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ORDER DENSE ESSENTIALITY AND BEHAVIOR OF ORDER DENSE INJECTIVITY

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Abstract. In this paper, we study the categorical and algebraic properties, such as limits and colimits of the category **Pos**-S with respect to order dense embeddings. Injectivity with respect to this class of monomorphisms has been studied by the author and used to obtain information about injectivity relative to order embeddings. Then, we study three different kinds of essentiality, usually used in literature, with respect to the class of all order dense embeddings of S-posets, and investigate their relations to order dense injectivity. We will see, among other things, that although all of these essential extensions are not necessarily equivalent, they behave equivalently with respect to order dense injectivity. More precisely, it is proved that order dense injectivity well behaves regarding these essentialities. Finally, a characterization of these essentialities over pogroups is given.

1. Introduction

The study of injectivity with respect to different classes of monomorphisms is crucial in almost all categories. It is well known that there is no non-trivial injective object with respect to monomorphisms in the categories **Pos** of posets and **Pos**-S of S-posets. The most natural monomorphisms in ordered structures are order embeddings which are the equalizers (regular monomorphisms). That is why researchers study different types of injectivity with respect to different types of embeddings instead of general monomorphisms. For example, in [8], [12],

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[15], and [16] injectivity of S-posets with respect to order embeddings has been studied and [8] shows that there are enough regular injective S-posets. There are three main propositions given by Banaschewski in relation to injectivity that are called Well-Behavior Theorems of Injectivity. These propositions are about the relations between the notion of injectivity, absolute retract, essential extension, and injective hull. Also, there are three different definitions of essentiality usually used in literature with respect to a subclass of monomorphisms. Ebrahimi et al. in [5, 6] and Barzegar et al. in [2] studied the behavior of $\mathcal{M}$-injectivity with respect to these three different types of essentiality relative to a subclass $\mathcal{M}$ of monomorphisms of a category in general. In this paper, we study the categorical and algebraic properties, such as limits and colimits of the category **Pos**-S with respect to order dense embeddings. Injectivity with respect to this class of monomorphisms is related to regular injectivity and this motivates us to study this kind of injectivity. Also, three kinds of essentiality with respect to the class of all order dense embeddings of S-posets are studied and their relations to order dense injectivity are investigated. It is proved among other things, that although all of these essential extensions are not necessarily equivalent, they behave equivalently with respect to order dense injectivity. More precisely, it is proved that order dense injectivity well behaves regarding these essentialities. Finally, a characterization of these essentialities over pogroups is given.

First we briefly recall the definition and the categorical and algebraic ingredients of the category **Pos**-S of (right) S-posets needed in the sequel. For more information see [4], [7] and [9]. Recall that a monoid (semigroup) S is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A (*right*) *S-poset* over a pomonoid (or, a posemigroup S) is a poset A which is also an S-act whose action $\lambda : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order.

An *S-poset map* (or *S-poset morphism*) is an action preserving monotone map between S-posets. Moreover, regular monomorphisms (equalizers) are exactly *order embeddings* (briefly, *embeddings*); that is, (mono) morphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$.

Let A be an S-poset. A (*right*) *S-poset congruence* on A is a (right) S-act congruence θ, that is, an equivalence relation on A which is closed under S-action, with the property that the S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map. The set of all S-poset congruences on A is denoted by $Con(A)$. For a binary relation β on A, define a relation $\leq_\beta$ on A by $a \leq_\beta a'$ if and only if $a \leq a_1 \beta a'_1 \leq ... \leq a_n \beta a'_n \leq a'$ for some $a_1, a'_1, ..., a_n, a'_n \in A, n \in \mathbb{N}$. Then an S-act congruence θ on A is an S-poset congruence if and only if for every $a, a' \in A$, $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by $[a]_\theta \leq [a']_\theta$ if and only if $a \leq_\theta a'$. Clearly, $a \leq a'$ implies that $a \leq_\theta a'$.

Recall that the product of a family of S-posets is their cartesian product, with componentwise action and order. The coproduct is their disjoint union, with natural action and componentwise order. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. Also for a family $(A_i)_{i\in I}$ of S-posets each of which has a unique fixed element 0, the direct sum $\bigoplus_i A_i$ is defined to be the sub-S-poset of the product $\prod_i A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = 0$ for all $i \in I$ except finitely many number of indices.

Recall from [11] that, an order embedding $A \xrightarrow{m} B$ of S-posets is called *regular essential* if $f : B \to C$ is an order embedding whenever fm is an order embedding and is called *mono-essential* if $f : B \to C$ is a monomorphism whenever fm is a monomorphism.

We recall the following definition of sublimit and subcolimit in a category from [4].

Definition 1.1. Let the functor $D : I \to \mathbf{Pos}\text{-}S$ be an *ordered diagram* in the sense that $Mor(I)$ is a poset, where I is a small category. A family $(f_i : A \to Di)_{i\in I'}$, where $I' = \{i \in I : i = dom(d),$ for some $d \in Mor(I), d \neq \mathrm{id}\}$ is called a *subsource* for D. If also, for every pair of I-morphisms $d \leq d'$ in I where $d : i \to j$ and $d' : k \to j$, we have $Dd \circ f_i \leq Dd' \circ f_k$, then we call the family a *natural subsource* for D. Here, the order relation $\leq$ between S-poset maps is defined pointwise. A natural subsource $(f_i : A \to Di)_{i\in I'}$ which has the universal property that, for every natural subsource $(g_i : B \to Di)_{i\in I'}$ there exists a unique S-poset map $h : B \to A$ such that $f_i \circ h = g_i$ for all $i \in I'$, is said to be a *sublimit* of the ordered diagram D.

In particular, a sublimit of a discrete diagram

$$d \circlearrowleft \bullet \qquad \bullet \circlearrowright d'$$

in which $d, d' \neq \mathrm{id}$ but $Dd = \mathrm{id}_{Di}, Dd' = \mathrm{id}_{Dj}$, is called a *subproduct* of Di and Dj, which coincides with the product of Di and Dj. A sublimit of a diagram

$$\bullet \underset{d'}{\overset{d}{\rightrightarrows}} \bullet$$

in which $d \leq d'$ is called a *subequalizer* of Dd and Dd', and a sublimit of a diagram

$$\bullet \xrightarrow{d} \bullet \xleftarrow{d'} \bullet$$

in which $d \leq d'$ is called a *subpullback* of Dd and Dd'. Note that a subequalizer or subpullback of Dd' and Dd may be different from that of Dd and Dd'.

Dually, subcolimits, and in particular, subcoproducts, subcoequalizers, and subpushouts are defined. Notice that the universal property given above guarantees that sublimits and subcolimits are unique.

