# ON LOCAL ANTIMAGIC CHROMATIC NUMBER OF GRAPHS

S. SHAEBANI

ABSTRACT. A *local antimagic labeling* of a connected graph $G$ with at least three vertices, is a bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that for any two adjacent vertices $u$ and $v$ of $G$, the condition $\omega_f(u) \neq \omega_f(v)$ holds; where $\omega_f(u) = \sum_{x \in N(u)} f(xu)$. Assigning $\omega_f(u)$ to $u$ for each vertex $u$ in $V(G)$, induces naturally a proper vertex coloring of $G$; and $|f|$ denotes the number of colors appearing in this proper vertex coloring. The *local antimagic chromatic number* of $G$, denoted by $\chi_{la}(G)$, is defined as the minimum of $|f|$, where $f$ ranges over all local antimagic labelings of $G$. In this paper, we explicitly construct an infinite class of connected graphs $G$ such that $\chi_{la}(G)$ can be arbitrarily large while $\chi_{la}(G \vee \bar{K}_2) = 3$, where $G \vee \bar{K}_2$ is the join graph of $G$ and the complement graph of $K_2$. The aforementioned fact leads us to an infinite class of counterexamples to a result of [Local antimagic vertex coloring of a graph, *Graphs and Combinatorics* **33** (2017), 275–285].

## 1. INTRODUCTION

Unless otherwise stated, we consider connected finite simple graphs that have at least three vertices. Let $G$ be a graph and $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ be a bijection. For each vertex $u$ in $V(G)$, we define $\omega_f(u)$ to be the sum of the labels of all incident edges to $u$; more precisely, $\omega_f(u) = \sum_{x \in N(u)} f(xu)$. Whenever there is no ambiguity on $f$, we use the symbol $\omega(u)$ instead of $\omega_f(u)$.

Let $G$ be a graph and $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ be a bijection. If $\omega_f(u) \neq \omega_f(v)$ for any two distinct vertices $u$ and $v$ in $V(G)$, then $f$ is called an *antimagic labeling* of $G$ [5]. Hartsfield and Ringel conjectured that every connected graph with at least three vertices admits an antimagic labeling [5]. They also made a weaker conjecture that every tree with at least three vertices admits an antimagic labeling. By several authors, these two conjectures were partially shown to be true, but they are still unsolved. For the best and most interesting results were obtained so far, one can see [7] for trees and [3] for general graphs. Also, for a detailed and interesting review on these conjectures one can see chapter 6 of [4].

Arumugam, Premalatha, Bača, and Semaničová-Feňovčíková in [1], and independently, Bensmail, Senhaji, and Lyngsie in [2], posed a new definition as a relaxation of the notion of antimagic labeling. They called a bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ a *local antimagic labeling* of $G$ if for any two adjacent vertices $u$ and $v$ in $V(G)$, the condition $\omega_f(u) \neq \omega_f(v)$ holds. They conjectured that every connected graph with at least three vertices admits a local antimagic labeling. This conjecture was solved partially in [2]. Finally, Haslegrave proved this conjecture by means of probabilistic tools [6].

Based on the notion of local antimagic labeling, Arumugam, Premalatha, Bača, and Semaničová-Feňovčíková introduced a new graph coloring parameter. Let $G$ be a connected graph with at least three vertices, and $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ be a local antimagic labeling of $G$. For any two adjacent vertices $u$ and $v$ we have $\omega_f(u) \neq \omega_f(v)$; so, assigning $\omega_f(u)$ to $u$ for each vertex $u$ in $V(G)$, induces naturally a proper vertex coloring of $G$ which is called a *local antimagic vertex coloring* of $G$. Let $|f|$ denote the number of colors appearing in this proper vertex coloring. More precisely, $|f| = |\{\omega_f(u) : u \in V(G)\}|$. The *local antimagic chromatic number* of $G$, denoted by $\chi_{la}(G)$, is defined as the minimum of $|f|$, where $f$ ranges over all local antimagic labelings of $G$ [1].

