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A GENERALIZATION OF PRIME HYPERIDEALS IN KRASNER HYPERRINGS

L. KAMALI ARDEKANI* AND B. DAVVAZ

Abstract. In this paper, we give a characterization of new generalization of prime hyperideals in Krasner hyperrings by introducing 2-absorbing hyperideals. We study fundamental properties of 2-absorbing hyperideals on Krasner hyperrings and investigate some related results.

1. Introduction

Prime ideals play a significant role in commutative ring theory. Because of this importance, the concept of 2-absorbing ideals in a commutative ring was introduced by Badawi [2] as a generalization of prime ideals. After this, [8, 9, 10] have continued these studies and obtained several results. Recently, this notion is generalized to the hypercase by introducing the 2-absorbing hyperideals in a multiplicative hyperring [1]. In this paper, we introduce the notion of the 2-absorbing hyperideals on Krasner hyperrings and give some properties of such hyperideals.

Let us first recall some preliminary definitions.

Assume that H is a non-empty set and $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote $A \circ B = \bigcup\limits_{a \in A,\ b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called

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∗Corresponding author.

a *semihypergroup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that $\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense. There are comprehensive references for hyperrings, for example see [3, 7]. In fact, different kinds of hyperrings are defined which one of them is *Krasner hyperring* described as follows [6]:

A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ satisfying the following axioms: (1) $(R, +)$ is a canonical hypergroup which means that (i) $(R, +)$ is a semihypergroup, i.e., $x + (y + z) = (x + y) + z$, for all $x, y, z \in R$, (ii) $x + y = y + x$, for all $x, y \in R$, (iii) There exists $0 \in R$ such that $0 + x = \{x\}$, for all $x \in R$, (iv) For all $x \in R$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, (we write $-x$ for x' and we call it the opposite of x), (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in R$; (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$; (3) The multiplication is distributive with respect to the hyperoperation $+$. Throughout this paper, by a hyperring we mean a Krasner hyperring.

The meaning of *center* of hyperring $(R, +, \cdot)$ is $Z(R) = \{x \in R \mid x \cdot y = y \cdot x, \text{ for all } y \in R\}$ and R is called *commutative* if $Z(R) = R$ i.e., $(R, \cdot)$ is a commutative semigroup. A hyperring $(R, +, \cdot)$ is called *hyperfield* if $(R, \cdot)$ is a commutative monoid and all nonzero elements of R are multiplicatively invertible. The identity element of the monoid $(R, \cdot)$ is called *unit element* of hyperring $(R, +, \cdot)$. For example, suppose that $\mathbb{K} := \{0, 1\}$ is a commutative monoid with the multiplication $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. The hyperaddition is given by $0 + 1 = 1 + 0 = 1$, $0 + 0 = 0$ and $1 + 1 = \{0, 1\}$. Then, $\mathbb{K}$ is a hyperfield called the *Krasner hyperfield* with unit element 1 [5]. A hyperring $(R, +, \cdot)$ is called *hyperdomain*, if R is a commutative hyperring with unit element and $xy = 0$ implies that $x = 0$ or $y = 0$, for all $x, y \in R$.

A non-empty subset A of a hyperring $(R, +, \cdot)$ is called *subhyperring* of R if $(A, +, \cdot)$ is itself a hyperring. A non-empty subset I of a hyperring R is called a *hyperideal* if and only if (1) $u, v \in I$ imply that $u - v \subseteq I$, for all $u, v \in I$, (2) $u \in I$ and $r \in R$ imply that $r \cdot u \in I$ and $u \cdot r \in I$. Remember here that $(I :_R x) = \{y \in R \mid y \cdot x \in I\}$, for all $x \in R$, is a hyperideal. A hyperideal I is called *prime* if $xy \in I$ implies that $x \in I$ or $y \in I$. A prime hyperideal P is said to be a *minimal prime hyperideal over an ideal* I if it is minimal among all prime ideals containing I. Note that we do not exclude I even if it is a prime ideal. A prime hyperideal is said to be a *minimal prime hyperideal* if it is a minimal prime ideal over the zero hyperideal. By applying the argument similar in spirit to the proof of Theorem 2.1 of [4], one can easily show that if I and P are hyperideals of R such that $I \subseteq P$ and P is a minimal prime hyperideal of I, then, for all $x \in P$, there is $y \in R \setminus P$ and a nonnegative integer n such that $yx^n \in I$.

A *good homomorphism* between two hyperrings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ is a map $f : R_1 \longrightarrow R_2$ such that for all $x, y \in R_1$, we have $f(x +_1 y) = f(x) +_2 f(y)$, $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ and $f(0) = 0$. Let $f : R_1 \longrightarrow R_2$ be a good homomorphism. The kernel of f is defined as $kerf = \{x \in R_1 \mid f(x) = 0\}$. It is trivial that $kerf$ is a hyperideal of R_1. Note that a prime hyperideal of a commutative hyperring R can be described as the kernel of a homomorphism from R to the Krasner hyperfield $\mathbb{K}$ [5].

