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FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring. In this paper we consider some relations between filter regular sequence, regular sequence and system of parameters over R-modules. Also we obtain some new results about cofinitness and cominimaxness of local cohomology modules.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H_I^i(M) = \varinjlim_{n\geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [5] or [3] for more details about local cohomology. The concept of filter regular sequence plays an important role in this paper. We say that a sequence $x_1, \ldots, x_n$ of elements of I, is an I-filter regular sequence on M, if

$$\operatorname{Supp}_R\Big(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M}\Big) \subseteq V(I),$$

for all $i = 1, \ldots, n$. Also, we say that an element $x \in I$ is an I- filter regular sequence on M if $\operatorname{Supp}_R(0 :_M x) \subseteq V(I)$. The concept of an I-filter regular sequence on M is a generalization of the concept of a filter

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regular sequence which has been studied in [18]. Both concepts coincide if I is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. In 1969, A. Grothendieck conjectured that if I is an ideal of R and M is a finitely generated R-module, then the R-modules $\operatorname{Hom}_R(R/I, H_I^i(M))$ are finitely generated for all $i \geq 0$. R. Hartshorne has provided a counterexample to this conjecture in [6]. Also he defined a module T to be I-cofinite if $\operatorname{Supp} T \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, T)$ is finitely generated for each $i \geq 0$ and he asked the following question.

For which rings R and ideals I are the modules $H_I^i(M)$ I-cofinite for all i and all finitely generated modules M?

Hartshorne proved that if I is an ideal of the complete regular local ring R and M a finitely generated R-module, then $H_I^i(M)$ is I-cofinite in two following cases:
(i) I is principal ideal, (see [6], Corollary 6.3),
(ii) I is prime ideal with $\dim R/I = 1$, (see [6], Corollary 7.7).
This subject was studied by several authors afterwards, (see [4], [11], [9], [19], [1] and [10]).

Some important results of this paper are as follows:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ be a finitely generated R-module of dimension $d \geq 1$. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be an $\mathfrak{m}$-filter regular sequence for M. Then the following statements are holds:*

(1) $x_1, \ldots, x_d$ is a system of parameters for M.
(2) For each $1 \leq i \leq d$, the R-module $H_{\mathfrak{m}}^i(M)$ is $(x_1, \ldots, x_i)$-cofinite.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Then for every finitely generated R-module $M \neq 0$ of dimension d, the following statements are equivalent:*

(1) $H_{\mathfrak{m}}^d(M)$ is I-cofinite.
(2) $H_{\mathfrak{m}}^d(M) \cong H_I^d(M)$.

Theorem 1.3. *Let R be a Noetherian ring, I an ideal of R and $M \neq 0$ be a finitely generated R-module such that $\dim \dfrac{M}{IM} \leq 1$. If $t \geq 1$ and $x_1, \ldots, x_t \in I$ is an I-filter regular sequence for M, then for each $0 \leq i \leq t-1$, the R-module $H_I^i(M)$ is $(x_1, \ldots, x_t)$-cofinite and $\operatorname{Hom}_R\left(\dfrac{R}{(x_1, \ldots, x_t)}, H_I^t(M)\right)$ is finitely generated.*

For each R-module L, we denote by $\mathrm{Assh}_R L$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Also, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$ and we denote $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$. Finally, for each R-module L, we denote by $\mathrm{mAss}_R L$, the minimal elements of $\mathrm{Ass}_R L$. For any unexplained notation and terminology we refer the reader to [3] and [12].

2. Main results

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ be a finitely generated R-module of dimension $d \geq 1$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be an $\mathfrak{m}$-filter regular sequence for M. Then*

(1) $x_1, \dots, x_d$ is a system of parameters for M.
(2) For each $1 \leq i \leq d$, the R-module $H^i_{\mathfrak{m}}(M)$ is $(x_1, \dots, x_i)$-cofinite.