2. Main results

2.1. Categorical properties of order dense embeddings.

To study mathematical notions, such as injectivity, tensor product, and flatness in any category $\mathcal{A}$, one needs to have some categorical and algebraic information about the pair $(\mathcal{A}, \mathcal{M})$, where $\mathcal{M}$ is a class of (mono)morphisms. In this section, after recalling the definition of order dense embeddings, we study some categorical and algebraic properties of the category **Pos**-S with respect to these monomorphisms. We study the composition, limit, and colimit properties in the following three subsections.

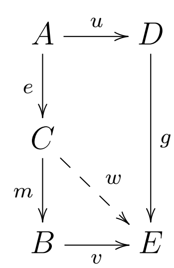
Definition 2.1. A sub-S-poset A of an S-poset B is called *order dense* in B if for each $b \in B$ there exists $a \in A$ with $b \leq a$. By an *order dense embedding*, we mean an order embedding $f : A \to B$ such that $f(A)$ is an order dense sub-S-poset of B. Also, a sub-S-poset A of an S-poset B is called *down closed* in B if for each $a \in A$ and $b \in B$ with $b \leq a$ one has $b \in A$. By a *down closed embedding*, we mean an order embedding $f : A \to B$ such that $f(A)$ is a down closed sub-S-poset of B.
The class of all order dense embeddings is denoted by $\mathcal{M}_{od}$.

2.1.1. *Composition properties of order dense embeddings.* Here, we investigate some composition properties of the class $\mathcal{M}_{od}$ of order dense embeddings. These properties and the ones given in the rest of this section are what normally used to study injectivity, and of course other mathematical notions.

Remark 2.2. (1) Notice that all isomorphisms are order dense and the composition of an isomorphism with an order dense embedding is an order dense embedding. Also, each surjective S-poset map is order dense and the composition of two order dense embeddings is an order dense embedding.

(2) The class $\mathcal{M}_{od}$ is right cancellable, in the sense that for S-poset maps f and g if $gf \in \mathcal{M}_{od}$ then $g \in \mathcal{M}_{od}$.
(3) The class $\mathcal{M}_{od}$ is not left cancellable. For example, consider the S-poset maps $\mathbf{2} \overset{f}{\hookrightarrow} \mathbf{2}\dot{\cup}\mathbf{1} \overset{g}{\to} \mathbf{3}$ with trivial actions of a pomonoid S on $\mathbf{2}, \mathbf{2}\dot{\cup}\mathbf{1}$ and $\mathbf{3}$ where $\mathbf{1} = \{c\}$ and $\mathbf{2} = \{a, b\}, \mathbf{3} = \{a, b, c\}$ are one, two and three element chains, respectively, f is inclusion and g is given by $g(a) = a, g(b) = g(c) = c$. Then $gf \in \mathcal{M}_{od}$ but f is not in $\mathcal{M}_{od}$.

Proposition 2.3. *Let $f : A \to B$ be an S-poset map. Then there are S-poset maps e, m such that:*
(1) *$f = me$ with $e \in \mathcal{M}_{od}$, and*
(2) *for any commutative rectangular*



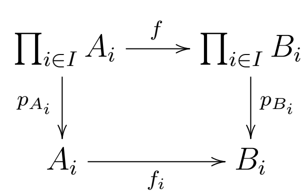
in **Pos**-*S, there is an S-poset map $w : C \to E$ with $gu = we$ and $vm = w$.*

Proof. Take $f : A \to B$, and let $C = \downarrow f(A)$ where $\downarrow f(A) = \{b \in B : \exists a \in A, b \leq f(a)\}$. Define $e : A \to C$ by $e(a) = f(a)$ for $a \in A$, and take $m : C \to B$ to be the inclusion map. Then $f = me$. This proves (1). To see (2), define $w : C \to E$ by $w(f(a)) = gu(a)$, $w(b) = v(b), b \in \downarrow f(A) - f(A)$. Then clearly w is well-defined. It is clear, by the definition of w, that $gu = we$ and $vm = w$. □

2.1.2. *Limits of order dense embeddings.* Here, we will investigate the behavior of order dense embeddings with respect to limits.

Proposition 2.4. *$\mathcal{M}_{od}$ is closed under products.*

Proof. Let $(f_i : A_i \to B_i)_{i \in I}$ be a family of order dense embeddings. Consider the commutative diagram



which f exists by the universal property of products. It is easily proved that f is an order embedding. So we show that f is an order dense map. We have to show that for each $b = (b_i)_{i\in I} \in \prod_{i\in I} B_i$ there exists an element $a \in \prod_{i\in I} A_i$ such that $b \leq f(a)$. Since each f_i is order dense and for each $i \in I$, $b_i \in B_i$, there exists $a_i \in A_i, b_i \leq f_i(a_i)$. Now $b = (b_i)_{i\in I} \leq f(a) = (f_i(a_i))_{i\in I}$. Hence f is order dense. □

Recall that the *pullback* of S-poset maps $f : A \to C$ and $g : B \to C$ is the sub-S-poset $P = \{(a, b) : f(a) = g(b)\}$ of $A \times B$, together with the restricted projection maps. By substituting "=" in the definition of pullback P by "≤ ", the *subpullback* of f and g is obtained (see [4]).

In the definition of pullback, if A and B are disjoint S-posets, the pullback of f and g is different from that of g and f.

Recall that a class of morphisms of a category is called *subpullback stable* if subpullbacks transfer those morphisms. In the next result, we prove this property for order dense embeddings of S-posets.

Proposition 2.5. *The class $\mathcal{M}_{od}$ is subpullback stable.*

Proof. Consider the subpullback diagram

$$\begin{array}{ccc} P & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

where P is the sub-S-poset $\{(a, b) : f(a) \leq g(b)\}$ of $A \times B$, and subpullback maps $p_A : P \to A, p_B : P \to B$ are restrictions of the projection maps. Assume that $f, g \in \mathcal{M}_{od}$. We show that $p_A, p_B \in \mathcal{M}_{od}$. By [10], p_A and p_B are order embeddings. Now, let $b \in B$. Then $g(b) \in C$. Since f is order dense there exists $a \in A$ such that $g(b) \leq f(a)$. Now, there exists $b' \in B$ such that $f(a) \leq g(b')$ since g is order dense. Thus $g(b) \leq g(b')$ and then $b \leq b'$ since g is an order embedding. Therefore, $(a, b') \in P$ and $b \leq b' = p_B(a, b')$ which shows that p_B is an order dense embedding. Similarly, $p_A \in \mathcal{M}_{od}$. □

Definition 2.6. The subclass $\mathcal{M}$ of monomorphisms is *closed under $\mathcal{M}$-subpullbacks* if in the following subpullback diagram

$$\begin{array}{ccc} P & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

with $g, f \in \mathcal{M}$ one has $gp_B, fp_A \in \mathcal{M}$.