2. 2-absorbing hyperideals in Krasner hyperrings

In this section, we treat to the introducing 2-absorbing hyperideals on Krasner hyperrings and investigate more results with respect to such hyperideals.

Definition 2.1. A proper hyperideal I of a hyperring $(R, +, \cdot)$ is called a 2-*absorbing hyperideal* if $a \cdot b \cdot c \in I$ implies that $a \cdot b \in I$ or $a \cdot c \in I$ or $b \cdot c \in I$, for all $a, b, c \in R$.

Example 2.2. Let $(G, \odot)$ be a group and $H = G \cup \{0, u, v\}$, where 0 is an absorbing element under multiplication and u, v are distinct orthogonal idempotents with

$$\begin{array}{ll} u \odot v = v \odot u = 0; & u \odot u = u; \\ v \odot v = v; & a \odot 0 = 0 \odot a = 0, \text{ for all } a \in H; \\ u \odot g = g \odot u = u; & v \odot g = g \odot v = v, \text{ for all } g \in G. \end{array}$$

If we define hyperoperation $\oplus$ on H as follows:

$$\begin{array}{l} a \oplus 0 = 0 \oplus a = \{a\}; \quad a \oplus a = \{0, a\}, \text{ for all } a \in H; \\ a \oplus b = b \oplus a = H \setminus \{0, a, b\}, \text{ for all } a, b \in H \setminus \{0\} \text{ and } a \neq b. \end{array}$$

Then, $(H, \oplus, \odot)$ is a Krasner hyperring [3]. Put $I = \{0, u\}$ and $J = \{0\}$. Obviously, I and J are 2-absorbing hyperideals. The hyperideal I is prime but J is not prime, because $u \odot v = 0 \in J$ while $u, v \notin J$.

Example 2.3. Let $(R, +, \cdot)$ be a hyperdomain and

$$M = \left\{ \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \mid x_1, x_2 \in R \right\}.$$

Put $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in R \right\}$ and define the hyperoperation $\oplus$ and the operation $\odot$ on M as

$$\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} x_1' & x_2' \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix} \mid \ y_i \in x_i + x_i', 1 \leq i \leq 2 \right\}$$

and

$$\begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} x_1' & x_2' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_1 \cdot x_1' & x_1 \cdot x_2' \\ 0 & 0 \end{pmatrix}.$$

Then, $(M, \oplus, \odot)$ is a Krasner hyperring and I is a 2-absorbing hyperideal of M.

Note that by the same argument of Theorem 2.8 of [1], one can show that a nonzero proper hyperideal I of a hyperring R is a 2-absorbing hyperideal if and only if whenever $I_1 \cdot I_2 \cdot I_3 \subseteq I$, for some hyperideals I_1, I_2, I_3 of R, then $I_1 \cdot I_2 \subseteq I$ or $I_2 \cdot I_3 \subseteq I$ or $I_1 \cdot I_3 \subseteq I$.

From now on, the hyperring $(R, +, \cdot)$ is commutative with unit element. Also, we may use xy instead of $x \cdot y$.

Theorem 2.4. *Let I be a 2-absorbing hyperideal of R. Then, one of the following statements is valid:*

(1) *$\sqrt{I} = P$ is a prime hyperideal of R and $P^2 \subseteq I$;*
(2) *$\sqrt{I} = P_1 \cap P_2$, $P_1 P_2 \subseteq I$ and $(\sqrt{I})^2 \subseteq I$, where P_1 and P_2 are the only distinct prime hyperideals of R that are minimal over I.*

Proof. We prove this statement in three steps:

Step 1: $\sqrt{I}$ is a 2-absorbing hyperideal of R.
Suppose that $x, y, z \in R$ such that $xyz \in \sqrt{I}$. By assumption, $(xyz)^2 \in I$. Thus, $x^2y^2z^2 \in I$ and this implies that $(xy)^2 = x^2y^2 \in I$ or $(xz)^2 = x^2z^2 \in I$ or $(yz)^2 = y^2z^2 \in I$. Therefore, at least one of xy, xz and yz belongs to $\sqrt{I}$.

