Journal of Algebraic Systems
Vol. 8, No. 1, (2020), pp 1-5

MULTIPLICATION MODULES THAT ARE FINITELY GENERATED

Y. TOLOOEI*

ABSTRACT. Let R be a commutative ring with identity and M be a unitary R-module. An R-module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It is shown that over a Noetherian domain R with $\dim(R) \leq 1$, multiplication modules are cyclic or isomorphic to an invertible ideal of R. Moreover, we give a characterization of finitely generated multiplication modules.

1. Introduction

All rings in this article are commutative with identity and modules are unitary. For a ring R, we denote by $\dim(R)$ the classical Krull dimension of R and for a module M, we denote by $\mathrm{Ann}(M)$ the annihilator of M. A ring R is called *semilocal* whenever $R/\mathrm{J}(R)$ is a semisimple ring. A module M is called *multiplication module* whenever for every submodule N of M there exists an ideal I of R such that $N = IM$. Examples of multiplication modules are every ring, every cyclic module, every ideal in a Dedekind domain [7, page 38] and every ideal in a regular ring. In [4], equivalent conditions for multiplication modules to be finitely generated is given, and here we give more equivalent conditions for finitely generated multiplication modules. Multiplication modules are generalized to the non-commutative ring by Tuganbaev [9]. Recently in [1], rings in which every ideal is multiplication as a multiplication ring, is studied. Also, Perez et al. [6],

MSC(2010): Primary: 13C10; Secondary: 13C13, 13C99.
Keywords: Multiplication module, Noetherian ring, faithful module.
Received: 14 July 2019, Accepted: 19 October 2019.
*Corresponding author.

generalized the multiplication modules, Azizi and Jayaram work on the multiplication module which they called principal multiplication modules [2] and Smith [8] work on the fully invariant multiplication modules. We bring here some results from [3] and [4] which will be useful throughout the paper.

Theorem 1.1. [3, Theorem 4.] *Let R be a semi-local ring. Then an R-module M is a multiplication module if and only if it is cyclic.*

Lemma 1.2. [4, Lemma 3.6] *Let R be a domain and M a faithful multiplication R-module. Then there exists an invertible ideal I of R such that $M \cong I$.*

Proposition 1.3. [4, Proposition 3.4.] *Let M be a faithful multiplication R-module. Then M is finitely generated R-module if and only if $M \neq PM$ for all minimal prime ideals P of R.*

2. Multiplication modules

Let M be an R-module. Torsion subset of M, denoted by $\mathrm{T}(M)$, is defined as $\mathrm{T}(M) = \{x \in M | \mathrm{Ann}(x) \neq 0\}$. Note that $\mathrm{T}(M)$ is not necessary a submodule of M, unless R is a domain. Recall that a ring R is a *Dedekind ring* if and only if every ideal of R is invertible. Also the Krull dimension of Dedekind domain is less than 1. In the following theorem, we characterize multiplication modules over a Noetherian domain R with the Krull dimension less than or equal to 1.

Theorem 2.1. *Let R be a Noetherian domain with $\dim(R) \leq 1$. Then M is a multiplication R-module if and only if either $M \cong I$ for some invertible ideal I of R or M is a cyclic R-module.*

Proof. $(\Rightarrow)$. Let M be a multiplication R-module. We consider the following two cases:
Case 1: If $\mathrm{T}(M) = 0$, then by Lemma 1.2, $M \cong I$ for some invertible ideal I of R.
Case 2: Let $\mathrm{T}(M) \neq 0$. We claim that $\mathrm{Ann}(M) \neq 0$. Let $0 \neq m \in \mathrm{T}(M)$, then $K = \mathrm{Ann}(m) \neq 0$ and also $Rm = LM$ for some non-zero ideal L of R. Thus $KLM = (0)$ and since R is a domain, $KL \neq (0)$ and so $\mathrm{Ann}(M) \neq 0$ (i.e., $M = \mathrm{T}(M)$). Let $I = \mathrm{Ann}(M)$. Since R is Noetherian, R/I is also a Noetherian ring with $\dim(R/I) = 0$, i.e R/I is an Artinian ring. On the other hand, M is an R/I-multiplication module. Thus by Theorem 1.1, M is a cyclic R/I-module. Hence it is a cyclic R-module, as desired.
$(\Leftarrow)$. It is evident. $\square$

