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SOME CLASSIFICATIONS OF MONOIDS BY VARIOUS NOTIONS OF INJECTIVITY OF ACTS

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Abstract. This paper is a continuation of recent researches concerning generalization of injectivity of acts over moniods, namely, C-injectivity and InD-injectivity. We introduce new kinds of injectivity, such as, LC-injectivity and CQ-injectivity. Classifications of monoids by properties of these kinds of injective acts are presented. It is proved that a monoid S is completely (cyclic) injective if and only if it is completely quasi (CQ-) injective. Some results on quasi-injective acts are proved. Also new characterizations for right hereditary monoids and right PP-monoids are given.

1. Introduction

Throughout this note, unless otherwise stated, S is a monoid. A *(right) S-act* A is a non-empty set on which S acts unitarily. Also by an S-act we mean a right S-act. An S-act A is called *injective*, if for any S-act B, any subact B' of B and any homomorphism $f \in Hom(B', A)$, there exists a homomorphism $\overline{f} \in Hom(B, A)$ which extends f, that is, $\overline{f}\mid_{B'} = f$.

(I)
$$\begin{tikzcd} B' \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[dl, "\exists \overline{f}"] \\ A & \end{tikzcd}$$

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If B' in the above diagram is cyclic, then A is called C-*injective*, which was studied in [14]. Moreover, if B is also cyclic in diagram **(I)**, then A is called CC-*injective* which was introduced in [15].

The concept of injectivity of acts with respect to (special) subclasses of monomorphisms of acts has been considered recently (for example, see [14], [15], [11] and [10]). In this paper, we will introduce some other types of injectivity and investigate the relationship between various kinds of injectivities. This investigation also leads to some new results on C-injectivity in Theorems 3.3 and 3.6.

An S-act D is *indecomposable*, if it is not a disjoint union of any two non-empty subacts. Clearly every cyclic act is indecomposable and by [6, Lemma 1.5.36], any homomorphic image of an indecomposable act is indecomposable. Also an S-act C is called *locally cyclic* if every finitely generated subact of C is contained within a cyclic subact, that is, for all $a, b \in C$, there exists $c \in C$ such that $a, b \in cS$. A right ideal of S is called a *locally principal right ideal* if it is locally cyclic as a right S-act. As an example, it is well-known that every direct limit of every sequence of monoids is locally cyclic. Locally cyclic acts have useful role in homological classification of monoids. For example, in [5], it is proved that every S-act has a projective cover if and only if every locally cyclic S-act has a projective cover. In [14], it is shown that all cyclic S-acts are injective if and only if all S-acts are C-injective. As every cyclic act is locally cyclic, it seems reasonable to ask whether a similar result holds for locally cyclic acts, that is, when every locally cyclic S-act is injective. This question has been answered in [10] for indecomposable acts and new kinds of injectivity of acts have been introduced by using the notion of indecomposability. It is shown that if every indecomposable S-act is injective, then every S-act is injective. This motivates us to define a new class of injective acts strictly between the new kinds of injective acts that are mentioned above, by means of locally cyclic acts.

To introduce the new kinds of injectivity we follow diagram **(I)** and define the following notions of injectivity. An S-act A is called *locally cyclic injective* or LC-*injective* for short, if for any S-act B, any locally cyclic subact B' of B and any homomorphism $f \in Hom(B', A)$, there exists a homomorphism $\overline{f} \in Hom(B, A)$ which extends f, that is, $\overline{f} \mid_{B'} = f$. Also as in [10], we call A is a *indecomposable domain injective* or InD-*injective* S-act, If B' is indecomposable as a substitute for locally cyclic in this definition. Also in particular, an S-act A is called *locally cyclic domain weakly injective* or LC-*weakly injective* for short, if for any locally principal right ideal K of S and any homomorphism $f \in Hom(K, A)$, there exists a homomorphism $\overline{f} \in Hom(S, A)$ which extends f, that is, $\overline{f}\mid_K = f$. Also if K is indecomposable instead of being locally cyclic, then A is called *indecomposable domain weakly injective* or InD-*weakly injective* for short.