Step 2: There are at most two distinct prime hyperideals of R that are minimal over I.
Suppose that P_1 and P_2 are distinct prime hyperideals of R that are minimal over I. Then, there are $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. Also, there exist $c_2 \in R \setminus P_1$, $c_1 \in R \setminus P_2$ and $m, n \in \mathbb{N}$ such that $c_2 x_1^n, c_1 x_2^m \in I$. This implies that $c_2 x_1, c_1 x_2 \in I \subseteq P_1 \cap P_2$, because I is a 2-absorbing hyperideal. Consequently, $c_1 \in P_1 \setminus P_2$ and $c_2 \in P_2 \setminus P_1$. Hence, $(c_1 + c_2) \cap P_1 = \emptyset$, since if $t \in (c_1 + c_2) \cap P_1$, then $c_2 \in -c_1 + t \subseteq P_1$ which contradicts $c_2 \notin P_1$. In the same way, $(c_1 + c_2) \cap P_2 = \emptyset$. Therefore, for all $t \in c_1 + c_2$ we have $tx_2 \notin P_1$ and $tx_1 \notin P_2$ which lead

to $tx_1, tx_2 \notin I$. On the other hand, $(c_1 + c_2)x_1x_2 \subseteq I$. Thus, for all $t \in c_1 + c_2$ we get $tx_1x_2 \in I$ and this implies that $x_1x_2 \in I$.

Now, suppose that P_3 is a prime hyperideal of R that is minimal over I and $P_3 \neq P_1, P_2$. Consequently, there are $y_1 \in P_1 \setminus (P_2 \cup P_3)$ and $y_2 \in P_2 \setminus (P_1 \cup P_3)$. Then, by the previous argument $y_1y_2 \in I \subseteq P_3$ which leads to $y_1 \in P_3$ or $y_2 \in P_3$, a contradiction.

Step 3: In this step, we prove the principle assertion of Theorem. Suppose that $x, y \in \sqrt{I}$. Then, $x^2, y^2 \in I$ and so $x(x + y)y \subseteq I$. Therefore, for all $t \in x+y$, we have $xt \in I$ or $xy \in I$ or $ty \in I$, because I is a 2-absorbing hyperideal. If $xt \in I$, then $xt \in x(x + y) = x^2 + xy$ and consequently $xy \in -x^2 + xt \subseteq I$. Similarly, $ty \in I$ yields $xy \in I$. Therefore, we have $(\sqrt{I})^2 \subseteq I$. By Steps (1) and (2), $\sqrt{I} = P$ is a prime hyperideal of R or $\sqrt{I} = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime hyperideals of R that are minimal over I. If $\sqrt{I} = P$, then $P^2 = (\sqrt{I})^2 \subseteq I$. If $\sqrt{I} = P_1 \cap P_2$, then for all $y \in \sqrt{I}$, $z_1 \in P_1 \setminus P_2$ and $z_2 \in P_2 \setminus P_1$ we have $y + z_1 \subseteq P_1 \setminus P_2$. By the same argument of Step 2, we get $z_1z_2 \in I$ and $(y + z_1)z_2 \subseteq I$. Thus, for all $s \in yz_2 + z_1z_2$, we have $yz_2 \in s - z_1z_2 \subseteq I$. Similarly, $yz_1 \in I$ and this implies that $P_1P_2 \subseteq I$. □

Theorem 2.5. *Let I be a hyperideal of R. Then, I is a 2-absorbing hyperideal of R if and only if $(I :_R x)$ is a prime hyperideal of R containing $\sqrt{I}$, for all $x \in \sqrt{I} \setminus I$.*

Proof. By Theorem 2.4, either $\sqrt{I} = P$ or $\sqrt{I} = P_1 \cap P_2$, where P is a prime hyperideal and P_1, P_2 are nonzero distinct prime hyperideals of R that are minimal over I. We prove the statement for the case $\sqrt{I} = P_1 \cap P_2$ and a similar argument implies the assertion for the case $\sqrt{I} = P$.

Suppose that I is a 2-absorbing hyperideal of R. According to Theorem 2.4, we conclude $xP_1, xP_2 \subseteq I$, for all $x \in \sqrt{I} \setminus I$. This means that $P_1, P_2 \subseteq (I :_R x)$ and consequently $\sqrt{I} \subseteq (I :_R x)$. Assume that $yz \in (I :_R x)$, where $y, z \in R$ and $x \in \sqrt{I} \setminus I$. Clearly, the statement is valid when $y \in P_1 \cup P_2$ or $z \in P_1 \cup P_2$. Then, we prove it for $y, z \notin P_1 \cup P_2$. In this case, we have $yz \notin P_1 \cap P_2 = \sqrt{I}$ that leads to $yz \notin I$. Hence, by assumption we get $y \in (I :_R x)$ or $z \in (I :_R x)$ which implies that $(I :_R x)$ is a prime hyperideal.