Proof. (1). By definition $x_i \notin \cup_{P \in \mathrm{Ass}\left(\frac{R}{(x_1,\dots,x_{i-1})}\right)\setminus\{\mathfrak{m}\}} P$ for each $1 \leq i \leq d$, and so $x_i \notin \cup_{P \in \mathrm{Assh}_R\left(\frac{R}{(x_1,\dots,x_{i-1})}\right)} P$. Therefore $x_1, \dots, x_d$ is a system of parameters for M.
(2). By [8, Proposition 1.2], $H^j_{(x_1,\dots,x_i)}(M) \cong H^j_{\mathfrak{m}}(M)$ for each $0 \leq j \leq i-1$ and $\dim \mathrm{Supp}\, H^j_{(x_1,\dots,x_i)}(M) \leq 0$. Hence by [1, Theorem 2.6], the R-module $H^j_{(x_1,\dots,x_i)}(M)$ is $(x_1, \dots, x_i)$-cofinite. Also for $j > i$, $H^j_{(x_1,\dots,x_i)}(M) = 0$. Thus by [15, Proposition 3.11], the R-module $H^i_{(x_1,\dots,x_i)}(M)$ is also $(x_1, \dots, x_i)$-cofinite. Since $H^{i-1}_{(x_1,\dots,x_i)}(M)$ is Artinian, it follows from Grothendick vanishing theorem [3, Proposition 6.1], $H^1_{Rx_{i+1}}\big(H^{i-1}_{(x_1,\dots,x_i)}(M)\big) = 0$. By [17], there exists an exact sequence as follows $0 \to H^1_{Rx_{i+1}}\big(H^{i-1}_{(x_1,\dots,x_i)}(M)\big) \to H^i_{(x_1,\dots,x_{i+1})}(M) \to H^0_{Rx_{i+1}}\big(H^i_{(x_1,\dots,x_i)}(M)\big) \to 0$. Note that this exact sequence shows

$$H^i_{(x_1,\dots,x_{i+1})}(M) \cong H^0_{Rx_{i+1}}\big(H^i_{(x_1,\dots,x_i)}(M)\big).$$

Also by [9], we have

$$H^i_{(x_1,\dots,x_{i+1})}(M) \cong H^i_{\mathfrak{m}}(M).$$

Therefore

$$H^i_{\mathfrak{m}}(M) \cong H^0_{Rx_{i+1}}\big(H^i_{(x_1,\dots,x_i)}(M)\big)$$

and there exists an exact sequence as $0 \to H^i_{\mathfrak{m}}(M) \to H^i_{(x_1,\dots,x_i)}(M)$. Since $\mathrm{Hom}_R\left(\dfrac{R}{(x_1,\dots,x_i)}, H^i_{(x_1,\dots,x_i)}(M)\right)$ is finitely generated (because $H^i_{(x_1,\dots,x_i)}(M)$ is $(x_1, \dots, x_i)$-cofinite), it follows that the $R-$module

$\operatorname{Hom}_R\Big(\dfrac{R}{(x_1,\dots,x_i)}, H^i_{\mathfrak{m}}(M)\Big)$ is also finitely generated. Now, by [16, Theorem 1.6] and by Artinianess of $H^i_{\mathfrak{m}}(M)$, we conclude that $H^i_{\mathfrak{m}}(M)$ is $(x_1,\dots,x_i)$-cofinite. □

Theorem 2.2. *Let $(R,\mathfrak{m})$ be a complete Noetherian local ring and $M \neq 0$ be a finitely generated R-module of dimension $d \geq 1$. Let $P \in \operatorname{Ass} M$ be such that $\dim \dfrac{R}{P} = t \geq 1$. Then for any $\mathfrak{m}$-filter regular sequence for M such as $x_1,\dots,x_t \in \mathfrak{m}$, $\operatorname{Rad}(P+(x_1,\dots,x_t)) = \mathfrak{m}$. In particular $x_1,\dots,x_t$ is a system of parameters for $\dfrac{R}{P}$.*

Proof. By Cohen's theorem every complete Noetherian ring is a homomorphic image of a Gorenstein local ring. Then by [2], we have

$$\{q \in \operatorname{Att}_R H^t_{\mathfrak{m}}(M) \mid \dim \frac{R}{q} = t\} = \{q \in \operatorname{Ass} M \mid \dim \frac{R}{q} = t\}.$$

Since $P \in \operatorname{Ass} M$ and $\dim \dfrac{R}{P} = t$, it follows that $P \in \operatorname{Att} H^t_{\mathfrak{m}}(M)$. By the previous Theorem, the R-module $H^t_{\mathfrak{m}}(M)$ is $(x_1,\dots,x_t)$-cofinite and so by [16, Theorem 1.6], $\operatorname{Rad}(P+(x_1,\dots,x_t)) = \mathfrak{m}$. □

Theorem 2.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Then for every finitely generated R-module $M \neq 0$ of dimension d, the following statements are equivalent.*

(1) $H^d_{\mathfrak{m}}(M)$ is I-cofinite.
(2) $H^d_{\mathfrak{m}}(M) \cong H^d_I(M)$.