Corollary 2.7. *The class of order dense embeddings is closed under $\mathcal{M}_{od}$-subpullbacks.*

Proof. Since the composition of order dense embeddings is an order dense embedding the result holds by the above proposition. $\square$

Let $\mathcal{A} : \mathbf{I} \to \mathbf{Pos}\text{-}S$ be a diagram in $\mathbf{Pos}$-S determining the S-posets A_α, for $\alpha \in I = Obj(\mathbf{I})$, and S-poset maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor(\mathbf{I})$. Recall that the limit of this diagram is $A = \varprojlim_\alpha A_\alpha := \bigcap_{\alpha \in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha\in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta} p_\alpha(a) = p_\beta(a)\}$ and p_α, p_β are the α, βth projection maps of the product. Also, the limit S-maps are $q_\alpha =: p_\alpha \mid_A : \varprojlim_\alpha A_\alpha \to A_\alpha$.

Proposition 2.8. *$\mathcal{M}_{od}$ is closed under limits.*

Proof. Let $\mathcal{A}, \mathcal{B} : \mathbf{I} \to \mathbf{Pos}\text{-}S$ be diagrams in $\mathbf{Pos}$-S determining the S-posets A_α, B_α, for $\alpha \in I = Obj(\mathbf{I})$, and S-poset maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, $g'_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta$ in $Mor(\mathbf{I})$. Consider limits of these diagrams with limit maps $q_\alpha : \varprojlim A_\alpha \to A_\alpha$, $q'_\alpha : \varprojlim B_\alpha \to B_\alpha$. Let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of order dense embeddings such that $g'_{\alpha\beta} f_\alpha = f_\beta g_{\alpha\beta}$. Let f denote $\varprojlim f_\alpha : \varprojlim A_\alpha \to \varprojlim B_\alpha$ which exists by the universal property of limits. We show that f belongs to $\mathcal{M}_{od}$. Consider the following diagram

$$\begin{array}{ccc} \varprojlim A_\alpha & \xrightarrow{\ f\ } & \varprojlim B_\alpha \\ {\scriptstyle q_\alpha}\downarrow & & \downarrow{\scriptstyle q'_\alpha} \\ A_\alpha & \xrightarrow[f_\alpha]{} & B_\alpha. \end{array}$$

Let $f(a) \le f(a'), a, a' \in \varprojlim A_\alpha$. Then $f_\alpha q_\alpha(a) = q'_\alpha f(a) \le q'_\alpha f(a') = f_\alpha q'_\alpha(a')$ and so $q_\alpha(a) \le q_\alpha(a')$ since for each α, f_α is an order embedding. Hence $(a_\alpha)_\alpha \le (a'_\alpha)_\alpha$ which means that f is an order embedding. Now, let $(b_\alpha)_\alpha \in \varprojlim B_\alpha$. Since for each α, f_α is order dense, for each α there exists $a_\alpha \in A_\alpha$ such that $b_\alpha \le f_\alpha(a_\alpha)$. Then $q'_\alpha((b_\alpha)) \le f_\alpha(q_\alpha(a_\alpha)_\alpha) = q'_\alpha f((a_\alpha)_\alpha)$ and hence $(b_\alpha)_\alpha \le f((a_\alpha)_\alpha)$. Therefore, f is an order dense embedding. $\square$

2.1.3. *Colimits of order dense embeddings.* This subsubsection is devoted to the study of the behavior of order dense embeddings with respect to colimits.

Proposition 2.9. *$\mathcal{M}_{od}$ is closed under coproducts.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u'_i} \\ \coprod_{i\in I} A_i & \xrightarrow[f]{} & \coprod_{i\in I} B_i \end{array}$$

in which $\{f_i : A_i \to B_i\}_{i\in I}$ is a family of order dense embeddings. We want to show that the coproduct morphism $f = \coprod_{i\in I} f_i : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ (which uniquely exists by the universal property of coproducts) is an order dense embedding. By Proposition 5 of [10], f is an order embedding thus it is enough to show that f is order dense. For, let $b \in \coprod_{i\in I} B_i$. Then there exists $i \in I$, $b_i \in B_i$ such that $b = u'_i(b_i) = (b_i, i)$. Since f_i is order dense, there exists $a_i \in A_i$ with $b_i \leq f_i(a_i)$. Now, $b = (b_i, i) = u'_i(b_i) \leq u'_i f_i(a_i) = f u_i(a_i) = f(a_i, i)$. Hence f is order dense. $\square$

The following is an immediate corollary of Proposition 2.4.

Corollary 2.10. *$\mathcal{M}_{od}$ is closed under direct sums.*

Definition 2.11. The category $\mathcal{A}$ is said to satisfy the $\mathcal{M}$-transferability property, for a subclass $\mathcal{M}$ of monomorphisms, if for all $f \in \mathcal{A}$ and $m \in \mathcal{M}$ with common domain there is a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle u} \\ \bullet & \xrightarrow[g]{} & \bullet \end{array}$$

with $u \in \mathcal{M}$.

Remark 2.12. The notion of $\mathcal{M}$-transferability property is used by universal algebrists whereas category theorists prefer "$\mathcal{M}$'s are preserved by pushouts" or "pushouts transfer $\mathcal{M}$'s" which is the same, provided that pushouts exist and $\mathcal{M}$ is left cancellable.

First recall from [4] that the pushout of S-poset maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct $B \sqcup C = (\{1\} \times B) \cup (\{2\} \times C)$ by the S-poset congruence $\theta(H)$ generated by $H = \{((1, f(a)), (2, g(a))) : a \in A\}$ with S-poset maps $q_B = \pi u_B : B \to (B \sqcup C)/\theta$, $q_C = \pi u_C : C \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the natural epimorphism, and $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are coproduct injections.

Recall the following lemma from [14].

Lemma 2.13. *In the category* **Pos**-S*, pushouts transfer order dense embeddings.*

Note that since the composition of order dense embeddings is an order dense embedding, one has the following corollary.

Corollary 2.14. *The pushout of order dense embeddings belongs to* $\mathcal{M}_{od}$.

Recalling the following lemma from [10] we have the following result.