Let $G_1$ and $G_2$ be two vertex disjoint graphs. The *join graph* of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and its edge set equals $E(G_1) \cup E(G_2) \cup \{ab : a \in V(G_1) \text{ and } b \in V(G_2)\}$.

Theorem 2.16 of [1] asserts that if a graph $G$ has at least four vertices, then $\chi_{la}(G) + 1 \leq \chi_{la}(G \vee \bar{K}_2)$, where $\bar{K}_2$ is the complement graph of a complete graph with two vertices. In this paper, we show that the mentioned theorem is incorrect. In this regard, we explicitly construct an infinite class of connected graphs $G$ such that $\chi_{la}(G)$ can be arbitrarily large and $\chi_{la}(G \vee \bar{K}_2) = 3$.

## 2. Main results

This section is devoted to constructing an infinite class of connected graphs $G$ such that $\chi_{la}(G)$ can be arbitrarily large while $\chi_{la}(G \vee \bar{K}_2) = 3$. Our procedure is to consider the complete bipartite graph $K_{1,n}$ that satisfies $\chi_{la}(K_{1,n}) = n + 1$ for each positive integer $n \geq 2$. We show that if $n$ is odd and $n+1$ is not divisible by 3, then $\chi_{la}(K_{1,n} \vee \bar{K}_2) = 3$.

**Theorem 2.1.** *Let $n$ be an odd integer such that $n + 1$ is not divisible by 3. Then, the join of the star graph $K_{1,n}$ and the complement of $K_2$, say $H := K_{1,n} \vee \bar{K}_2$, satisfies $\chi_{la}(H) = 3$.*

*Proof.* Let the vertex set of the star graph $K_{1,n}$ be $\{v, v_1, v_2, \ldots, v_n\}$ and $v$ be its central vertex. Also, let $x$ and $y$ be the two vertices of $\bar{K}_2$. Since $H$ has some triangles, we have $\chi_{la}(H) \geq \chi(H) \geq 3$. So, for proving $\chi_{la}(H) = 3$, it suffices to provide a local antimagic labeling of $H$ that induces a local antimagic vertex coloring using exactly three colors.

For $n = 1$, define $f : E(H) \to \{1, 2, 3, 4, 5\}$ by

$$f(vv_1) = 1, \quad f(vx) = 5, \quad f(vy) = 4,$$
$$f(v_1x) = 2, \quad f(v_1y) = 3.$$

In this case, we have

$$\omega(v) = 10, \quad \omega(v_1) = 6, \quad \omega(x) = \omega(y) = 7.$$

Therefore, $f$ is a local antimagic labeling of $H$ that induces a local antimagic vertex coloring using exactly three colors.

For $n \geq 3$, the aim is to construct a local antimagic labeling $f : E(H) \to \{1, 2, 3, \ldots, 3n + 2\}$ such that $\omega(v_1) = \omega(v_2) = \cdots = \omega(v_n)$ and $\omega(x) = \omega(y)$. In this regard, we first assign $f(vv_i) = i$ for each $i$ in $\{1, 2, \ldots, n\}$. Also, in our construction, $\{f(vx), f(vy)\} = \{n+1, n+2\}$. Therefore,

$$\omega(v) = \sum_{i=1}^{n+2} i = \frac{(n+2)(n+3)}{2}.$$

Also, we must have

$$\omega(x) = \omega(y) = \frac{1}{2} \sum_{i=n+1}^{3n+2} i = \frac{(n+1)(4n+3)}{2},$$

and

$$\omega(v_1) = \omega(v_2) = \cdots = \omega(v_n) = \frac{9n+11}{2}.$$

This shows that since $n + 1$ is not divisible by 3, the desired $f$ will be a local antimagic labeling of $H$ and it induces a local antimagic vertex coloring of $H$ with three colors. We make a partition $\{A_1, A_2, \ldots, A_n\}$ of the set $\{n+3, n+4, \ldots, 3n+2\}$ such that for each $i$ in $\{1, 2, \ldots, n\}$, the set $A_i$ has two elements and $A_i = \{f(v_i x), f(v_i y)\}$. Also, $A_i$ has one element in $\{n+3, n+4, \ldots, 2n+2\}$ and one element in $\{2n+3, 2n+4, \ldots, 3n+2\}$. In this regard, our suitable partition is as the following;

$$
A_i = \begin{cases} \{2n + 4 - 2i \ , \ \frac{5n+3}{2} + i\} & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \\[2mm] \{3n + 4 - 2i \ , \ \frac{3n+3}{2} + i\} & \text{if} \quad \frac{n+3}{2} \leq i \leq n. \end{cases}
$$