Theorem 2.2. *Let M be a multiplication R-module. For every maximal ideal P of R, $PM \neq M$ if and only if $M_P \neq 0$. Furthermore, for every prime ideal P of R, $M_P \neq 0$ implies $PM \neq M$.*

Proof. ($\Rightarrow$). Let $PM \neq M$ and $x \in M \setminus PM$. If

$$Ann$$

(x)$\nsubseteq P$ then

$$Rx = (\mathrm{Ann}(x) + P)x = Px \subseteq PM,$$

which is a contradiction. Hence $\mathrm{Ann}(x) \subseteq P$ and it conclude that $0 \neq R_p x \subseteq M_P$.
($\Leftarrow$). Let $M_P \neq 0$ and $0 \neq \overline{x} \in M_P$, hence $\mathrm{Ann}(x) \subseteq P$. Suppose that $Rx = IM$ for some ideal I of R. Now, if $PM = M$ then

$$Rx = IM = I(PM) = P(IM) = P(Rx) = Px.$$

So $(1-p)x = 0$ for some $p \in P$ and hence $(1-p) \in$Ann$(x) \subseteq P$, but this is a contradiction. Therefore $PM \neq M$. □

Theorem 2.3. *Let M be a multiplication R-module. If $M_P \neq 0$ for every maximal* (*minimal*) *ideal P of R, then M is finitely generated.*

Proof. If $M_P \neq 0$ for every maximal (minimal) ideal P of R, by Theorem 2.2, $PM \neq M$ and by [4, Theorem 3.1] (Proposition 1.3), M is a finitely generated $R/$ Ann$(M)-$module, so M is a finitely generated R-module.

□

Recall that an R-module M is called *locally free* (*locally cyclic*) if M_P is a free (cyclic) R_P-module for any maximal (prime) ideal P of R (See [7, Exercise 2.21]). Note that for every R-module M if $I =$Ann(M), then M is a faithful R/I-module, hence we can assume that every multiplication module is a faithful module. Barnard [3] proved that if M is finitely generated R-module, then M is a multiplication module if and only if M is a locally cyclic R-module [3, Proposition 5]. The following lemma shows that a faithful multiplication module is locally free. We note that the following lemma exists in the literature, however we give here its proof for the sake of completeness and convenience of the reader.

Lemma 2.4. *If M is a faithful multiplication R-module, then M is a locally free R-module.*

Proof. Let P be a maximal ideal of R. If $M_P = 0$ then it is free with empty basis. So, let $M_P \neq 0$. Thus by Theorem 2.2, $PM \neq M$.

Let $x \in M \setminus PM$. Therefore there exists an ideal I of R such that $IM = Rx$. It is easy to see that $I \nsubseteq P$ thus $I_P = R_P$ and we have,

$$R_P x = R_P(IM)_P = R_P I_P M_P = R_P M_P = M_P.$$

We shall show that $\mathrm{Ann}_{R_P}(x) = 0$. Suppose that $a/b \in \mathrm{Ann}_{R_P}(x)$. So

$$ax/b = 0 \Longrightarrow \exists u \in R \setminus P \text{ such that } uax = 0.$$

Hence $0 = Ruax = Rua(IM) = (Iua)M$. Since M is faithful, $Iua = 0$ and so $0 = I_P ua = R_P ua = R_p a$. Hence $a/b = 0$, as wanted.