Proof. $1 \to 2$ Let $H^d_{\mathfrak{m}}(M)$ be I-cofinite module. Then $H^d_{\mathfrak{m}}(M) \otimes_R \hat{R}$ is also $I\hat{R}$-cofinite. Hence by [16, Theorem 1.6], for each $P \in \operatorname{Att}_{\hat{R}}\big(H^d_{\mathfrak{m}\hat{R}}(\hat{M})\big) = \operatorname{Assh}_{\hat{R}}(\hat{M})$, $\operatorname{Rad}(I\hat{R}+P) = \mathfrak{m}\hat{R}$ and so $H^d_{I\hat{R}}(\dfrac{\hat{R}}{P}) \neq 0$. Therefore $H^d_{I\hat{R}}(\hat{R}) \otimes_{\hat{R}} \dfrac{\hat{R}}{P} \neq 0$ and $P \in \operatorname{Att}_R H^d_{I\hat{R}}(\hat{R})$. Consequently $\operatorname{Att}_{\hat{R}} H^d_{\mathfrak{m}\hat{R}}(\hat{R}) \subseteq \operatorname{Att}_{\hat{R}} H^d_{I\hat{R}}(\hat{R}) \subseteq \operatorname{Att} H^d_{\mathfrak{m}\hat{R}}(\hat{R})$ and so $\operatorname{Att}_{\hat{R}}\big(H^d_{\mathfrak{m}\hat{R}}(\hat{R})\big) = \operatorname{Att}_{\hat{R}}\big(H^d_{I\hat{R}}(\hat{R})\big)$. Now by [7], $H^d_{\mathfrak{m}\hat{R}}(\hat{R}) \cong H^d_{I\hat{R}}(\hat{R})$. Hence we have the following:

$$H^d_{\mathfrak{m}}(R) \cong H^d_{\mathfrak{m}\hat{R}}(\hat{R}) \cong H^d_{I\hat{R}}(\hat{R}) \cong H^d_I(R)$$

$(2 \to 1)$. By [15], $H^d_I(M)$ is I-cofinite. Since $H^d_I(M) \cong H^d_{\mathfrak{m}}(M)$, it follows that $H^d_{\mathfrak{m}}(M)$ is I-cofinite. □

Corollary 2.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d and I be an ideal of R such that $H^d_{\mathfrak{m}}(R)$ is I-cofinite. Then $\operatorname{ara}(I) = d$.*

Proof. The module $H^d_{\mathfrak{m}}(R)$ is I-cofinite, hence $H^d_I(R) \cong H^d_{\mathfrak{m}}(R) \neq 0$ and so $\operatorname{ara}(I) \geq \operatorname{cd}(I, R) = d$. On the other hand by [14, Corollary 2.8], $\operatorname{ara}(I) \leq d$. □

Definition 2.5. Let I be an ideal of R. The arithmetic rank of I, denoted by $\operatorname{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I.

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 0$ and $x_1, \dots, x_{d-1} \in \mathfrak{m}$ be such that $I = (x_1, \dots, x_{d-1})$. Then $\operatorname{Hom}_R\left(\frac{R}{I}, H^d_{\mathfrak{m}}(R)\right)$ is not finitely generated.*

Proof. By [16, Theorem 1.6], the R-module $\operatorname{Hom}_R(\frac{R}{I}, H^d_{\mathfrak{m}}(R))$ is finitely generated if and only if $H^d_{\mathfrak{m}}(R)$ is I-cofinite. But in this case $\operatorname{ara}(I) = d$. On the other hand $\operatorname{ara}(I) \leq d - 1$ which is a contradiction. □