Lemma 2.15. *Multiple pushouts transfer order embeddings.*

Proposition 2.16. *Multiple pushouts transfer order dense embeddings. Also, multiple pushout of order dense embeddings is an order dense embedding.*

Proof. Let $(Q, (A_i \overset{q_i}{\to} Q)_{i\in I})$ be the multiple pushout of a family $\{f_i : A \to A_i | i \in I\}$ of order dense embeddings. Recall that $Q = (\coprod A_i)/\theta$, where $\theta = \theta(H)$ is the S-poset congruence on $\coprod A_i$ generated by $H = \{(u_i(f_i(a)), u_j(f_j(a)))|a \in A, i, j \in I\}$, and $q_i = \pi u_i, i \in I$, where $\pi : \coprod A_i \to Q$ and $u_i : A_i \to \coprod A_i$ are the natural epimorphism and coproduct injections, respectively. We take $\alpha \in I$ and prove that q_α is an order dense embedding. By Lemma 2.15, it is enough to show that q_α is an order dense map. Let $[x]_\theta \in (\coprod A_i)/\theta$, then there exists $i \in I$ and $a_i \in A_i$ such that $[x]_\theta = [(i, a_i)]_\theta$. If $i = \alpha$, we have $[x]_\theta = [(\alpha, a_\alpha)]_\theta \leq [(\alpha, a_\alpha)]_\theta = q_\alpha(a_\alpha)$, and so the result holds. Otherwise, since f_i is an order dense embedding, there exists $a \in A$ such that $a_i \leq f_i(a)$ and hence $[(i, a_i)]_\theta = q_i(a_i) \leq q_i(f_i(a)) = q_\alpha(f_\alpha(a))$. Therefore, q_α is an order dense embedding. □

Definition 2.17. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for every set indexed family $\{m_i : A \to A_i | i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all $m_i, i \in I$; that is there are $d_i : A_i \to B$ with $d_i m_i = m$.

Proposition 2.18. **Pos**-S *has* $\mathcal{M}_{od}$*-bounds.*

Proof. Let $\{h_\alpha : A \to B_\alpha | \alpha \in I\}$ be a set indexed family of order dense embeddings and $h : A \to B = (\coprod_\alpha B_\alpha)/\theta$ be the multiple pushout of $h_\alpha, \alpha \in I$. Then h factors over all $h_\alpha, \alpha \in I$, and is an order dense embedding, by Proposition 2.16. □

Definition 2.19. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property, if the morphism m in the definition of $\mathcal{M}$-bounds factors over all $m_i, i \in I$, through members of $\mathcal{M}$; that is $d_i, i \in I$, belongs to $\mathcal{M}$.

Proposition 2.20. **Pos**-S *has* $\mathcal{M}_{od}$*-amalgamation property.*

Proof. Since, by Proposition 2.16, multiple pushouts transfer order dense embeddings, we get the result. □

Finally, we study directed colimit of order dense embeddings in **Pos**-S. Recall that a directed system of S-posets and S-poset maps is a family $(A_i)_{i\in I}$ of S-posets indexed by an up-directed set I endowed by a family $(\psi_{ij} : A_i \to A_j)_{i\leq j\in I}$ of S-poset maps such that given $i \leq j \leq k \in I$, $\psi_{ik} = \psi_{jk}\psi_{ij}$, and $\psi_{ii} = \text{id}$. Also the pair $(\underrightarrow{lim} A_i, \{\alpha_i : A_i \to \underrightarrow{lim} A_i\})$ or in abbreviation, $\underrightarrow{lim} A_i$ is called the *directed colimit* (or *direct limit*) of the directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ if for every $i \leq j \in I, \alpha_j\psi_{ij} = \alpha_i$, and for every $(B, f_i : A_i \to B)$ with $f_j\psi_{ij} = f_i, i \leq j \in I$, there exists a unique S-poset map $v : \underrightarrow{lim} A_i \to B$ such that $v\alpha_i = f_i$, for every $i \in I$.

Recall from [3] that the directed colimit of a directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ of S-posets exists, and may be represented as $(A/\theta, (\psi_i = \gamma_\theta u_i : A_i \to A/\theta)_{i\in I})$, where $\gamma_\theta : \coprod A_i \to (\coprod A_i)/\theta$ is the natural epimorphism and u_i is coproduct injection and
(i) $A = \coprod A_i$;
(ii) $a\theta a' (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) = \psi_{jk}(a')$;
(iii) $[a]_\theta \leq [a']_\theta (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) \leq \psi_{jk}(a')$;
(iv) for each $i \in I$ and $a \in A_i, \psi_i(a) = [a]_\theta$.

Proposition 2.21. **Pos**-S *has* $\mathcal{M}_{od}$*-directed colimits.*

Proof. Let $h : A \to \underrightarrow{lim} B_\alpha = \coprod_\alpha B_\alpha/\rho$ be a directed colimit of $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta})$ in **Pos**-S and $\{h_\alpha : A \to B_\alpha\}_{\alpha\in I}$ be a family of order dense embeddings, with directed S-poset maps $g_{\alpha\beta} : B_\alpha \to B_\beta$ $(\alpha \leq \beta)$ and the colimit maps $g_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$. Recall that, for each $\beta \in I, h = \underrightarrow{lim} h_\alpha = g_\beta h_\beta$. By [10], h is an order embedding. Also, h is order dense. For, let $b \in \underrightarrow{lim}_\alpha B_\alpha$, then there exists $x_\alpha \in B_\alpha$ such that $b = [x_\alpha]_\rho$ and, since h_α is order dense, there exists an element $a \in A$ with $x_\alpha \leq h_\alpha(a)$. Then $b = [x_\alpha]_\rho = g_\alpha(x_\alpha) \leq g_\alpha h_\alpha(a) = h(a)$ as required. □

Theorem 2.22. *Let I be an up-directed set and $\{h_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a directed family of order dense embeddings. Then the directed colimit homomorphism $h : \underrightarrow{lim} A_i \to \underrightarrow{lim} B_i$ is an order dense embedding.*

Proof. Let $(\underrightarrow{lim} A_i, f_i), (\underrightarrow{lim} B_i, g_i)$ be directed colimits of directed systems $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ and $((B_i)_{i\in I}, (\varphi_{ij})_{i\leq j})$ with the colimit maps

$f_i = \gamma_\theta u_i : A_i \to \varinjlim_i A_i = \coprod_{\alpha\in I} A_i/\theta$, $g_i = \gamma_{\theta'} u'_i : B_i \to \varinjlim_i B_i = \coprod_{i\in I} B_i/\theta'$, respectively and $\{h_i : A_i \to B_i | i \in I\}$ be a directed family of order dense embeddings such that for every $i \leq j, f_j\psi_{ij} = f_i$ and $g_j\varphi_{ij} = g_i$. Then $g_j h_j \psi_{ij} = g_j \varphi_{ij} h_i = g_i h_i$. Thus $h = \varinjlim h_i$ exists by the universal property of colimits. Consider $\varinjlim A_i = (\coprod A_i)/\theta$ and $\varinjlim B_i = (\coprod B_i)/\theta'$. By Theorem 3 of [10], it is enough to show that h is an order dense map. Let $[x]_{\theta'} \in (\coprod B_i)/\theta'$. Then $[x]_{\theta'} = [(\alpha, b_\alpha)]_{\theta'}$ for some $\alpha \in I$ and $b_\alpha \in B_\alpha$. Since h_α is order dense, there exists $a_\alpha \in A_\alpha, b_\alpha \leq h_\alpha(a_\alpha)$. Then $[x]_{\theta'} = [(\alpha, b_\alpha)]_{\theta'} = g_\alpha(b_\alpha) \leq g_\alpha h_\alpha(a_\alpha) = hf_\alpha(a_\alpha)$, as desired. □

Definition 2.23. We say that a category $\mathcal{A}$ fulfills the $\mathcal{M}$-*chain* condition if for every directed system $((A_\alpha)_{\alpha\in I}, (f_{\alpha\beta})_{\alpha\leq\beta\in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha\in I}$ with $g_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 2.24. **Pos**-S *fulfills the* $\mathcal{M}_{od}$-*chain condition.*

Proof. Take $A = \varinjlim_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be the colimit maps. Then applying Proposition 2.21, we get the result. □

2.2. **Order dense essentiality and the behavior of order dense injectivity.** In this subsection, the notion of essentiality with respect to order dense embeddings in the category of S-posets is studied. There are three different definitions of essentiality usually used in literature with respect to a subclass of monomorphisms. We study these kinds of essentiality with respect to the class of all order dense embeddings of S-posets, and investigate their relations to order dense injectivity. More precisely, it is proved that order dense injectivity well behaves regarding all of these essentialities.