Now, suppose that $(I :_R x)$ is a prime hyperideal, for all $x \in \sqrt{I} \setminus I$. In order to prove that I is a 2-absorbing hyperideal, assume that $xyz \in I$, where $x, y, z \in R$. Then, $yz \in (I :_R x)$. Obviously, at least one of x, y, z belongs to $(P_1 \cup P_2) \setminus I$. For proving the assertion, without loss of generality suppose that $x \in (P_1 \cup P_2) \setminus I$. In this case either

$x \in \sqrt{I} \setminus I$ or $x \in (P_1 \cup P_2) \setminus \sqrt{I}$. If $x \in \sqrt{I} \setminus I$, then by the hypothesis we get $y \in (I :_R x)$ or $z \in (I :_R x)$. Consequently, $yx \in I$ or $zx \in I$ which implies that I is a 2-absorbing hyperideal. If $x \in (P_1 \cup P_2) \setminus \sqrt{I}$, then $(I :_R x_1) = P_2$ and $(I :_R x_2) = P_1$, for all $x_1 \in P_1 \setminus P_2$ and $x_2 \in P_2 \setminus P_1$. Similar to the previous argument, we find that I is a 2-absorbing hyperideal.

□

Theorem 2.6. *Let I be a 2-absorbing hyperideal and $P = P_1$ and P_2 be prime hyperideals of R. Then,*

(1) *If $\sqrt{I} = P$, then $(I :_R x)$ is a 2-absorbing hyperideal of R, for all $x \in R \setminus P$ with $\sqrt{(I :_R x)} = P$ and $\Omega = \{(I :_R x) \mid x \in R\}$ is a totally ordered set;*

(2) *If $\sqrt{I} = P_1 \cap P_2$, then $(I :_R x)$ is a 2-absorbing hyperideal of R, for all $x \in R \setminus (P_1 \cup P_2)$ with $\sqrt{(I :_R x)} = P_1 \cap P_2$ and $\Omega = \{(I :_R x) \mid x \in R \setminus (P_1 \triangle P_2)\}$ is a totally ordered set;*

(3) *If $\sqrt{I} = P_1 \cap P_2$, then $(I :_R x) = P_2$, for all $x \in P_1 \setminus P_2$ and $(I :_R x) = P_1$, for all $x \in P_2 \setminus P_1$.*

Proof. The proof is similar to Theorem 2.5 of [1]. □

Theorem 2.7. *Let I be a 2-absorbing hyperideal of R such that $I \neq \sqrt{I}$. Then,*

(1) *If $x \in \sqrt{I} \setminus I$ and $y \in R$ such that $yx \notin I$, then $(I :_R yx) = (I :_R x)$;*

(2) *If $x, y \in \sqrt{I} \setminus I$, then $(I :_R fx + dy) = (I :_R x)$, for all $f, d \in R$ such that $fd \notin (I :_R x)$. In particular, $(I :_R x + y) = (I :_R x)$.*

Proof. (1) Suppose that $c \in (I :_R yx)$, where $x \in \sqrt{I} \setminus I$ and $y \in R$. Then, $cy \in (I :_R x)$ which means that $c \in (I :_R x)$, by Theorem 2.5. Therefore, $(I :_R yx) \subseteq (I :_R x)$. It is clear that $(I :_R x) \subseteq (I :_R yx)$ and consequently the statement is valid.

(2) Suppose that $x, y \in \sqrt{I} \setminus I$. Then, $(I :_R x) \subset (I :_R y)$ or $(I :_R y) \subset (I :_R x)$, by Theorem 2.6. In order to establish the assertion, without loss the generality, assume that $(I :_R x) \subset (I :_R y)$ which leads to $(I :_R x) \subset (I :_R y) \subseteq (I :_R dy)$ and $(I :_R x) \subseteq (I :_R fx)$. Therefore, for all $t \in (I :_R x)$ we get $t(dy + fx) \subseteq I$ and so $(I :_R x) \subseteq (I :_R dy + fx)$. For proving equality, suppose that there exists $s \in dy + fx$ such that $(I :_R x) \neq (I :_R s)$. By applying Theorem 2.6, there exists $z \in (I :_R y) \cap (I :_R s)$ such that $z \notin (I :_R x)$, because $(I :_R x) \subseteq (I :_R y)$ and $(I :_R x) \subseteq (I :_R dy+fx)$. Since $zs \in z(dy+fx)$, hence $zfx \in -zdy+zs \subseteq I$ which means that $zf \in (I :_R x)$. Therefore, $z \in (I :_R x)$ or $f \in (I :_R x)$ and this is a contradiction. □

Definition 2.8. Let I be a nonzero proper hyperideal of R and

$$Z_R(R/I) = \{r + I \in R/I \mid \exists\ s \in R \setminus I \text{ such that } rs \in I\}.$$

Then, I is called *Primal* if $Z_R(R/I)$ is a prime hyperideal of R containing I.