It is obvious that for each $i$ in $\{1, 2, \ldots, n\}$, we have

$$
\omega(v_i) = i + f(v_i x) + f(v_i y) = \tfrac{9n+11}{2}.
$$

Accordingly, the following $n + 1$ sets

$$
\{f(vx), f(vy)\}, \{f(v_1 x), f(v_1 y)\}, \ldots, \{f(v_n x), f(v_n y)\}
$$

are determined. For completing the proof, it is sufficient to determine the exact values of each of

$$
f(vx), f(vy), f(v_1 x), f(v_1 y), \ldots, f(v_n x), f(v_n y),
$$

in such a way that $\omega(x) = \omega(y)$. In this regard, we consider the following four cases.

**Case 1.** The case that $\frac{n+1}{2} \overset{4}{\equiv} 0$.

In this case, we have $n \geq 7$. First we determine $f(v_i x)$ and $f(v_i y)$ for each $i$ in $\{1, 2, \ldots, \frac{n+1}{2}\}$; as follows.

$$
f(v_i x) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 1 \quad \text{or} \quad i \overset{4}{\equiv} 0) \\[2mm] 2n + 4 - 2i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 2 \quad \text{or} \quad i \overset{4}{\equiv} 3) \end{cases}
$$

and

$$
f(v_i y) = \begin{cases} 2n + 4 - 2i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 1 \quad \text{or} \quad i \overset{4}{\equiv} 0) \\[2mm] \frac{5n+3}{2} + i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 2 \quad \text{or} \quad i \overset{4}{\equiv} 3). \end{cases}
$$

If $i$ is a positive integer such that $i \overset{4}{\equiv} 1$ and $i + 4 \leq \frac{n+1}{2}$, then

$$
\sum_{j=i}^{i+3} f(v_j x) = 9n + 8 - 2i
$$

and

$$\sum_{j=i}^{i+3} f(v_j y) = 9n + 8 - 2i.$$

This shows that since $\frac{n+1}{2}$ is divisible by 4, we have

$$\sum_{j=1}^{\frac{n+1}{2}} f(v_j x) = \sum_{j=1}^{\frac{n+1}{2}} f(v_j y).$$

Now, we put $f(vx) = n + 2$ and $f(vy) = n + 1$. Also, for each $i$ in $\left\{ \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2} \right\}$ put

$$f(v_i x) = \begin{cases} \frac{3n+3}{2} + i & \text{if} \quad i \in \left\{ \frac{n+3}{2}, \frac{n+5}{2} \right\} \\ \\ 3n + 4 - 2i & \text{if} \quad i = \frac{n+7}{2} \end{cases}$$

and

$$f(v_i y) = \begin{cases} 3n + 4 - 2i & \text{if} \quad i \in \left\{ \frac{n+3}{2}, \frac{n+5}{2} \right\} \\ \\ \frac{3n+3}{2} + i & \text{if} \quad i = \frac{n+7}{2}. \end{cases}$$

We have

$$f(vx) + \sum_{j=\frac{n+3}{2}}^{\frac{n+7}{2}} f(v_j x) = f(vy) + \sum_{j=\frac{n+3}{2}}^{\frac{n+7}{2}} f(v_j y).$$

Therefore,

$$f(vx) + \sum_{j=1}^{\frac{n+7}{2}} f(v_j x) = f(vy) + \sum_{j=1}^{\frac{n+7}{2}} f(v_j y).$$