As we mentioned, in the category **Pos**-S, one has the following different definitions for *essentialness*. An order dense embedding $M \stackrel{m}{\to} X$ is called:

($\mathcal{M}_{e1}$-essential) $M \stackrel{m}{\to} X \stackrel{f}{\to} Y \in \mathcal{M}_{od} \Rightarrow f \in \mathcal{M}_{od}$.

($\mathcal{M}_{e2}$-essential) $M \stackrel{m}{\to} X \stackrel{f}{\to} Y \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$.

($\mathcal{M}_{e3}$-essential) $M \stackrel{m}{\to} X \stackrel{f}{\to} Y \in \mathcal{M}_{od} \Rightarrow f \in \mathcal{M}ono$,

where $\mathcal{M}ono$ and $\mathcal{M}_{ei}$, for $i = 1, 2, 3$, are the classes of all monomorphisms and $\mathcal{M}_{ei}$-essential monomorphism, respectively, in **Pos**-S. One can easily see that $\mathcal{M}_{e1}$ and $\mathcal{M}_{e2}$ are subclasses of $\mathcal{M}_{e3}$. In the next section, it will be shown that $\mathcal{M}_{e1}$ is a proper subclass of $\mathcal{M}_{e2}$ and $\mathcal{M}_{e2} = \mathcal{M}_{e3}$.

Lemma 2.25. *The composition of $\mathcal{M}_{ei}$-essential monomorphisms is an $\mathcal{M}_{ei}$-essential monomorphism, for i=1, 2, 3.*

Lemma 2.26. *Suppose that A is an order dense sub-S-poset of B and B is an order dense sub-S-poset of C. Then A is $\mathcal{M}_{ei}$-essential, for $i = 1, 2, 3$, in C if and only if A is $\mathcal{M}_{ei}$-essential in B and B is $\mathcal{M}_{ei}$-essential in C.*

As we know the $\mathcal{M}$-Banaschewski's condition with respect to a subclass $\mathcal{M}$ of monomorphisms in a category plays an important role in the behavior of $\mathcal{M}$-injectivity. In the following, we investigate the mentioned condition for the classes $\mathcal{M}_{ei}$, for $i = 1, 2, 3$, in **Pos-**S and then study the behavior of order dense injectivity of S-posets.

Definition 2.27. We say that the category **Pos**-S fulfills $\mathcal{M}_{ei}$-Banaschewski's condition, for $i = 1, 2, 3$, if for every $f \in \mathcal{M}_{od}$, there is an S-poset map g with $gf \in \mathcal{M}_{ei}$.

Lemma 2.28. *The category* **Pos-**S *fulfills $\mathcal{M}_{ei}$-Banaschewski's condition, for $i = 1, 2, 3$.*

Proof. We only prove that $\mathcal{M}_{e1}$-Banaschewski's condition is satisfied in **Pos-**S. Recall from [11] that the category **Pos-**S fulfills Banaschewski's condition for regular essential monomorphisms. Thus for every order dense embedding $f : A \to B$ there exists an S-poset map $g : B \to C$ such that gf is regular essential. Now, we claim that gf is $\mathcal{M}_{e1}$-essential. Let $h : C \to D$ be an S-poset map such that hgf is an order dense embedding. Since gf is regular essential, h is an order embedding as desired. It is clear that h is order dense. □

Now, we recall the definition of order dense injective S-posets.

Definition 2.29. (1) We call an S-poset A *order dense injective* or briefly *od-injective* if it is injective with respect to order dense embeddings $B \to C$.

Recall the followings from [14].

Remark 2.30. (1) Clearly one can take B in the above definition of order dense injectivity to be an order dense sub-S-poset of C.

(2) If A is a regular injective S-poset then it is order dense injective, but the converse is not necessarily true.

(3) An S-poset P is regular injective if and only if it is dc-injective (injective with respect to down closed embeddings) as well as order dense injective.

(4) Every non-trivial order dense injective S-poset has a zero which is the bottom element, but the converse is not true in general. To see this, take a poset $P = \{a, b, c, d, e, f\}$ with the order $a \leq b, c \leq d$, and $a, b, c, d, e \leq f$ as an S-poset with trivial actions. Then by [13], P is down closed injective since it has a top element. If P which has a bottom element a is order dense injective it must be regular injective by (3), but it is not regular injective since it is not a complete poset (See [1]).

(5) An S-poset A is order dense injective if and only if it is a retract of each of its extensions in which it is order dense.

Theorem 2.31. (First Theorem of Well-Behavior)
For a pomonoid S and every S-poset A, the followings are equivalent:
(i) *A is order dense injective.*
(ii) *A is order dense absolute retract.*
(iii) *A has no proper $\mathcal{M}_{e3}$-essential extensions.*
(iv) *A has no proper $\mathcal{M}_{e2}$-essential extensions.*
(v) *A has no proper $\mathcal{M}_{e1}$-essential extensions.*

Proof. (i)$\Leftrightarrow$(ii) is clear by Remark 2.30(5).

(ii)$\Rightarrow$(iii) Let B be an $\mathcal{M}_{e3}$-essential extension of A. Then, A is order dense in B and so by (ii), there exists an order dense retraction $g : B \to A$. Thus $g\mid_A$, which is equal to id_A, is an order dense embedding. But, A is $\mathcal{M}_{e3}$-essential in B, so g has to be a monomorphism. Now, for $b \in B$, $g(g(b)) = g(b)$ since $g(b) \in A$, and this implies $g(b) = b$ which gives $b \in A$. Thus $B = A$.

