \in S$. It is a well-known fact that $Cay(G, S)$ is connected if and only if S generates the group G, see [2].

Let Γ be a simple graph with the adjacency matrix $A(\Gamma)$. The characteristic polynomial $\chi_\lambda(\Gamma)$ of $A(\Gamma)$ is defined as [3]:

$$\chi_\lambda(\Gamma) = |\lambda I - A|$$

and the roots of this polynomial are called the spectrum of Γ.

The study of the spectrum of Cayley graphs is closely related to the irreducible characters of the group under consideration. If G is abelian, then the spectrum of $\Gamma = Cay(G, S)$ can be determined as follows.

Theorem 3.9. [2] *Let S be a symmetric subset of abelian group G. Then the eigenvalues of the adjacency matrix of $Cay(G, S)$ are given by*

$$\lambda_\varphi = \sum_{s \in S} \varphi(s),$$

where $\varphi \in Irr(G)$.

The following theorem is implicitly contained in [10, 11].

Theorem 3.10. *Let G be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of the graph $\Gamma = Cay(G, S)$. Then the eigenvalues of A are given by*

$$[\lambda_\chi]^{\chi(1)^2}, \ \chi \in Irr(G),$$

where $\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$.

By a circulant matrix, we mean a square $n \times n$ matrix whose rows are a cyclic permutation of the first row. A circulant matrix with the first row $[c_0, c_1, \ldots, c_{n-1}]$ is denoted by $[[c_0, c_1, c_2, \ldots, c_{n-1}]]$. By a circulant graph, we mean a graph whose adjacency matrix is circulant. We recall [3] that for $\omega = e^{\frac{2\pi}{n}i}$, the $n-$th root of unity, all eigenvalues of circulant matrix $[[c_0, c_1, c_2, \ldots, c_{n-1}]]$ are given by

$$\lambda_j = c_0 + c_{n-1}\omega^j + c_{n-2}\omega^{2j} + \cdots + c_1\omega^{(n-1)j},\ 0 \leq j \leq n-1. \quad (3.3)$$

Let A, B be two arbitrary sets. In what follows we assume that

$$\delta_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \not\subseteq B \end{cases}.$$

Let $C_g = g^G \cup (g^{-1})^G$. It is clear that every normal subset of G is the union of some conjugacy classes. In other words, if S is a symmetric normal subset of G, then $S \subseteq \bigcup_{g\in G} C_g$ and all eigenvalues of Cayley graph $Cay(G,S)$ are as follows:

$$\lambda_\chi = \frac{1}{2\chi(1)} \sum_{g\in G} \sum_{s\in C_g} \delta_{C_g}(S)|C_g|[\chi(s) + \chi(s^{-1})],$$

where $\chi \in Irr(G)$.

Example 3.11. Consider the cyclic group $\mathbb{Z}_n$ as following cases:
Case 1. n is odd, thus $C_i = \{x^i, x^{-i}\}$ $(1 \leq i \leq \frac{n-1}{2})$ are normal symmetric subsets of $\mathbb{Z}_n$ and so

$$S \subseteq \bigcup_{i=1}^{\frac{n-1}{2}} C_i.$$

For $0 \leq j \leq n-1$, $\chi_j(x^i) = \omega^{ij}$ are all irreducible characters of $\mathbb{Z}_n = \langle x \rangle$ and $\omega = e^{\frac{2\pi}{n}i}$. Hence

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}).$$

Case 2. n is even, hence all normal symmetric subsets are

$$C_i = \{x^i, x^{-i}\}\ (1 \leq i \leq \frac{n}{2} - 1)\ and\ C_{\frac{n}{2}} = \{x^{n/2}\}.$$

Therefore,

$$S \subseteq \bigcup_{i=1}^{\frac{n}{2}} C_i.$$

Similar to the last case, we have

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}) + (-1)^{\frac{n}{2}j}\delta_{C_{\frac{n}{2}}}(S).$$

The *cartesian product* $\Gamma_1 \boxtimes \Gamma_2$ of two graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices $(u, v), (x, y) \in V(\Gamma_1 \boxtimes \Gamma_2)$ are adjacent if and only if either $u = x$ and $(v, y) \in E(\Gamma_2)$ or $(u, x) \in E(\Gamma_1)$ and $v = y$.

Theorem 3.12. [3] *Let $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_m$ be eigenvalues of graphs Γ_1 and Γ_2, respectively. Then, for $1 \leq i \leq n$ and $1 \leq j \leq m$, all eigenvalues of $\Gamma_1 \boxtimes \Gamma_2$ are $\lambda_i + \mu_j$.*

Consider the presentation of Frobenius group introduced in the begining of this section. Let S be a minimal normal symmetric subset of $F_{q,p}$ and $\Gamma = Cay(F_{q,p}, S)$.

Theorem 3.13. [5] *By above notation, we have*

$$Spec(\Gamma, S) = \begin{pmatrix} 2q & q(\omega^j + \omega^{-j}) & 0 \\ 1 & 1 & rp^2 \end{pmatrix} (1 \leq j \leq p-1),$$

where $S = \langle b^G \cup (b^{-1})^G \rangle$ and ω is a pq-th root of unity.