Now, it is turn to determine the exact values of

$$f\left( xv_{\frac{n+9}{2}} \right), f\left( yv_{\frac{n+9}{2}} \right), f\left( xv_{\frac{n+11}{2}} \right), f\left( yv_{\frac{n+11}{2}} \right), \ldots, f(xv_n), f(yv_n).$$

Consider the following assignments;

$$f(v_i x) = \begin{cases} \frac{3n+3}{2} + i & \text{if} \quad \frac{n+9}{2} \leq i \leq n \quad \text{and} \quad (i \stackrel{4}{\equiv} 0 \quad \text{or} \quad i \stackrel{4}{\equiv} 3) \\ \\ 3n + 4 - 2i & \text{if} \quad \frac{n+9}{2} \leq i \leq n \quad \text{and} \quad (i \stackrel{4}{\equiv} 1 \quad \text{or} \quad i \stackrel{4}{\equiv} 2) \end{cases}$$

and

$$f(v_i y) = \begin{cases} 3n + 4 - 2i & \text{if} \quad \frac{n+9}{2} \leq i \leq n \quad \text{and} \quad (i \stackrel{4}{\equiv} 0 \quad \text{or} \quad i \stackrel{4}{\equiv} 3) \\ \\ \frac{3n+3}{2} + i & \text{if} \quad \frac{n+9}{2} \leq i \leq n \quad \text{and} \quad (i \stackrel{4}{\equiv} 1 \quad \text{or} \quad i \stackrel{4}{\equiv} 2). \end{cases}$$

Since $\frac{n+1}{2}$ is divisible by 4, the number of vertices in $\left\{v_i \mid \frac{n+9}{2} \leq i \leq n\right\}$ is divisible by 4. Also, $\frac{n+9}{2} \overset{4}{\equiv} 0$. Now, if $\{i, i+1, i+2, i+3\} \subseteq \left\{\frac{n+9}{2}, \frac{n+11}{2}, \ldots, n-1, n\right\}$ and $i \overset{4}{\equiv} 0$, we have

$$\sum_{j=i}^{i+3} f(v_j x) = \sum_{j=i}^{i+3} f(v_j y).$$

Accordingly,

$$\sum_{j=\frac{n+9}{2}}^{n} f(v_j x) = \sum_{j=\frac{n+9}{2}}^{n} f(v_j y).$$

We conclude that

$$f(vx) + \sum_{j=1}^{n} f(v_j x) = f(vy) + \sum_{j=1}^{n} f(v_j y);$$

and the proof is completed in this case.

**Case 2.** The case that $\frac{n+1}{2} \overset{4}{\equiv} 2$.
For each $i$ in $\left\{1, 2, \ldots, \frac{n+1}{2}\right\}$, we define $f(v_i x)$ and $f(v_i y)$ as follows;

$$f(v_i x) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 1 \quad \text{or} \quad i \overset{4}{\equiv} 0) \\[2em] 2n + 4 - 2i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 2 \quad \text{or} \quad i \overset{4}{\equiv} 3) \end{cases}$$

and

$$f(v_i y) = \begin{cases} 2n + 4 - 2i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 1 \quad \text{or} \quad i \overset{4}{\equiv} 0) \\[2em] \frac{5n+3}{2} + i & \text{if} \quad 1 \leq i \leq \frac{n+1}{2} \quad \text{and} \quad (i \overset{4}{\equiv} 2 \quad \text{or} \quad i \overset{4}{\equiv} 3). \end{cases}$$

If $\{i, i+1, i+2, i+3\} \subseteq \left\{1, 2, \ldots, \frac{n+1}{2}\right\}$ and $i \overset{4}{\equiv} 1$, then

$$\sum_{j=i}^{i+3} f(v_j x) = \sum_{j=i}^{i+3} f(v_j y).$$

Because of $\frac{n+1}{2} \overset{4}{\equiv} 2$, we have

$$\sum_{i=1}^{\frac{n+1}{2}} f(v_i y) = 3 + \sum_{i=1}^{\frac{n+1}{2}} f(v_i x).$$