By [9, Lemma 3.4], every locally cyclic act is indecomposable. Thus we have the following implications:

$$\text{InD-injectivity} \Longrightarrow \text{LC-injectivity} \Longrightarrow \text{C-injectivity}$$

and,

$$\text{weakly injectivity} \Longrightarrow \text{InD-weak injectivity} \Longrightarrow \text{LC-weak injectivity}$$
$$\Longrightarrow \text{principally weak injectivity}$$

In a series of examples in [7] (Examples 2.4, 3.1, 3.3 and 3.17), it is shown that the above implications are strict. Also, it is well-known that if A is an indecomposable $S-$act and $a, b \in A$, then there exist $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \in S,\ a_1, a_2, \ldots, a_n \in A_S$ such that

$$a = a_1s_1,\ a_1t_1 = a_2s_2,\ a_2t_2 = a_3s_3, \ldots,\ a_nt_n = b$$

(see for example, [9]). If S is a group, by substituting elements in the above sequence, we have $b \in aS$, for an arbitrary element $b \in A$. So A is generated by any element $a \in A$. Thus any indecomposable act A over a group S is cyclic. Therefore over groups the notions of InD-injectivity, LC-injectivity and C-injectivity of acts are equivalence.

In this paper we shall study the relation between these kinds of injectivity and other kinds of injectivity over a monoid with a (right) zero.

A monoid S is called *right hereditary* if all right ideals of S are projective. Also S is called a *right PP-monoid* if all principal right ideals of S are projective. In the first section we study monoids over which factors of injective acts are InD-weakly injective. It is shown that such monoids are exactly right hereditary monoids. By a similar proof we show that a monoid over which every factor of an injective act is principally weakly injective, is a right PP-monoid.

An S-act A is called *quasi injective* if any homomorphism from any subact of A can be extended to A. In Section 3, we will study quasi injective acts. An S-act A is called CQ-injective if it is quasi injective relative to all inclusions from cyclic subacts of A, that is, any homomorphism from any cyclic subact of A can be extended to A. We shall show that if all S-acts are CQ-injective, then all S-acts are C-injective.

The reader is referred to [6] for notations and preliminaries and basic results related to acts.

First we have a few useful results on the new concepts that are defined above. Some of them are proved in [10] for InD-injectivity. The proofs of these results are easy and will be omitted.

Proposition 1.1. *The following statements are equivalent:*

(i) *An S-act C is LC-injective (resp. InD-injective).*
(ii) *For any S-act B containing C, and any family $\{c_i\}_{i\in I}$ in C that generates a locally cyclic (resp. an indecomposable) S-act, there exists an S-morphism $f \in Hom(B, C)$ such that $f(c_i) = c_i$, for any $i \in I$.*
(iii) *For any family $\{c_i\}_{i\in I}$ in C that generates a locally cyclic (resp. an indecomposable) S-act, there exists an S-morphism $f \in Hom(E(C), C)$ such that $f(c_i) = c_i$, for any $i \in I$, where $E(C)$ is the injective envelope of C.*

Proposition 1.2. *Every LC-injective S-act (and so every InD-injective S-act) contains a zero.*

Note that there is a right $S-$act with a zero that is not LC-injective. For instance, let S be a monoid with a left zero which is not injective as a right $S-$act (for example consider the monoid S of non-negative integers with multiplication, then the identity morphism of S cannot be extended to a homomorphism from $\frac{1}{2}S$ to S). Then by [14, Proposition 6], the cyclic right $S-$act S is not C-injective. Thus it is not LC-injective.

Corollary 1.3. *Every locally cyclic (resp. indecomposable) LC-injective (resp. InD-injective) S-act is injective.*

Proposition 1.4. *Suppose that S is a monoid. Then the following statements hold.*

(i) *Any retract of an LC-injective (resp. InD-injective) act is also LC-injective (resp. InD-injective).*
(ii) *If $A = \prod_{i\in I} A_i$, where each A_i is an S-act, then A is LC-injective (resp. InD-injective) if and only if A_i is LC-injective (resp. InD-injective), for every $i \in I$.*
(iii) *If $A = \coprod_{i\in I} A_i$, where each A_i is an S-act, then A is LC-injective (resp InD-injective) and each A_i contains a zero, if and only if A_i is LC-injective (resp InD-injective), for each $i \in I$.*

Moreover, similar results are true for retracts, products and coproducts of LC-weakly injective (InD-weakly injective) S-acts.

2. Different Kinds of Injectivity Over Monoids

In this section we prove some results about different kinds of (weak) injectivity over a monoid. By $E(A)$ we mean the injective envelope of an S-act A.

Proposition 2.1. *Let S be a monoid and A be an S-act. Then A is C-injective (resp. LC-injective, InD-injective) if and only if $E(B') \subseteq A$, for each cyclic (resp. locally cyclic, indecomposable) subact B' of A.*

Proof. We prove the result for C-injective S-acts. The proofs of other cases are similar. Let B' be a cyclic subact of A. Since A is C-injective, the inclusion map from B' into A can be extended to a homomorphism $g : E(B') \longrightarrow A$. Moreover, since B' is essential in $E(B')$, g is a monomorphism. Thus $E(B') \subseteq A$.