4. Spectrum of Cayley graphs of order p^2q

In this section, we compute the spectra of Cayley graphs on groups of order p^2q. To do this, we compute the normal symmetric subsets of G and then by applying Theorem 3.10, we compute the spectrum of $Cay(G, S)$ in terms of minimal normal symmetric generating subset S.

Theorem 4.1. *If $\Gamma_1 = Cay(G, S_1)$ and $\Gamma_2 = Cay(H, S_2)$, then $\Gamma = Cay(G \times H, S)$ is Cayley graph if and only if $S = \{(s_1, 1_G), (1_H, s_2); s_1 \in S_1, s_2 \in S_2)\}$.*

Proof. See [7]. $\square$

By using Example 3.11 and Theorem 3.12, one can compute the spectrum of Cayley graph $\Gamma_i = Cay(G_i, S_i)$, where $1 \leq i \leq 4$. Hence, to compute the spectrum of Cayley graphs of order p^2q, it is sufficient to determine the spectrum of $\Gamma = Cay(G, S)$, where $G \in \{G_5, H_4, H_5, H_{5+i}\}$. First, suppose $G = G_5$, $l_1 = \frac{p(p-1)}{2}$, $l_2 = \frac{q-1}{2p}$, $l_3 = \frac{p^2-1}{2}$ and $l = l_1 + l_2 + l_3$. Then the non-trivial symmetric subsets of G_5 are

$$\begin{aligned} C_i &= (a^{ip})^G \cup (a^{-ip})^G \ (1 \leq i \leq l_1), \\ C_{l_1+j} &= (b^{v_j})^G \cup (b^{-v_j})^G \ (1 \leq j \leq l_2), \\ C_{l_1+l_2+n} &= (a^n)^G \cup (a^{-n})^G \ (1 \leq n \leq l_3), \\ C_{l+k} &= (a^{kp}b^{v_i})^G \cup ((a^{kp}b^{v_i})^{-1})^G (1 \leq k \leq l_1). \end{aligned}$$

Hence, $S \subseteq \bigcup_{i=1}^{l+l_1} C_i$ and so we have

$$\begin{aligned} \lambda_{1_G} &= |S|, \\ \lambda_{\chi_m} &= \sum_{j=1}^{l_1} \delta_{C_j}(S)(\epsilon^{jmp} + \epsilon^{-jmp}) + 2p \sum_{j=1}^{l_2} \delta_{C_{l_1+j}}(S) \\ &+ (q-1) \sum_{n=1}^{l_3} \delta_{C_{l_1+l_2+n}}(S)(\epsilon^{mn} + \epsilon^{-mn}) \\ &+ p \sum_{k=1}^{l_1} \delta_{C_{l+k}}(S)(\epsilon^{kmp} + \epsilon^{-kmp}), \\ \lambda_{\varphi_m} &= 2\sum_{j=1}^{l_1} \delta_{C_j}(S) + \sum_{j=1}^{l_2} \delta_{C_{l_1+j}}(S)(A + \bar{A}) + \sum_{k=1}^{l_1} \delta_{C_{l+k}}(S)(B + \bar{B}), \\ \lambda_{\eta_{r,t}} &= p(\sum_{j=1}^{l_1} \delta_{C_i}(S)(\tau^{rj} + \tau^{-rj}) \\ &+ \sum_{j=1}^{l_2} \delta_{C_{l_1+j}}(S)(C + \bar{C}) + \sum_{k=1}^{l_1} \delta_{C_{l+k}}(S)(D + \bar{D})), \end{aligned}$$

where $\epsilon = e^{\frac{2\pi i}{p^2}}, \tau = e^{\frac{2\pi i}{p}}, \gamma = e^{\frac{2\pi i}{q}}, A = \sum_{u \in U} \gamma^{v_i v_j u}, B = \sum_{u \in U} \gamma^{v_i v_j u}, C = \sum_{j=0}^{p-1} \gamma^{v_i t z^j}$ and $D = \sum_{u \in U} \gamma^{v_i u t}$.

Theorem 4.2. [5] *The minimal normal symmetric generating subset of groups G_5 is $S = (a^{kp}b^{v_i})^G \cup ((a^{kp}b^{v_i})^{-1})^G$.*

Corollary 4.3. [5] *Let $\Gamma_i = Cay(G_i, S_i)$, where $G_1, \ldots, G_5$ are groups introduced in Section 1 and S_i be a minimal normal symmetric generating subset of G_i. Then*

(1) *All eigenvalues of* Γ_1 *are*

$$[\omega^j + \omega^{-j}]^1,$$

where $\omega = e^{\frac{2\pi}{p^2q}i}$ *and* $0 \le j \le p^2q - 1$.

(2) *All eigenvalues of* Γ_2 *are*

$$[\alpha^j + \alpha^{-j} + \zeta^i + \zeta^{-i}]^1,$$

where $\alpha = e^{\frac{2\pi i}{pq}}$, $\zeta = e^{\frac{2\pi i}{p}}$, $0 \le j \le p-1$, $0 \le i \le pq-1$.