By setting the following four assignments

$$f\left(xv_{\frac{n+3}{2}}\right) = \frac{3n+3}{2} + \frac{n+3}{2}, \qquad\qquad f(vx) = n + 2,$$

$$f\left(yv_{\frac{n+3}{2}}\right) = 3n + 4 - 2\left(\frac{n+3}{2}\right), \qquad f(vy) = n + 1,$$

we obtain

$$f(vx) + \sum_{i=1}^{\frac{n+3}{2}} f(v_i x) = f(vy) + \sum_{i=1}^{\frac{n+3}{2}} f(v_i y).$$

Now, we determine the exact values of

$$f\left(xv_{\frac{n+5}{2}}\right), f\left(yv_{\frac{n+5}{2}}\right), f\left(xv_{\frac{n+7}{2}}\right), f\left(yv_{\frac{n+7}{2}}\right), \dots, f(xv_n), f(yv_n).$$

Let us regard the following assignments;

$$f(v_i x) = \begin{cases} \frac{3n+3}{2} + i & \text{if} \quad \frac{n+5}{2} \le i \le n \quad \text{and} \quad (i \stackrel{4}{\equiv} 0 \quad \text{or} \quad i \stackrel{4}{\equiv} 3) \\[2ex] 3n + 4 - 2i & \text{if} \quad \frac{n+5}{2} \le i \le n \quad \text{and} \quad (i \stackrel{4}{\equiv} 1 \quad \text{or} \quad i \stackrel{4}{\equiv} 2) \end{cases}$$

and

$$f(v_i y) = \begin{cases} 3n + 4 - 2i & \text{if} \quad \frac{n+5}{2} \le i \le n \quad \text{and} \quad (i \stackrel{4}{\equiv} 0 \quad \text{or} \quad i \stackrel{4}{\equiv} 3) \\[2ex] \frac{3n+3}{2} + i & \text{if} \quad \frac{n+5}{2} \le i \le n \quad \text{and} \quad (i \stackrel{4}{\equiv} 1 \quad \text{or} \quad i \stackrel{4}{\equiv} 2). \end{cases}$$

If $i \stackrel{4}{\equiv} 0$ and $\{i, i+1, i+2, i+3\} \subseteq \{\frac{n+5}{2}, \frac{n+7}{2}, \dots, n\}$, then

$$\sum_{j=i}^{i+3} f(v_j x) = \sum_{j=i}^{i+3} f(v_j y).$$

Thus, since $\frac{n+5}{2} \stackrel{4}{\equiv} 0$ and the number of vertices in $\left\{v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \dots, v_n\right\}$ is divisible by 4, we obtain that

$$\sum_{j=\frac{n+5}{2}}^{n} f(v_j x) = \sum_{j=\frac{n+5}{2}}^{n} f(v_j y).$$

Accordingly,

$$f(vx) + \sum_{j=1}^{n} f(v_j x) = f(vy) + \sum_{j=1}^{n} f(v_j y);$$

which is desired in this case.

**Case 3.** The case that $\frac{n+1}{2}$ is odd and $\frac{n+3}{2}$ is divisible by 3.
In this case, $\frac{n-1}{2}$ is even. Also, since $\frac{n+3}{2}$ is divisible by 3, both of $\frac{n}{3}$ and $\frac{n-3}{6}$ are integers. We define

$$f\left(v_1 x\right) = 2n + 4 - 2, \qquad f(vx) = n + 1,$$

$$f\left(v_1 y\right) = \frac{5n+3}{2} + 1, \qquad f(vy) = n + 2.$$

For each $i$ with $2 \leq i \leq \frac{n}{3}$, set $f(v_i x)$ and $f(v_i y)$ as the following;

$$f(v_i x) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad 2 \leq i \leq \frac{n}{3} \text{ and } i \text{ is even} \\ \\ 2n + 4 - 2i & \text{if} \quad 2 \leq i \leq \frac{n}{3} \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_i y) = \begin{cases} 2n + 4 - 2i & \text{if} \quad 2 \leq i \leq \frac{n}{3} \text{ and } i \text{ is even} \\ \\ \frac{5n+3}{2} + i & \text{if} \quad 2 \leq i \leq \frac{n}{3} \text{ and } i \text{ is odd.} \end{cases}$$