(v)$\Rightarrow$(ii) let B be an order dense extension of A. Then, by Lemma 2.28, there exists an S-poset map $g : B \to C$ such that $g\mid_A$ is $\mathcal{M}_{e1}$-essential. Applying (v), $g\mid_A$ is an isomorphism. Now, $g(g\mid_A)^{-1}$ is a retraction and so A is an order dense retract of B. □

Definition 2.32. Let A be an S-poset. Then
(1) By a *maximal $\mathcal{M}_{ei}$-essential extension* of A, for $i = 1, 2, 3$, we mean an $\mathcal{M}_{ei}$-essential extension B of A such that every S-poset map $h : B \to C$ from B to an $\mathcal{M}_{ei}$-essential extension C of A, for which $h\mid_A$ is the inclusion map, is an isomorphism.
(2) By a *minimal order dense injective extension of* A we mean an order dense extension B of A such that B is order dense injective, and every order dense embedding $k : C \to B$ from an order dense injective order dense extension C of A which maps A identically is an isomorphism.

Lemma 2.33. *If B is an $\mathcal{M}_{ei}$-essential extension of A, for i=* 1, 2, 3, *and A is order dense embedded into some regular injective S-poset E, then B can be order dense embedded into E as well.*

Considering $\mathcal{P}$, the set of all $\mathcal{M}_{ei}$-essential extensions of A, for $i = 1, 2, 3$, as a poset with the inclusion as its order, Zorn's lemma gives the existence of a maximal element in $\mathcal{P}$ which is clearly a maximal $\mathcal{M}_{ei}$-essential extension of A. This guarantees the existence of order dense injective hull of S-posets.

Proposition 2.34. *Every S-poset has a maximal $\mathcal{M}_{ei}$-essential extension, for i= 1, 2, 3.*

Definition 2.35. By an *$\mathcal{M}_{ei}$-injective hull* of an S-poset A, for $i = 1, 2, 3$, we mean an $\mathcal{M}_{ei}$-essential extension of A which is order dense injective.

Remark 2.36. For an S-poset A, $\mathcal{M}_{ei}$-injective hull of A, for $i = 1, 2, 3$, is unique up to isomorphism (if it exists). To see this, let B and C both be $\mathcal{M}_{ei}$-injective hull of A. Then there exists an S-poset map $h : B \to C$ such that $h\mid_A = \mathrm{id}_A$, because C is order dense injective and A is order dense in B. From the fact that A is $\mathcal{M}_{ei}$-essential in B, we get that h is an $\mathcal{M}_{ei}$-essential monomorphism. But, by Theorem 2.31, B has no proper $\mathcal{M}_{ei}$-essential extension, since it is order dense injective. So, h is an isomorphism, as required.

Now, we give the third theorem of well-behavior of order dense injectivity, which is about the relation between $\mathcal{M}_{ei}$-injective hulls and $\mathcal{M}_{ei}$-essential extensions, for $i = 1, 2, 3$.

Theorem 2.37. *Let S be a pomonoid and A be an S-poset. The followings are equivalent for an order dense extension B of A:*
(i) *B is the $\mathcal{M}_{e1}$-injective hull of A.*
(ii) *B is the $\mathcal{M}_{e3}$-injective hull of A.*
(iii) *B is the $\mathcal{M}_{e2}$-injective hull of A.*
(iv) *B is a maximal $\mathcal{M}_{e1}$-essential extension of A.*
(v) *B is a maximal $\mathcal{M}_{e3}$-essential extension of A.*
(vi) *B is a maximal $\mathcal{M}_{e2}$-essential extension of A.*
(vii) *B is a minimal order dense injective extension of A.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), (vi) $\Rightarrow$ (vii) are clear.

(i) $\Rightarrow$ (iv) Suppose B is the $\mathcal{M}_{e1}$-injective hull of A. Let C be an order dense extension of B and an $\mathcal{M}_{e1}$-essential extension of A. Then, applying Lemma 2.26, C is an $\mathcal{M}_{e1}$-essential extension of B. But, by Theorem 2.31, B being order dense injective has no proper $\mathcal{M}_{e1}$-essential extension and so $C = B$.

(iv) $\Rightarrow$ (i) Suppose B is a maximal $\mathcal{M}_{e1}$-essential extension of A. Then it has no proper $\mathcal{M}_{e1}$-essential extension by Lemma 2.26 and the result holds by Theorem 2.31.

(iv) $\Rightarrow$ (vii) Suppose B is a maximal $\mathcal{M}_{e1}$-essential extension of A. Then it has no proper $\mathcal{M}_{e1}$-essential extension by Lemma 2.26 and so it is order dense injective by Theorem 2.31. Let $k : C \to B$ be an order dense embedding from an order dense injective order dense extension C of A which maps A identically. Since A is $\mathcal{M}_{e1}$-essential in B, it is concluded by Lemma 2.26 that the same is true for $k(C)$ and then, since $k(C) \cong C$ is order dense injective, applying Theorem 2.31, we get $B = k(C)$. Therefore, k is an isomorphism.

(vii) $\Rightarrow$ (i) By Proposition 2.34, there exists a sub-S-poset E of B which is a maximal $\mathcal{M}_{e1}$-essential extension of A. Then by Theorem 2.31, E is order dense injective and so $E = B$. Hence B is an $\mathcal{M}_{e1}$-essential extension of A. Therefore, B is the $\mathcal{M}_{e1}$-injective hull of A. □

The second theorem of well-behavior of order dense injectivity is about the existence of $\mathcal{M}_{ei}$-injective hulls, for $i = 1, 2, 3$, which is proved in the following for S-posets.

Corollary 2.38. (Second Theorem of Well-Behavior) *Each S-poset has the $\mathcal{M}_{ei}$-injective hull, for $i = 1, 2, 3$.*

Proof. By Proposition 2.34 and Theorem 2.37, the proof is obvious. □

Definition 2.39. Let A be an S-poset. Then
(1) By a *largest $\mathcal{M}_{ei}$-essential extension* of A, for $i = 1, 2, 3$, we mean an $\mathcal{M}_{ei}$-essential extension B of A such that for each $\mathcal{M}_{ei}$-essential extension C of A there exists an S-poset map $h : C \to B$ such that $h\mid_A$ is the order dense inclusion map.
(2) By a *smallest order dense injective extension of A* we mean an order dense injective order dense extension B of A such that for each order dense injective order dense extension C of A there exists a monomorphism $g : B \to C$ such that $g\mid_A$ is the order dense inclusion map.

Note 2.40. If gf is $\mathcal{M}_{ei}$-essential and g is a monomorphism then g is $\mathcal{M}_{ei}$-essential, for $i = 1, 2, 3$.

Some other conditions can be added to the equivalent conditions given in the preceding theorem.

Theorem 2.41. *The following conditions are equivalent to the conditions of Theorem 2.37:*
(viii) *B is a largest $\mathcal{M}_{e1}$-essential extension of A.*
(ix) *B is a largest $\mathcal{M}_{e2}$-essential extension of A.*
(x) *B is a largest $\mathcal{M}_{e3}$-essential extension of A.*
(xi) *B is an smallest order dense injective extension of A.*

Proof. Recalling the notations of Theorem 2.37, we have:

(i)⇒(viii) It is clear by Lemma 2.33.