Theorem 2.9. *Let I be a 2-absorbing hyperideal of R such that $I \neq \sqrt{I}$. Then, I is a Primal hyperideal of R.*

Proof. First, we show that $Z_R(R/I) = Q/I$, where $Q = \bigcup_{x \in (\sqrt{I} \setminus I)} (I :_R x)$. For this purpose, suppose that $a + I \in Q/I$. Then, there exists $x \in \sqrt{I} \setminus I$ such that $a \in (I :_R x)$. Therefore, $ax \in I$ which follows that $a + I \in Z_R(R/I)$. For proving $Z_R(R/I) \subseteq Q/I$, assume that $a + I \in Z_R(R/I)$, where $a \notin I$. Then, there is $b \in R \setminus I$ such that $ab \in I$. By Theorem 2.4, we can distinguish two cases:

Case 1: $\sqrt{I} = P$ is a hyperideal of R. Then, we have $ab \in P$ and consequently $a \in P \setminus I$ or $b \in P \setminus I$. Therefore, $a \in (I :_R a)$ or $a \in (I :_R b)$ which implies that $a + I \in Q/I$.

Case 2: $\sqrt{I} = P_1 \bigcap P_2$, where P_1 and P_2 are the only distinct prime hyperideals of R that are minimal over I. If $a \in \sqrt{I} \setminus I$ or $b \in \sqrt{I} \setminus I$, then by applying the same argument as for Case (1), we find $a + I \in Q/I$. Now, suppose that $a, b \notin \sqrt{I} \setminus I$. Therefore, a belongs to $P_1 \setminus P_2$ or $P_2 \setminus P_1$ and consequently $a \in (I :_R b)$, by Theorem 2.5. Hence, $a + I \in Q/I$ which leads to $Z_R(R/I) \subseteq Q/I$.

Thus in both cases, we have $Z_R(R/I) = Q/I$ as desired. Moreover, since, $I \neq \sqrt{I}$, then Theorem 2.6 implies that $\Omega = \{(I :_R x) \mid x \in \sqrt{I} \setminus I\}$ is a set of linear ordered (prime) hyperideals of R. Therefore, $Z_R(R/I) = \bigcup_{(I:_R x) \in \Omega} ((I :_R x)/I)$ is a hyperideal of R/I. $\square$

Theorem 2.10. *Let R' be a commutative hyperring with unit element and $\varphi : R \longrightarrow R'$ be a good homomorphism.*

1. *If I' is a 2-absorbing hyperideal of R', then $\varphi^{-1}(I')$ is a 2-absorbing hyperideal of R;*
2. *If φ is an epimorphism and I is a 2-absorbing hyperideal of R containing $ker\varphi$, then $\varphi(I)$ is a 2-absorbing hyperideal of R'.*

Proof. (1) Suppose that $abc \in \varphi^{-1}(I')$, then $\varphi(a)\varphi(b)\varphi(c) \in I'$. Therefore, at least one of the $\varphi(ab)$, $\varphi(bc)$ and $\varphi(ac)$ belongs to I' which implies that $ab \in \varphi^{-1}(I')$ or $bc \in \varphi^{-1}(I')$ or $ac \in \varphi^{-1}(I')$.

(2) Assume that $a', b', c' \in R'$ such that $a'b'c' \in \varphi(I)$. Then, there are $a, b, c \in R$ such that $\varphi(a) = a'$, $\varphi(b) = b'$ and $\varphi(c) = c'$. Therefore, $\varphi(abc) = a'b'c' \in \varphi(I)$ which deduce that there is $i \in I$ such that

$(abc - i) \cap ker\varphi \neq \emptyset$. Consider $t \in (abc - i) \cap ker\varphi$. We conclude that $abc \in t + i \subseteq ker\varphi + I \subseteq I$. This implies that $ab \in I$ or $ac \in I$ or $bc \in I$. Consequently, at least one of the $a'b'$, $a'c'$ and $b'c'$ belongs to $\varphi(I)$. □

The following corollary is deduced directly from Theorem 2.10.

Corollary 2.11. *Let I and J be distinct proper hyperideals of R. If $J \subseteq I$ and I is a 2-absorbing hyperideal of R, then I/J is a 2-absorbing hyperideal of R/J.*

Theorem 2.12. *Let R_1, R_2 be Krasner hyperrings and $R = R_1 \times R_2$.*

(1) *If I_1 (I_2, respectively) is a 2-absorbing hyperideal of R_1 (R_2, respectively), then $I_1 \times R_2$ ($R_1 \times I_2$, respectively) is a 2-absorbing hyperideal of R;*
(2) *If J is a 2-absorbing hyperideal of R, then either $J = I_1 \times R_2$ ($J = R_1 \times I_2$, respectively), where I_1 (I_2, respectively) is a 2-absorbing hyperideal of R_1 (R_2, respectively) or $I = I_1 \times I_2$, where I_1 (I_2, respectively) is a prime hyperideal of R_1 (R_2, respectively).*

Proof. (1) It is straightforward.