It is obvious that if $i$ is an even integer with $2 \leq i \leq \frac{n}{3}$, then

$$f(yv_i) + f(yv_{i+1}) = f(xv_i) + f(xv_{i+1}) + 3.$$

So,

$$\sum_{i=2}^{\frac{n}{3}} f(yv_i) = \frac{n-3}{2} + \sum_{i=2}^{\frac{n}{3}} f(xv_i).$$

Now, for each $i$ with $\frac{n+3}{3} \leq i \leq \frac{n+1}{2}$, define $f(v_i x)$ and $f(v_i y)$ as the following;

$$f(v_i x) = \begin{cases} 2n + 4 - 2i & \text{if} \quad \frac{n+3}{3} \leq i \leq \frac{n+1}{2} \text{ and } i \text{ is even} \\ \\ \frac{5n+3}{2} + i & \text{if} \quad \frac{n+3}{3} \leq i \leq \frac{n+1}{2} \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_i y) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad \frac{n+3}{3} \leq i \leq \frac{n+1}{2} \text{ and } i \text{ is even} \\ \\ 2n + 4 - 2i & \text{if} \quad \frac{n+3}{3} \leq i \leq \frac{n+1}{2} \text{ and } i \text{ is odd.} \end{cases}$$

If $i$ is an even integer with $\frac{n+3}{3} \leq i \leq \frac{n+1}{2}$, then we have

$$f(xv_i) + f(xv_{i+1}) = f(yv_i) + f(yv_{i+1}) + 3.$$

Therefore,

$$\sum_{i=\frac{n+3}{3}}^{\frac{n+1}{2}} f(xv_i) = \frac{n+3}{4} + \sum_{i=\frac{n+3}{3}}^{\frac{n+1}{2}} f(yv_i).$$

Finally, let us regard the following assignments for $f(v_ix)$ and $f(v_iy)$ when $i$ is an integer with $\frac{n+3}{2} \le i \le n$;

$$f(v_ix) = \begin{cases} 3n + 4 - 2i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is even} \\[2mm] \frac{3n+3}{2} + i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_iy) = \begin{cases} \frac{3n+3}{2} + i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is even} \\[2mm] 3n + 4 - 2i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is odd.} \end{cases}$$

Again, for each even integer $i$ with $\frac{n+3}{2} \le i \le n$ we have

$$f(xv_i) + f(xv_{i+1}) = f(yv_i) + f(yv_{i+1}) + 3.$$

Thus,

$$\sum_{i=\frac{n+3}{2}}^{n} f(xv_i) = \frac{3(n-1)}{4} + \sum_{i=\frac{n+3}{2}}^{n} f(yv_i).$$

We conclude that

$$f(vx) + \sum_{i=1}^{n} f(v_ix) = f(vy) + \sum_{i=1}^{n} f(v_iy);$$

which completes the proof in this case.

**Case 4.** The case that $\frac{n+1}{2}$ is odd and $\frac{n-1}{2}$ is divisible by 3. In this case, we define

$$f(v_1x) = 2n + 4 - 2, \qquad f(vx) = n + 2,$$

$$f(v_1y) = \frac{5n+3}{2} + 1, \qquad f(vy) = n + 1.$$

For each $i$ with $2 \le i \le \frac{n+2}{3}$, put $f(v_ix)$ and $f(v_iy)$ as the following;

$$f(v_ix) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad 2 \le i \le \frac{n+2}{3} \text{ and } i \text{ is even} \\[2mm] 2n + 4 - 2i & \text{if} \quad 2 \le i \le \frac{n+2}{3} \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_i y) = \begin{cases} 2n + 4 - 2i & \text{if} \quad 2 \le i \le \frac{n+2}{3} \text{ and } i \text{ is even} \\ \\ \frac{5n+3}{2} + i & \text{if} \quad 2 \le i \le \frac{n+2}{3} \text{ and } i \text{ is odd.} \end{cases}$$