(viii)⇒(xi) Take $E_{e1}(A)$ to be the $\mathcal{M}_{e1}$-injective hull of A which exists by Corollary 2.38. Since $E_{e1}(A)$ is an $\mathcal{M}_{e1}$-essential extension of A and B is a largest $\mathcal{M}_{e1}$-essential extension of A, we obtain an S-poset map $h : E_{e1}(A) \to B$ such that $h\mid_A$ is the order dense inclusion map. Now, since A is $\mathcal{M}_{e1}$-essential in $E_{e1}(A)$, h is an order dense embedding and so, since B is an $\mathcal{M}_{e1}$-essential extension of A, Note 2.40 implies that h is $\mathcal{M}_{e1}$-essential. But, $E_{e1}(A)$ is order dense injective, and so, by Theorem 2.31, has no proper $\mathcal{M}_{e1}$-essential extension. Hence, h is an isomorphism. Therefore, B is order dense injective. So, B is evidently an smallest order dense injective extension of A.

(xi)⇒(i) Suppose $E_{e1}(A)$ is the $\mathcal{M}_{e1}$-injective hull of A which exists by Corollary 2.38. Then, since $E_{e1}(A)$ is order dense injective and B is a smallest order dense injective extension of A, there exists a monomorphism $g : B \to E_{e1}(A)$ such that $g\mid_A$ is the order dense inclusion map. Also, since B is an $\mathcal{M}_{e1}$-essential extension, we get that g is $\mathcal{M}_{e1}$-essential (by Note 2.40). But, B is order dense injective and so has no proper $\mathcal{M}_{e1}$-essential extension. Thus, g is an isomorphism. Hence, B is an $\mathcal{M}_{e1}$-essential extension and so it is an $\mathcal{M}_{e1}$-injective hull of A. □

2.3. **Interrelations of different kinds of essentiality and more on order dense injective hulls.** This subsection is devoted to study the interrelations of different classes of essentialities. Moreover, some characterizations regarding the $\mathcal{M}_{ei}$-essential extensions and also the $\mathcal{M}_{ei}$-injective hull of S-posets, for $i = 1, 2, 3$, under some conditions are presented.

Theorem 2.42. *For an order dense embedding $f : A \to B$ in the category* **Pos-**S*, the following implications hold:*

$\mathcal{M}_{e1}$*-essential* ⇒ $\mathcal{M}_{e2}$*-essential* ⇔ $\mathcal{M}_{e3}$*-essential.*

Proof. We only prove $\mathcal{M}_{e3}$-essential ⇒ $\mathcal{M}_{e2}$-essential. Let an order dense embedding $f : A \to B$ be $\mathcal{M}_{e3}$-essential and $g : B \to C$ be an S-poset map for which gf is a monomorphism. Consider an $\mathcal{M}_{e1}$-injective hull $h : B \to E_{e1}(B)$ where h is $\mathcal{M}_{e1}$-essential, which exists by Theorem 2.38. Clearly, $hf \in \mathcal{M}_{e3}$. Let (Q, p, q) be the pushout of

g and h in the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle q} \\ E_{e1}(B) & \xrightarrow[p]{} & Q \end{array}$$

Using Lemma 2.13, pushouts transfer order dense embeddings, we get that q is an order dense embedding. Then $phf = qgf$ is a monomorphism. Since $hf \in \mathcal{M}_{e3}$ and p is a monomorphism. Thus $qg = ph$ is a monomorphism and hence g is a monomorphism and hence f is $\mathcal{M}_{e2}$-essential. □

Remark 2.43. The class of $\mathcal{M}_{e1}$-essential extensions of an S-poset A is strictly smaller than the class of its $\mathcal{M}_{e2}$-essential extensions. For, take a non-empty S-poset A without zero element. We claim that the inclusion map $i : A \to A \sqcup \{0\}$ is $\mathcal{M}_{e2}$-essential but not $\mathcal{M}_{e1}$-essential, where $A \sqcup \{0\}$ obtained by adjoining a zero element 0 to A. To prove i is $\mathcal{M}_{e2}$-essential, let $g : A \sqcup \{0\} \to B$ be an S-poset map such that $g|_A$ is a monomorphism. Then we show that g itself is one-one. Indeed, if $g(a) = g(0)$ for some $a \in A$, then for every $s \in S, g(as) = g(a)s = g(0)s = g(0s) = g(0) = g(a)$ and so $as = a$ which means that a is a zero element of A, a contradiction. Finally, it remains to show that i is not $\mathcal{M}_{e1}$-essential. For this, let $\{0\} \oplus A$ be the S-poset obtained by adjoining a zero bottom element 0 to A. Note that the identity map $\mathrm{id} : A \sqcup \{0\} \to \{0\} \oplus A$ is not an order embedding whereas $\mathrm{id} \circ i$ is clearly an order dense embedding.

One can find the characterization for $\mathcal{M}_{e2}$ and $\mathcal{M}_{e1}$-essentialities in the following two theorems.

Theorem 2.44. *For an order dense embedding $f : A \to B$, the followings are equivalent:*

(i) *f is an $\mathcal{M}_{e2}$-essential.*

(ii) *For every epimorphism $g : B \to C$ such that gf is a monomorphism, g is itself a monomorphism.*

(iii) *For every S-poset congruence θ on B such that for the canonical epimorphism $\pi : B \to B/\theta, \pi f$ is a monomorphism, one gets $\theta = \Delta_B$.*

(iv) *For every monogenic congruence θ on B such that for the canonical epimorphism $\pi : B \to B/\theta, \pi f$ is a monomorphism, one gets $\theta = \Delta_B$.*

Proof. It is clear that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).
(iv) ⇒ (i) Let $g : B \to C$ be an S-poset map such that gf is a monomorphism, and $g(b_1) = g(b_2)$. Then, since $\theta(b_1, b_2) \subseteq \ker g$, by

decomposition theorem, g can be factorized through $B/\theta(b_1, b_2)$, and hence πf is a monomorphism, where $\pi : B \to B/\theta(b_1, b_2)$. So, by (iv), $\theta(b_1, b_2) = \Delta_B$, and thus $b_1 = b_2$. □

Theorem 2.45. *For an order dense embedding $f : A \to B$, the followings are equivalent:*

(i) *f is an $\mathcal{M}_{e1}$-essential.*

(ii) *For every epimorphism $g : B \to C$ such that gf is an order embedding, g is itself an order embedding.*

(iii) *For every S-poset congruence θ on B such that for the canonical epimorphism $\pi : B \to B/\theta, \pi f$ is an order embedding, one gets $\theta \subseteq \leq_B$.*

(iv) *For every monogenic congruence θ on B such that for the canonical epimorphism $\pi : B \to B/\theta(b_1, b_2), \pi f$ is an order embedding, one gets $\theta(b_1, b_2) \subseteq \leq_B$.*

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
(iv) $\Rightarrow$ (i) Let $g : B \to C$ be an S-poset map such that gf is an order embedding, and $g(b_1) \leq g(b_2)$. Then, since $\theta(b_1, b_2) \subseteq K_g$, where K_g is the subkernel of an S-poset map g, by decomposition theorem, g can be factorized through $B/\theta(b_1, b_2)$, and hence πf is an order embedding, where $\pi : B \to B/\theta(b_1, b_2)$. So, by (iv), $\theta(b_1, b_2) \subseteq \leq_B$, and thus $b_1 \leq b_2$. □

Recalling the definitions of regular-essentialness and mono-essentialness, one has the following corollary.