(2) Suppose that J is a proper 2-absorbing hyperideal of R. Then, $J = I_1 \times I_2$, where for $i = 1, 2$ we have I_i is a hyperideal of R_i. Assume that $I_2 = R_2$ and $R' = R/(\{0\} \times R_2)$. Therefore, $I_1 \neq R_1$ and $J' = J/(\{0\} \times R_2)$ is a 2-absorbing hyperideal of R', by Corollary 2.11. It follows that I_1 is a 2-absorbing hyperideal of R_1, since $R' \cong R_1$ and $I_1 \cong J'$. In the same way, $I_1 = R_1$ implies that I_2 is a 2-absorbing hyperideal of R_2.

For completing the proof it is enough to show that if $I_1 \neq R_1$ and $I_2 \neq R_2$, then I_i is a prime hyperideal of R_i, where $i = 1, 2$. Assume that at least one of I_i is not prime, e.g. I_1. Therefore, there are $a, b \in R_1$ such that $ab \in I_1$ but $a, b \notin I_1$. Putting $x = (a, 1)$, $y = (1, 0)$ and $z = (b, 1)$, we give $xyz = (ab, 0) \in J$ while $xy = (a, 0), xz = (ab, 1), yz = (b, 0)$ do not belong to J and this is a contradiction to the assumption. □

Theorem 2.13. *Let I be a hyperideal of R and S be a multiplicatively closed subset of R. In addition, let $S^{-1}R$ be the hyperring of quotients of R.*

(1) *If I is a 2-absorbing hyperideal of R and $S \cap I = \emptyset$, then $S^{-1}I$ is a 2-absorbing hyperideal of $S^{-1}R$;*
(2) *If $S^{-1}I$ is a 2-absorbing hyperideal of $S^{-1}R$ and $S \cap Z_R(R/I) = \emptyset$, then I is a 2-absorbing hyperideal of R.*

Proof. (1) Suppose that $a, b, c \in R$ and $s, t, k \in S$ such that $(a/s)(b/t)$ $(c/k) \in S^{-1}I$. Then, there exists $u \in S$ such that $uabc \in I$. Hence, $uab \in I$ or $uac \in I$ or $bc \in I$, by hypothesis. If $uab \in I$, then $(a/s)(b/t)$ $= (uab)/(ust) \in S^{-1}I$. Also, $uac \in I$ implies that $(a/s)(c/k) = (uac)/(usk) \in S^{-1}I$ and $bc \in I$. Therefore, $(b/t)(c/k) \in S^{-1}I$. By the above result, $S^{-1}I$ is a 2-absorbing hyperideal.

(2) Suppose that $a, b, c \in I$ such that $abc \in I$. In this case, we have $(abc)/1 = (a/1)(b/1)(c/1) \in S^{-1}I$. Hence, $(a/1)(b/1) \in S^{-1}I$ or $(b/1)(c/1) \in S^{-1}I$ or $(a/1)(c/1) \in S^{-1}I$, since $S^{-1}I$ is a 2-absorbing hyperideal. If $(a/1)(b/1) \in S^{-1}I$, then there exists $u \in S$ such that $uab \in I$. This implies that $ab \in I$, since $S \cap Z_R(R/I) = \emptyset$.

Similarly, $(b/1)(c/1) \in S^{-1}I$ ($(a/1)(c/1) \in S^{-1}I$, respectively) which leads to $bc \in I$ ($ac \in I$, respectively). Consequently, I a 2-absorbing hyperideal. ☐

Definition 2.14. A proper hyperideal I of R is called *irreducible* precisely if I can not be expressed as the intersection of two strictly larger hyperideals of R.

The following theorem shows the relationship between irreducible and 2-absorbing hyperideals.

Theorem 2.15. *Let I be an irreducible hyperideal of R and $P = P_1$, P_2 be distinct prime hyperideals of R.*

(1) *If $\sqrt{I} = P$, then I is a 2-absorbing hyperideal if and only if $P^2 \subseteq I$ and $(I :_R x) = (I :_R x^2)$, for all $x \in R \setminus P$;*
(2) *If $\sqrt{I} = P_1 \cap P_2$, then I is a 2-absorbing hyperideal if and only if $P_1P_2 \subseteq I$ and $(I :_R x) = (I :_R x^2)$, for all $x \in R \setminus P_1 \cap P_2$.*

Proof. (1) For proving the necessity part, it is only necessary to check $(I :_R x^2) \subseteq (I :_R x)$, for all $x \in R \setminus P$. Because, it is clear $(I :_R x) \subseteq (I :_R x^2)$ and $P^2 \subseteq I$, by Theorem 2.4.

Suppose that $y \in (I :_R x^2)$. Then, $yx \in I$ or $x^2 \in I$. If $x^2 \in I$, then $x \in P$ and this is a contradiction. Then, $yx \in I$ which implies that $y \in (I :_R x)$ and consequently $(I :_R x^2) \subseteq (I :_R x)$ as desired.