For each even integer $i$ with $2 \le i \le \frac{n+2}{3}$, the following equality holds;

$$f(y v_i) + f(y v_{i+1}) = f(x v_i) + f(x v_{i+1}) + 3.$$

This implies that

$$\sum_{i=2}^{\frac{n+2}{3}} f(y v_i) = \frac{n-1}{2} + \sum_{i=2}^{\frac{n+2}{3}} f(x v_i).$$

For each $i$ with $\frac{n+5}{3} \le i \le \frac{n+1}{2}$, we define $f(v_i x)$ and $f(v_i y)$ as follows;

$$f(v_i x) = \begin{cases} 2n + 4 - 2i & \text{if} \quad \frac{n+5}{3} \le i \le \frac{n+1}{2} \text{ and } i \text{ is even} \\ \\ \frac{5n+3}{2} + i & \text{if} \quad \frac{n+5}{3} \le i \le \frac{n+1}{2} \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_i y) = \begin{cases} \frac{5n+3}{2} + i & \text{if} \quad \frac{n+5}{3} \le i \le \frac{n+1}{2} \text{ and } i \text{ is even} \\ \\ 2n + 4 - 2i & \text{if} \quad \frac{n+5}{3} \le i \le \frac{n+1}{2} \text{ and } i \text{ is odd.} \end{cases}$$

Now, for each even integer $i$ that $\frac{n+5}{3} \le i \le \frac{n+1}{2}$ we have

$$f(x v_i) + f(x v_{i+1}) = f(y v_i) + f(y v_{i+1}) + 3.$$

So, we obtain

$$\sum_{i=\frac{n+5}{3}}^{\frac{n+1}{2}} f(x v_i) = \frac{n-1}{4} + \sum_{i=\frac{n+5}{3}}^{\frac{n+1}{2}} f(y v_i).$$

Now, it is time to determine $f(v_i x)$ and $f(v_i y)$ for those integers $i$ that $\frac{n+3}{2} \le i \le n$. Let us assign

$$f(v_i x) = \begin{cases} 3n + 4 - 2i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is even} \\ \\ \frac{3n+3}{2} + i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

and

$$f(v_i y) = \begin{cases} \frac{3n+3}{2} + i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is even} \\ \\ 3n + 4 - 2i & \text{if} \quad \frac{n+3}{2} \le i \le n \text{ and } i \text{ is odd.} \end{cases}$$

Since the equality $f(xv_i) + f(xv_{i+1}) = f(yv_i) + f(yv_{i+1}) + 3$ holds for each even integer $i$ that $\frac{n+3}{2} \leq i \leq n$, we have

$$\sum_{i=\frac{n+3}{2}}^{n} f(xv_i) = \frac{3(n-1)}{4} + \sum_{i=\frac{n+3}{2}}^{n} f(yv_i).$$

Accordingly,

$$f(vx) + \sum_{i=1}^{n} f(v_i x) = f(vy) + \sum_{i=1}^{n} f(v_i y);$$

and therefore, the proof is completed in the final case.         $\square$

The following corollary is an immediate consequence of Theorem 2.1.

**Corollary 2.2.** *For each positive integer $n$, there exists a graph $G_n$ such that*

$$\min\left\{\frac{\chi_{la}(G_n)}{\chi_{la}(G_n \vee \bar{K}_2)}, \chi_{la}(G_n) - \chi_{la}(G_n \vee \bar{K}_2)\right\} > n.$$

*In other words,* $\min\left\{\dfrac{\chi_{la}(G)}{\chi_{la}(G \vee \bar{K}_2)}, \chi_{la}(G) - \chi_{la}(G \vee \bar{K}_2)\right\}$ *can be made arbitrarily large.*

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**Saeed Shaebani**

School of Mathematics and Computer Science, Damghan University, P.O. Box
36716-41167, Damghan, Iran.

Email: shaebani@du.ac.ir