(3) *All eigenvalues of* Γ_3 *are*

$$[\xi^i + 2q]^1, [\xi^i + q(\alpha^j + \alpha^{-j})]^1, [\xi^i]^t,$$

where $t = p(q-1), \alpha = e^{\frac{2\pi i}{q}}$, $\xi = e^{\frac{2\pi i}{p}}$, $1 \le j \le p-1$, $0 \le i \le p-1$.

(4) *All eigenvalues of* Γ_4 *are*

$$[2q]^1, [q(\alpha^j + \alpha^{-j})]^1, [0]^t,$$

where $t = p^2(q-1), \alpha = e^{\frac{2\pi i}{p^2}}$, $1 \le j \le p^2 - 1$.

(5) *All eigenvalues of* Γ_5 *are*

$$[p]^1, [A + \bar{A}]^{p^2-1}, [B + \bar{B}]^{p(q-1)}, [D + \bar{D}]^{p(p-1)(q-1)},$$

where A, B *and* D *are those given before Theorem 4.2.*

Theorem 4.4. *The minimal normal symmetric generating subset of* H_4 *is* $S = (a^i)^G \cup ((a^{-i}))^G$, $1 \le i \le q-1$.

Proof. The proof is similar to that of Theorem 4.2. $\square$

Corollary 4.5. *The spectrum of Cayley graph* $\Gamma = Cay(H_4, S)$, *where* S *is a minimal normal symmetric subset of* H_4 *is*

$$Spec(\Gamma, S) = \begin{pmatrix} 2p^2 & p^2(\gamma^{ij} + \gamma^{-ij}) & 0 \\ 1 & q-1 & (p^2-1)/q \end{pmatrix} (1 \le i, j \le q-1),$$

where $S = \langle (a^i)^G \cup (a^{-i})^G \rangle$, $\gamma = e^{\frac{2\pi i}{q}}$.

Theorem 4.6. *The minimal normal symmetric generating subset of group* $G = H_5$ *is* $S = (a^i)^G \cup ((a^{-i}))^G$, *where* $1 \le i \le q-1$.

Proof. It is easy to see that $(a^i)^1 = a^i \in \langle S \rangle$ and $a \in \langle S \rangle$. On the other hand, the number of q-Sylow subgroups is p^2. Hence $b, c \in \langle S \rangle$ and so $\langle S \rangle = G$. On the other hand $(b^i c^k)^G \subset \langle b, c \rangle$ implies that S is a minimal normal symmetric generating subset of H_5. $\square$

Corollary 4.7. *The spectrum of Cayley graph $\Gamma = Cay(H_5, S)$, where S is a minimal normal symmetric generating subset of H_5 is*

$$Spec(\Gamma, S) = \begin{pmatrix} 2p^2 & p^2(\gamma^{ij} + \gamma^{-ij}) & 0 \\ 1 & q-1 & (p^2-1)/q \end{pmatrix} (1 \leq i, j \leq q-1),$$

where $S = \langle (a^i)^G \cup (a^{-i})^G \rangle$ and $\gamma = e^{\frac{2\pi i}{q}}$.

Theorem 4.8. *The minimal normal symmetric generating subset of group $G = H_{5+i}$ is $S = (a^i)^G \cup ((a^{-i}))^G,\ 1 \leq i \leq q-1$.*

Proof. The proof is similar to the proof of Theorem 4.2. □

Corollary 4.9. *The spectrum of Cayley graph $\Gamma = Cay(H_{5+i}, S)$, where S is a minimal normal symmetric generating subset of H_{5+i} is*

$$Spec(\Gamma, S) = \begin{pmatrix} 2p^2 & p^2(\gamma^{ij} + \gamma^{-ij}) & 0 \\ 1 & q-1 & (p^2-1)/q \end{pmatrix} (1 \leq i, j \leq q-1),$$

where $S = \langle (a^i)^G \cup (a^{-i})^G \rangle,\ \gamma = e^{\frac{2\pi i}{q}}$.

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COMPUTING THE EIGENVALUES OF CAYLEY GRAPHS OF ORDER p^2q, WHERE p AND q ARE PRIMES

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محاسبه مقادیر ویژه گراف‌های کیلی از مرتبه حاصل‌ضرب سه عدد اول متمایز

مجتبی قربانی، عزیز سیدهادی و فاطمه نوروزی لرکی
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گراف G را متقارن می‌نامیم هرگاه گروه خودریختی آن روی مجموعه کمان‌های گراف انتقالی عمل کند. همچنین گراف کیلی $\Gamma = Cay(G, S)$ روی گروه G را یک گراف نرمال متقارن می‌نامیم هرگاه گروه $N_A(R(G)) = R(G)Aut(G, S)$ روی مجموعه کمان‌های Γ به‌طور انتقالی عمل کند. در این مقاله، همه گراف‌های نرمال متقارن همبند از درجه ۴ و از مرتبه p^q رده‌بندی می‌شوند که در آن p و q اعداد اول متمایز هستند.

کلمات کلیدی: گراف متقارن، گراف کیلی، گراف نرمال، گراف کمان-انتقالی.