Proposition 2.7. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $M \neq 0$ be a finitely generated R-module. Let N be submodule of M such that $\dim N = t \geq 1$. Then any $\mathfrak{m}$-filter regular sequence for M such as $x_1, \dots, x_t \in \mathfrak{m}$ is a system of parameters for N.*

Proof. Let $m \operatorname{Ass}_R N = \{P_1, \dots, P_n\}$, where $m \operatorname{Ass}_R N$ denotes the minimal elements of $\operatorname{Ass}_R N$. Then for each $1 \leq i \leq n$, $\dim \frac{R}{P_i} \leq \dim N = t$ and clearly $\dim \frac{R}{P_i} \geq 1$. Let $j = \dim \frac{R}{P_i}$. Then $j \leq t$ and by Theorem 2.2, $\operatorname{Rad}(P_i + (x_1, \dots, x_j)) = \mathfrak{m}$. Since $(x_1, \dots, x_j) \subseteq (x_1, \dots, x_t)$, it follows that $\operatorname{Rad}(P_i + (x_1, \dots, x_t)) = \mathfrak{m}$. We claim that $\operatorname{Rad}(\cap_{i=1}^n P_i + (x_1, \dots, x_t)) = \mathfrak{m}$. For this, let Q be a minimal prime of $\cap_{i=1}^n P_i + (x_1, \dots, x_t)$. Hence there exists $1 \leq j \leq n$ such that $P_j \subseteq Q$ and so $p_j + (x_1, \dots, x_n) \subseteq Q$. Therefore $\mathfrak{m} = \operatorname{Rad}(P_j + (x_1, \dots, x_t) \subseteq \operatorname{Rad}(Q) = Q \subseteq \mathfrak{m}$ and consequently $Q = \mathfrak{m}$. But $\cap_{i=1}^n P_i = \operatorname{Rad}(\operatorname{Ann} N)$ shows that

$\operatorname{Rad}(\operatorname{Ann} N + (x_1, \dots, x_t)) = \mathfrak{m}$ and so $\dim_R \frac{N}{(x_1, \dots, x_t)N} = 0$.

This completes the proof that $x_1, \dots, x_t$ is a system of parameters for N. □

Corollary 2.8. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, M be a finitely generated R-module and N be a submodule of M which is a Cohen-Macaulay with $\dim N = t$. If $x_1, \dots, x_t \in \mathfrak{m}$ is an $\mathfrak{m}$-filter regular sequence for M, then $x_1, \dots, x_t$ is a N-regular sequence.*

Proof. By Proposition 2.7, $x_1, \dots, x_t$ is a system of parameters for N. But N is a Maximal Cohen-Macaulay as an $\dfrac{R}{\operatorname{Ann} N}$-module. Also $x_1 + \operatorname{Ann} N, \dots, x_t + \operatorname{Ann} N$ is a system of parameters for $\dfrac{R}{\operatorname{Ann} N}$. On the other hand every maximal Cohen-Macaulay as an $\dfrac{R}{\operatorname{Ann} N}$-module is a balaneced big Cohen-Macaulay as an R-module. Set $y_i = x_i + \operatorname{Ann} N$ for each $1 \leq i \leq t$, then $y_1, \dots, y_t$ is an N-regualar sequence and this follows that $x_1, \dots, x_t$ is an N-regular sequence. □

Theorem 2.9. *Let R be a Noetherian ring, I an ideal of R and $M \neq 0$ be a finitely generated R-module such that $\dim \dfrac{M}{IM} \leq 1$. If $t \geq 1$ and $x_1, \dots, x_t \in I$ is an I-filter regular sequence for M, then for each $0 \leq i \leq t-1$, the R-module $H^i_I(M)$ is $(x_1, \dots, x_t)$-cofinite and $\operatorname{Hom}_R\left(\dfrac{R}{(x_1, \dots, x_t)}, H^t_I(M)\right)$ is finitely generated.*