Corollary 2.46. *An order embedding f is $\mathcal{M}_{e1}$-essential ($\mathcal{M}_{e2}$-essential) if and only if it is regular-essential (mono-essential) as well as order dense.*

As a corollary of Theorem 2.42, the above corollary and Proposition 2.2 of [11], one has the following proposition which gives a characterization of different kinds of essentiality with respect to the class $\mathcal{M}_{od}$, in terms of congruences.

Proposition 2.47. *Let $(A, \leq|_A)$ be an order dense sub-S-poset of an S-poset $(B, \leq)$. Then the followings hold:*

(i) *If B is an $\mathcal{M}_{e1}$-essential extension of A, then for any $\rho \in Con(B)$ with $\rho \neq \Delta_B$, one has $\leq_\rho|_A \neq \leq|_A$.*
Conversely, if for any preorder relation β on B with $\beta \neq \leq$ one has $\leq_\beta|_A \neq \leq|_A$, then B is an $\mathcal{M}_{e1}$-essential extension of A.

(ii) *B is an $\mathcal{M}_{e2}$-essential extension of A if and only if for any $\rho \in Con(B)$ with $\rho \neq \Delta_B$, one has $\rho\,|_A \neq \Delta_A$.*

(iii) *B is an $\mathcal{M}_{e3}$-essential extension of A if and only if for any $\rho \in Con(B)$ with $\rho \neq \Delta_B$, one has $\leq_\rho|_A \neq \leq|_A$.*

Recall from [1] that an extension E of a poset P is called *join dense* (*meet dense*) if for each element e of E, $e = \bigvee\{p \in P : p \leq e\}$ ($e = \bigwedge\{p \in P : e \leq p\}$). By Corollary 3.9 of [11] one gets the following characterization of $\mathcal{M}_{e1}$-essential extensions over pogroups.

Corollary 2.48. *Let S be a pogroup and A be an S-poset. Each order dense extension E of A is $\mathcal{M}_{e1}$-essential if and only if E is both meet and join dense.*

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References

1. B. Banaschewski and G. Bruns, Categorical characterization of the MacNeille completion, *Archiv der Mathematatik*, **18** (1967), 369–377.
2. H. Barzegar, M. M. Ebrahimi and M. Mahmoudi, Essentiality and Injectivity, *Appl. Categ. Structures*, **18**(1) (2010), 73–83.
3. S. Bulman-Fleming and V. Laan, Lazard's theorem for S-posets, *Math. Nachr.*, **278**(15) (2005), 1743–1755.
4. S. Bulman-Fleming and M. Mahmoudi, The category of S-posets, *Semigroup Forum*, **71**(3) (2005), 443–461.
5. M. M. Ebrahimi, M. Haddadi and M. Mahmoudi, Injectivity in a category: an overview on well behavior theorems, *Algebra, Groups, and Geometries*, **26**(4) (2009), 451–471.
6. M. M. Ebrahimi, M. Haddadi and M. Mahmoudi, Injectivity in a category: an overview on smallness conditions, *Categories and General Algebraic Structures with Applications*, **2**(1) (2014), 83–112.
7. M. M. Ebrahimi and M. Mahmoudi, The category of M-sets, *Ital. J. Pure Appl. Math.*, **9** (2001), 123–132.
8. M. M. Ebrahimi, M. Mahmoudi and H. Rasouli, Banaschewski's theorem for S-posets: regular injectivity and completeness, *Semigroup Forum*, **80**(2) (2010), 313–324.
9. M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.
10. H. Rasouli, Categorical properties of regular monomorphisms of S-posets, *European J. Pure Appl. Math.*, **7**(2) (2014), 166–178.
11. H. Rasouli and H. Barzegar, The regular injective envelope of S-posets, *Semigroup Forum*, **92**(1) (2015), 186–197.
12. L. Shahbaz and M. Mahmoudi, Various kinds of regular injectivity for S-posets, *Bull. Iran. Math. Soc.*, **40**(1) (2014), 243–261.
13. L. Shahbaz and M. Mahmoudi, Injectivity of S-posets with respect to down closed regular monomorphisms, *Semigroup Forum*, **91** (2015), 584–600.

14. L. Shahbaz, Order dense injectivity of S-posets, *Categories and General Algebraic Structures with Applications*, **3**(1) (2015), 43–63.
15. X. Zhang, Regular injectivity of S-posets over clifford pomonoids, *Southeast Asian Bulletin Math.*, **32** (2007), 1007–1015.
16. X. Zhang and V. Laan, On homological classification of pomonoids by regular weak injectivity properties of S-posets, *Central Eur. J. Math.*, **5**(1) (2007), 181–200.

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ORDER DENSE ESSENTIALITY AND BEHAVIOR OF ORDER DENSE INJECTIVITY

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اساسی بودن چگال ترتیبی و رفتار انژکتیو چگال ترتیبی

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در این مقاله، به مطالعه‌ی ویژگی‌های جبری و رسته‌ای مانند حد و هم‌حد در رسته‌ی S-**Pos** نسبت به نشانده‌های چگال ترتیبی می‌پردازیم. انژکتیوی نسبت به این کلاس از تکریختی‌ها توسط نویسنده مورد مطالعه قرارگرفته و برای به دست آوردن اطلاعاتی در مورد انژکتیوی نسبت به نشانده‌های ترتیبی مورد استفاده قرارگرفته است. سپس سه نوع اساسی بودن نسبت به کلاس نشانده‌های چگال ترتیبی در رسته‌ی S-**Pos** مطالعه می‌شود و ارتباط آنها با انژکتیوی چگال ترتیبی مطرح خواهد شد. مشاهده می‌شود با وجود این که تمامی این سه نوع اساسی بودن‌ها لزوماً با هم معادل نیستند ولی نسبت به انژکتیوی چگال ترتیبی رفتار یکسانی دارند. در نهایت، دسته‌بندی‌هایی از این اساسی بودن‌ها روی گروه‌های مرتب ارایه خواهد شد.

کلمات کلیدی: S-مجموعه‌ی مرتب، زیر S-مجموعه‌ی مرتب چگال ترتیبی، انژکتیوی چگال ترتیبی، اساسی بودن چگال ترتیبی.