For establishing the sufficiency part, assume that $x, y, z \in R$ such that $xyz \in I$ and $xy \notin I$. We show that either $xz \in I$ or $yz \in I$. From $xy \notin I$, it follows that $x \notin P$ or $y \notin P$ and so $(I :_R x) = (I :_R x^2)$ or $(I :_R y) = (I :_R y^2)$, respectively. Without loss of generality, suppose that $(I :_R x) = (I :_R x^2)$. For completing the proof as a contradiction, assume that $xz \notin I$ and $yz \notin I$. Consider $a \in \langle I + xz\rangle \cap \langle I + yz\rangle$ which follows that there are $a_1, a_2 \in I$ and $r_1, r_2 \in R$ such that $a \in (a_1 + r_1xz) \cap (a_2 + r_2yz)$. Consequently, $ax \in a_1x + r_1x^2z$ and $ax \in a_2x + r_2yzx \subseteq I$ which lead to $r_1x^2z \in -a_1x + ax \subseteq I$.

Therefore, $r_1z \in (I :_R x^2) = (I :_R x)$, by assumption. This implies that $a \in a_1 + r_1xz \subseteq I$. Then, $< I + xz > \cap < I + yz > \subseteq I$ and so $< I + xz > \cap < I + yz >= I$ which contradicts irreducibility of I.

(2) The proof is similar to Part (1). $\square$

In the process of proving the next theorem, we need the following lemma.

Lemma 2.16. *Let $P_1, P_2, ..., P_n$, where $n \geq 2$, be hyperideals of R such that at most two of them are not prime. Furthermore, let S be an additive canonical subhypergroup of R which is closed under multiplication and $S \subseteq \bigcup_{i=1}^{n} P_i$. Then, there exists $1 \leq j \leq n$ such that $S \subseteq P_j$.*

Proof. We prove this statement by induction on n. First, consider for $n = 2$ that is $S \subseteq P_1 \cup P_2$. As a contradiction, assume that $S \nsubseteq P_1$ and $S \nsubseteq P_2$. Then, there exists $a_j \in S \setminus P_j$, where $j = 1, 2$. Therefore, the hypothesis leads to $a_1 \in P_2$ and $a_2 \in P_1$. On the other hand, $a_1 + a_2 \subseteq S \subseteq P_1 \cup P_2$ and so for all $t \in a_1 + a_2$ we have t belongs to either P_1 or P_2. Since $a_1 \in \{a_1\} = a_1 + 0 \subseteq (a_1 + a_2) - a_2$, then there exists $t \in a_1 + a_2$ such that $a_1 \in t - a_2$. By the above results, if $t \in P_1$, then $a_1 \in P_1$. Also, if $t \in P_2$, then $a_2 \in -t - a_1 \subseteq P_2$, which is a contradiction in two cases. Thus we must have $S \subseteq P_1$ or $S \subseteq P_2$. Now, suppose that $k \geq 2$ and our assertion is valid for $n = k$. For completing the proof, assume that $n = k + 1$, where $k \geq 2$. Thus, we have $S \subseteq \bigcup_{i=1}^{k+1} P_i$ and since at most two of the P_i are not prime, we can assume that they have been indexed in such a way that P_{k+1} is prime. We claim that there is $1 \leq j \leq k$ such that $S \subseteq \bigcup_{\substack{i=1 \\ i \neq j}}^{k+1} P_i$. For proving this claim as a contradiction suppose that $S \nsubseteq \bigcup_{\substack{i=1 \\ i \neq j}}^{k+1} P_i$, for all $1 \leq j \leq k$. It follows that for all $1 \leq j \leq k$, there exists $a_j \in S \setminus \bigcup_{\substack{i=1 \\ i \neq j}}^{k+1} P_i$ which implies that $a_j \in P_j$, by hypothesis. Moreover, since $P_{k+1} \in Spec(R)$, we conclude that $a_1 \cdots a_k \notin P_{k+1}$. Consequently, $a_1 \cdots a_k \in \bigcap_{i=1}^{k} P_i \setminus P_{k+1}$ and $a_{k+1} \in P_{k+1} \setminus \bigcup_{i=1}^{k} P_i$. Now consider the element $b \in a_1 \cdots a_k + a_{k+1}$. If $b \in P_{k+1}$, then $a_1 \cdots a_k \in b - a_{k+1} \subseteq P_{k+1}$ and this is a contradiction. Therefore, b does not belong to P_{k+1}. Also, we can not have $b \in P_j$, where $1 \leq j \leq k$, for that would imply $a_{k+1} \in b - a_1 \cdots a_k \subseteq P_j$, again a contradiction. But $b \in S$, since for all $1 \leq j \leq k$ we have $a_j \in S$, which leads to a contradiction to the hypothesis that $S \subseteq \bigcup_{i=1}^{k+1} P_i$. It follows that the statement is valid. In fact, there is $1 \leq j \leq k+1$ such that $S \subseteq \bigcup_{\substack{i=1 \\ i \neq j}}^{k+1} P_i$. By applying the inductive hypothesis, we deduce that $S \subseteq P_i$, where $1 \leq i \leq k+1$. □