Proof. For each $0 \leq i \leq t-1$, we have $H^i_{(x_1,\dots,x_t)}(M) \cong H^i_I(M)$. Then

$$\operatorname{Supp} H^i_{(x_1,\dots,x_t)}(M) = \operatorname{Supp} H^i_I(M) \subseteq \operatorname{Supp} \frac{M}{IM}$$

and for each $0 \leq i \leq t-1$, $\dim \operatorname{Supp} H^i_{(x_1,\dots,x_t)}(M) \leq 1$. By [1], clearly the R-module $H^i_{(x_1,\dots,x_{t-1})}$ is $(x_1, \dots, x_t)$-cofinite. Since $H^i_{(x_1,\dots,x_t)}(M) = 0$ for all $i \geq t+1$, it follows from [15], that $H^t_{(x_1,\dots,x_t)}(M)$ is also $(x_1, \dots, x_t)$-cofinite. Consequently for each $i \geq 0$, the R-module $H^i_{(x_1,\dots,x_t)}(M)$ is $(x_1, \dots, x_t)$-Cofinite. Now, let $x_{t+1} \in I$ be such that $x_1, \dots, x_{t+1}$ is I-filter regular sequence. Since $x_{t+1} \in I$ and $H^{t-1}_{(x_1,\dots,x_t)}(M) \cong H^{t-1}_I(M)$ is I-torsion, then $H^1_{Rx_{t+1}}\big(H^{t-1}_{(x_1,\dots,x_t)}(M)\big) = 0$. On the other hand by [17], the following exact sequence is hold: $0 \to H^1_{Rx_{t+1}}\big(H^{t-1}_{(x_1,\dots,x_t)}(M)\big) \to H^t_{(x_1,\dots,x_{t+1})}(M) \to H^0_{Rx_{t+1}}\big(H^t_{(x_1,\dots,x_t)}(M)\big) \to 0$. But, $H^t_{(x_1,\dots,x_t)}(M) \cong H^t_I(M)$ and so by the above exact sequence, $H^t_I(M) \cong H^0_{Rx_{t+1}}\big(H^t_{(x_1,\dots,x_t)}(M)\big)$. Since $Rx_{t+1} \subseteq I$, it follows that

$$H^0_I\big(H^t_{(x_1,\dots,x_t)}(M)\big) \subseteq H^0_{Rx_t}\big(H^t_{(x_1,\dots,x_t)}(M)\big).$$

Also, $H^0_{Rx_{t+1}}\big(H^t_{(x_1,\dots,x_t)}(M)\big) \cong H^t_I(M)$ is I-torsion and hence

$$H^0_{Rx_{t+1}}\big(H^t_{(x_1,\dots,x_t)}(M)\big) \subseteq H^0_I\big(H^t_{(x_1,\dots,x_t)}(M)\big).$$

Then

$$H^t_I(M) \cong \Gamma_{Rx_{t+1}}\big(H^t_{(x_1,\dots,x_t)}(M)\big) = \Gamma_I\big(H^t_{(x_1,\dots,x_t)}(M)\big).$$

Finally from the exact sequence

$$0 \to H^t_I(M) \cong H^0_I\big(H^t_{(x_1,\dots,x_t)}(M)\big) \to H^t_{(x_1,\dots,x_t)}(M)$$

and $(x_1,\dots,x_t)$-cofinitness of $H^t_{(x_1,\dots,x_t)}(M)$, we conclude that $\operatorname{Hom}_R\Big(\dfrac{R}{(x_1,\dots,x_t)}, H^t_I(M)\Big)$ is finitely generated. □

Lemma 2.10. *Let M be an R-module and I be an ideal of R such that $\operatorname{Supp} M \subseteq V(I)$. Let $x \in I$ be such that $0 :_M x$ and M/xM are I-cominimax. Then so is M.*

Proof. The proof is similar to the proof of [15, Corollary 3.4]. □

Theorem 2.11. *With the assumption of Theorem 2.9, the R-module $H^t_I(M)$ is $(x_1,\dots,x_t)$-cominimax.*

Proof. We prove by induction on t. If $t = 1$, then we set $N = \dfrac{M}{\Gamma_I(M)}$ and so x_1 is an N-regular element and $H^1_I(N) \cong H^1_I(M)$.

Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{x_1} N \longrightarrow \frac{N}{x_1N} \longrightarrow 0$$

which implies that the following exact sequence

$$\dots \longrightarrow H^0_I(\frac{N}{x_1N}) \longrightarrow H^1_I(N) \xrightarrow{x_1} H^1_I(N) \longrightarrow H^1_I(\frac{N}{x_1N})$$

Clearly the R-module $0 :_{H^1_I(N)} x_1$ is finitely generated, and Rx_1-cominimax. Set

$$T = \{P \in \operatorname{Supp} H^1_I(N) \mid \dim \frac{R}{P} = 1\}.$$

Then $\big(H^1_I(N)\big)_P$ for all $P \in T$ is Artinian and Rx_1-cofinite. Also $T \subseteq \operatorname{Assh} \dfrac{M}{IM}$ and so is finite. By argument in [1, Theorem 2.6], $\dfrac{H^1_I(N)}{x_1H^1_I(N)}$ is minimax. Also $\dfrac{H^1_I(N)}{x_1H^1_I(N)}$ and $0 :_{H^1_I(N)} x_1$ are Rx_1-cominimax and hence $H^1_I(N)$ is also Rx_1-cominimax.

Now, let $t \geq 2$. Clearly $x_1,\dots,x_t$ is I-filter regular sequence over the R-module $\dfrac{M}{\Gamma_I(M)}$. Now $H^t_I(M) \cong H^t_I\big(\dfrac{M}{\Gamma_I(M)}\big)$ and $\dfrac{M}{\Gamma_I(M)}$ is a finitely generated I-torsion free R-module. We therefore assume in addition that $\Gamma_I(M) = 0$. Since $x_1 \notin \cup_{P\in \operatorname{Ass} M\setminus V(I)} P = \cup_{P \in \operatorname{Ass}(M)} P$, it follows that $(x_1,\dots,x_t) \nsubseteq \cup_{P\in\operatorname{Ass} M} P$.

Set $T := \{P \in \operatorname{Supp} H_I^{t-1}(M) \cup \operatorname{Supp} H_I^t(M) \mid \dim \frac{R}{P} = 1\}$. Hence $T \subseteq \operatorname{Assh}_R \frac{M}{IM}$, and so T is a finite set. Let $T = \{P_1, \ldots, P_n\}$. Then for each $i \geq 0$, $\operatorname{Supp} H^i_{IR_{P_k}}(M_{P_k}) \subseteq \{P_k R_{P_k}\}$, where $k = 1, 2, \ldots, n$. By [1], for each $t-1 \leq k \leq t$, $H^i_{IR_{P_k}}(M_{P_k})$ is R_{P_k}-Artinian and $(x_1, \ldots, x_t)R_{P_k}$-cofinite. Also

$$V\big((x_1, \ldots, x_t)R_{P_k}\big) \cap \operatorname{Att}_{R_{P_k}} H^i_{IR_{P_k}}(M_{P_k}) \subseteq V(P_k R_{P_k}).$$

Set

$$U := \cup_{i=t-1}^{t} \cup_{k=1}^{n} \left\{q \in \operatorname{Spec}(R) \mid qR_{P_k} \in \operatorname{Att}_{R_{P_k}} \left(H^i_{IR_{P_k}}(M_{P_k})\right)\right\}.$$

Therefore $U \cap V(x_1, \ldots, x_t) \subseteq T$. Since $(x_1, \ldots, x_t) \nsubseteq (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \operatorname{Ass} M} P)$, it follows that there exists an element $z_1 \in (x_1, \ldots, x_t)$ such that $x_1 + z_1 \notin (\cup_{q \in U \setminus V(I)} q) \cup (\cup_{P \in \operatorname{Ass} M} P)$.

Assume that $y_1 = x_1 + z_1$, then $(x_1, \ldots, x_t) = (y_1, x_2, \ldots, x_t)$ and $y_1 \in I$ is an I-filter regular sequence.

Now if $(x_1, \ldots, x_t) = (y_1, x_2, \ldots, x_t) \subseteq \cup_{P \in \left(\operatorname{Ass} \frac{R}{y_1 R}\right) \setminus V(I)} P$, then there exists $P \in (\operatorname{Ass} \frac{R}{y_1 R}) \setminus V(I)$ such that $(x_1, \ldots, x_t) \subseteq P$.