□

Theorem 2.5. *Let M be a faithful multiplication R-module. Then the following statements are equivalent:*
(a) *M is finitely generated.*
(b) *$M_P \neq 0$ for every prime ideal P of R.*
(c) *$M_P \neq 0$ for all maximal ideal P of R.*
(d) *$M_p \cong R_p$ for every $P \in$Spec(R).*
(e) *Hom$(M, R/P) \neq 0$ for every maximal ideal P of R.*

Proof. $(a) \Rightarrow (b) \Rightarrow (d)$ Let P be a prime ideal of R and $M = Rx_1 + \cdots + Rx_n$. If $M_P = 0$, then there exists $s_1, \ldots, s_n \in R \setminus P$ such that $s_1x_1 = s_2x_2 = \cdots = s_nx_n = 0$. Thus $0 \neq s_1s_2 \ldots s_n \in \mathrm{Ann}(M)$, that is a contradiction. Therefore $M_P \neq 0$ for every prime ideal P of R. Hence, by Theorem 1.1 M_P is cyclic R_P-module and by Lemma 2.4, M_P is a free R_P-module and thus $M_P \cong R_P$.
$(d) \Rightarrow (b) \Rightarrow (c)$ are clear.
$(c) \Rightarrow (a)$ It follows by Theorem 2.3.
$(c) \Rightarrow (e)$ By Theorem 2.2, $PM \neq M$ for every maximal ideal P of R. Therefore $\mathrm{Hom}(M, R/P) \neq 0$.
$(e) \Rightarrow (c)$ Let $\mathrm{Hom}(M, R/P) \neq 0$ for a maximal ideal P of R. Then it is clear that $PM \neq M$ and thus by Theorem 2.2, we get $M_P \neq 0$. □

Note that there exists an example of a multiplication module that is not cyclic and satisfy in Theorem 2.5. Let R be a domain and Q be the fraction field of R. An ideal I of R is called *invertible ideal* whenever $IJ = R$ for some subset J of Q. Let $I := \langle 3, 2 + \sqrt{-5} \rangle \subset \mathbb{Z}[\sqrt{-5}]$. It is easy to check that $3^{-1}(2 - \sqrt{-5}) \in I^{-1}$ and $II^{-1} = R$. Thus I is an invertible ideal and so it is multiplication module which is not cyclic.

The following corollary is proved by Nauom [5] and it is extended to duo rings by Tuganbaev [10].

Corollary 2.6. *Every faithful multiplication R-module M is a flat R-module.*

Proof. Let M be a faithful multiplication R-module. By Lemma 2.4, M is locally free and by [7, Exercise 4.14], M is flat. □

Corollary 2.7. *Every faithful multiplication module over a Noetherian ring is projective.*

Proof. Note that by [4, Corollary 3.3], a faithful multiplication module over a Noetherian ring is a finitely generated module. Now, the result follows from Lemma 2.4 and [7, Exercise 4.15]. □

Acknowledgments

The author would like to thank the referee for his/her careful reading of the paper and valuable suggestions to improve the presentation of this paper.

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Yaser Tolooei
Department of Mathematics, Faculty of Science, Razi University, Kermanshah, 67149-67346, Iran.
`y.toloei@razi.ac.ir`

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Y. TOLOOEI

مدول‌های ضربی متناهی‌مولد

یاسر طلوعی
گروه ریاضی، دانشکده علوم، دانشگاه رازی، کرمانشاه، ایران

فرض کنید R یک حلقه جابه‌جایی یک‌دار و M یک R-مدول یکانی باشد. R-مدول M را ضربی می‌نامند هرگاه برای هر زیرمدول N از M یک ایده‌آل از R مانند I وجود داشته باشد به گونه‌ای‌که $N = IM$. ثابت می‌کنیم که برای یک دامنه نوتری R با $\dim(R) \leq ۱$، مدول‌های ضربی یا دوری هستند یا با یک ایده‌آل وارون‌پذیر از R یکریخت هستند. به‌علاوه، ما یک مشخصه‌سازی برای مدول‌های ضربی متناهی‌مولد ارائه خواهیم کرد.

کلمات کلیدی: مدول ضربی، حلقه نوتری، مدول وفادار.