Theorem 2.17. *Let $I_1, I_2, ..., I_n$ be 2-absorbing hyperideals of R and I be a hyperideal of R such that $I \subseteq I_1 \cup I_2 \cup ... \cup I_n$. Then, there exists $1 \leq i \leq n$ such that $I^2 \subseteq I_i$.*

Proof. First, we show that there exists $1 \leq i \leq n$ such that $\sqrt{I} \subseteq \sqrt{I_i}$. By Theorem 2.4, we can assume that they have been indexed in such a way that $\sqrt{I_i} = p_i$ and $\sqrt{I_j} = p_{j,1} \cap p_{j,2}$, for all $1 \leq i \leq k$ and $k+1 \leq j \leq n$, where $p_i, p_{j,1}, p_{j,2}$ are prime hyperideals of R. Then, $\sqrt{I} \subseteq p_1 \cup p_2 \cup \cdots \cup p_k \cup (p_{k+1,1} \cap p_{k+1,2}) \cup \cdots \cup (p_{n,1} \cap p_{n,2})$ which follows that $\sqrt{I} \subseteq p_1 \cup p_2 \cup \cdots \cup p_k \cup p_{k+1,t_{k+1}} \cup \cdots \cup p_{n,t_n}$, where $t_{k+1}, \cdots, t_n \in \{1,2\}$. Therefore by applying Lemma 2.16, we find that $\sqrt{I} \subseteq p_i$ or $\sqrt{I} \subseteq p_{j,t_s}$, for some $1 \leq i \leq k$, $k+1 \leq j \leq n$ and $t_s \in \{1,2\}$. If $\sqrt{I} \subseteq p_{j,t_s}$, where $k+1 \leq j \leq n$, $t_s \in \{1,2\}$, then $\sqrt{I} \subseteq p_{j,t_s} \subseteq \bigcup_{j=k+1}^{n} p_{j,t_s}$. We may assume that $\sqrt{I} \subseteq \bigcap_{j=k+1}^{s} p_{j,1}$ and $\sqrt{I} \nsubseteq \bigcup_{j=s+1}^{n} p_{j,1}$, where $k+1 \leq s \leq n$. On the other hand, $\sqrt{I} \subseteq p_{k+1,2} \cup \cdots \cup p_{s,2} \cup p_{s+1,1} \cup \cdots \cup p_{n,1}$. Therefore, $\sqrt{I} \subseteq p_{j,2}$, for some $k+1 \leq j \leq s$, by Lemma 2.16. Hence, $\sqrt{I} \subseteq p_{j,1} \cap p_{j,2} = \sqrt{I_j}$, where $k+1 \leq j \leq s$. Then, in general there is $1 \leq i \leq n$ such that $\sqrt{I} \subseteq \sqrt{I_i}$ which leads to $I^2 \subseteq (\sqrt{I})^2 \subseteq (\sqrt{I_i})^2$. By applying Theorem 2.4, we get $I^2 \subseteq I_i$. □

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Leili Kamali Ardekani
Faculty of Engineering, Ardakan University, P.O. Box 184, Ardakan, Iran.
Email: `l.kamali@ardakan.ac.ir`

Bijan Davvaz
Department of Mathematics, Yazd University, Yazd, Iran.
Email: `davvaz@yazd.ac.ir`

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A GENERALIZATION OF PRIME HYPERIDEALS IN KRASNER HYPERRINGS

L. KAMALI ARDEKANI AND B. DAVVAZ

تعمیمی از ابرایده‌آل‌های اول روی ابرحلقه‌های کراسنر

لیلی کمالی اردکانی[۱] و بیژن دواز[۲]
[۱]دانشکده فنی و مهندسی، دانشگاه اردکان، اردکان، ایران
[۲]دانشکده ریاضی، دانشگاه یزد، یزد، ایران

با توجه به نقش مهم ابرایده‌آل‌های اول در نظریه‌ی ابرحلقه‌ها، در این مقاله به معرفی ابرایده‌آل‌های ۲-جاذب به عنوان تعمیمی جدید از ابرایده‌آل‌های اول روی ابرحلقه‌های کراسنر پرداخته می‌شود. برخی خواص و نتایج اساسی از ابرایده‌آل‌های ۲-جاذب روی ابرحلقه‌های کراسنر مورد مطالعه و بررسی قرار گرفته است که یکی از مهمترین آن‌ها شرایطی را روی ابرایده‌آل ۲-جاذب I از ابرحلقه‌ی کراسنر R بیان می‌کند که تحت آن $(I :_R x) = \{y \in R \mid y \cdot x \in I\}$ یک ابرایده‌آل ۲-جاذب است.

کلمات کلیدی: ابرایده‌آل اول، ابرایده‌آل ۲-جاذب، ابرحلقه‌ی کراسنر.