Since $I \nsubseteq P$, it follows that $\frac{x_1}{1}, \ldots, \frac{x_t}{1} \in PR_P$ is a R_P-reguler sequence and so $\operatorname{grade}\Big((\frac{x_1}{1}, \ldots, \frac{x_t}{1}, R_P)\Big) = t$. On the other hand $PR_P \in \operatorname{Ass} \frac{R}{y_1 R}$ and $(y_1, x_2, \ldots, x_t)R_P \subseteq PR_P$.

Then $\operatorname{grade}\big((y_1, x_2, \ldots, x_t)R_P, R_P\big) = 1$ if $t \geq 2$, and so $(y_1, x_2, \ldots, x_t) \nsubseteq \cup_{P \in \operatorname{Ass} \frac{R}{y_1 R}} P$. Hence there exists an element $z_2 \in (y_1, x_2, \ldots, x_t)$ such that $x_2 + z_2 \notin \cup_{P \in \operatorname{Ass} \frac{R}{y_1 R}} P$. Again, we put $y_2 = x_2 + z_2$, then $(y_1, x_2, \ldots, x_t) = (y_1, y_2, x_3, \ldots, x_t)$. By the similer argument in the above, we see that there exist elements $y_1, \ldots, y_t \in I$ such that $(x_1, \ldots, x_t) = (y_1, \ldots, y_t)$ and $y_1, \ldots, y_t$ is an I-filter regular sequene for M.

The exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow \frac{M}{y_1 M} \longrightarrow 0$$

induces a short exact sequence of local cohomology modules

$$0 \longrightarrow \frac{H_I^{t-1}(M)}{y_1 H_I^{t-1}(M)} \longrightarrow H_I^{t-1}(\frac{M}{y_1 M}) \longrightarrow 0 :_{H_I^t(M)} y_1 \longrightarrow 0$$

By a similar proof in [1], we see that $\frac{H_I^{t-1}(M)}{y_1 H_I^{t-1}(M)}$ is a minimax R-module.

Now, by induction hypothesis and since $y_2, \ldots, y_t$ is an I-filter regular sequence for $\frac{M}{y_1 M}$, we conclude that the R-module $H_I^{t-1}(\frac{M}{y_1 M})$ is $(y_2, \ldots, y_t)$-cominimax. Also, we note that $(y_2, \ldots, y_t) \subseteq (y_1, \ldots, y_t)$ and also $\operatorname{Supp} H_I^{t-1}(\frac{M}{y_1 M}) \subseteq V(y_1, \ldots, y_t)$. Therefore $H_I^{t-1}(\frac{M}{y_1 M})$ is $(y_1, \ldots, y_t)$-cominimax. Consequently by the above exact sequence $0 :_{H_I^t(M)} y_1$ is also $(y_1, \ldots, y_t)$-cominimax. On the other hand by argument in [1, Theorem 2.6], the R-module $\frac{H_I^t(M)}{y_1 H_I^t(M)}$ is minimax and hence is $(y_1, \ldots, y_t)$-cominimax.

Finally, $y_1 \in (y_1, \ldots, y_t) = (x_1, \ldots, x_t)$ and the R-modules $0 :_{H_I^t(M)} y_1$ and $\frac{H_I^t(M)}{y_1 H_I^t(M)}$ are both $(x_1, \ldots, x_t)$-cominimax. Thus by lemma 2.9, the R-module $H_I^t(M)$ is also $(x_1, \ldots, x_t)$-cominimax. $\square$

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FILTER REGULAR SEQUENCES AND LOCAL COHOMOLOGY MODULES

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رشته‌های صافی منظم و مدول‌های کوهومولوژی موضعی

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فرض کنید R یک حلقه جابجایی و نوتری با عنصر همانی ناصفر باشد. در این مقاله برخی روابط بین رشته‌های صافی منظم، رشته‌های منظم و دستگاه پارامتری را روی R-مدول‌ها بررسی می‌کنیم. همچنین نتایج جدیدی را در ارتباط با هم‌متناهی بودن و هم‌می‌نیماکس بودن مدول‌های کوهومولوژی موضعی به‌دست می‌آوریم.

کلمات کلیدی: رشته‌های صافی منظم، رشته‌های منظم، دستگاه پارامتری، مدول کوهومولوژی موضعی.