For the converse, suppose that D is a cyclic S-act contained in an S-act B and $f : D \longrightarrow A$ is a homomorphism. Then $B' = f(D)$ is a cyclic subact of A and so by assumption $E(B') \subseteq A$. Thus f can be considered as a homomorphism from D into $E(B')$. Hence there is an extension $\overline{f}$ of f from B into $E(B')$ which is also an extension into A. So A is C-injective. □

We have already seen the behavior of product and coproduct of a family of S-acts under different kinds of injectivity. Now we study the behavior of direct sum under such injectivities. In the category of acts over a monoid S the product of a family of objects is their cartesian product which is an S-act under the componentwise action of S. As an important subset of the product we have direct sum of a family of S-acts that will be used frequently in the rest of this paper.

Let $\{A_i\}_{i\in I}$ be a family of S-acts such that each A_i has a fixed element θ_i. The direct sum of these acts is denoted by $\bigoplus_{i\in I} A_i$ and is defined by the set

$$\{x \in \prod_{i\in I} A_i | \ \pi_j(x) \neq \theta_j \ \text{ for at most finite number of } \ j \in I\},$$

where π_j is projection to j-th component of x, for each $j \in I$. Then clearly $\bigoplus_{i\in I} A_i$ with the componentwise action of S is a subact of $\prod_{i\in I} A_i$. Clearly, when I is a finite set then the direct sum of A_i's is their product.

Note that the direct sum $\bigoplus_{i\in I} A_i$ of a family of S-acts, $\{A_i\}_{i\in I}$, depends on the choice of zeros of A_i's. If S is a monoid and 0 is a zero of S, then every S-act A contains an element θ such that for any $s \in S$, $\theta s = \theta$. Besides, if for each $a \in A$, $a0 = \theta$, then A is called a *centered (right) S-act.* Clearly, in this case θ is unique. Noting this, to define

$\bigoplus_{i\in I} A_i$ we always consider the centered acts. But our results are also true if we define $\bigoplus_{i\in I} A_i$ by choosing a zero of each A_i.

Recall that a monoid S is right Noetherian if all right ideals of S are finitely generated. Clearly, S is right Noetherian if and only if S satisfies the ascending chain condition for right ideals.

Lemma 2.2. *Let S be a monoid with a zero. Then S is right Noetherian if and only if $E(\bigoplus_{i\in I} A_i) \subseteq \bigoplus_{i\in I} E(A_i)$ for every family of injective S-acts $\{A_i\}_{i\in I}$.*

Proof. If S is right Noetherian, then by [12, Theorem 2], $\bigoplus_{i\in I} E(A_i)$ is an injective S-act, which contains $\bigoplus_{i\in I} A_i$. Thus $E(\bigoplus_{i\in I} A_i) \subseteq \bigoplus_{i\in I} E(A_i)$.

Let $\{A_i\}_{i\in I}$ be a family of injective S-acts and $A = \bigoplus_{i\in I} A_i$. Then by assumption, $A \subseteq E(A) \subseteq \bigoplus_{i\in I} E(A_i) = \bigoplus_{i\in I} A_i = A$. So $A = \bigoplus_{i\in I} A_i$ is injective. Thus by [12, Theorem 2], S is right Noetherian. □

In general direct sum of injective acts is not injective, but for C-injective (LC-injective) acts we have the following proposition.

Proposition 2.3. *Let A_i, $i \in I$ be S-acts, and $A = \bigoplus_{i\in I} A_i$. Then the following statements hold.*

(i) *A is C-injective if and only if A_i is C-injective, for every $i \in I$.*
(ii) *If A is LC-injective then A_i is LC-injective, for every $i \in I$. On the other hand if S is right Noetherian and each A_i is LC-injective, then A is LC-injective.*

Proof. Part (i) can be easily seen by [11, Theorem 3.30 and Proposition 3.32]. Also proof of the first part of (ii) is easy.

Second part of (ii). Let S be right Noetherian and A_i be LC-injective for each $i \in I$. Suppose that a family $\{x_j = (a_{ji})_{i\in I}\}_{j\in J}$ of elements of A generates a locally cyclic subact of A. Then it is easy to see that $\{a_{ji}\}_{j\in J}$ generates a locally cyclic subact of A_i. Since each A_i is LC-injective, by Proposition 1.1 there exists a homomorphism $f_i \in Hom(E(A_i), A_i)$ which is fixed on $\{a_{ji}\}_{j\in J}$. Consider the map $f = \bigoplus_{i\in I} f_i$ from $\bigoplus_{i\in I} E(A_i)$ into A with $f((b_i)_{i\in I}) = (f_i(b_i))_{i\in I}$. Then f is a well-defined homomorphism and is fixed on $\{x_j\}_{j\in J}$. Also by Lemma 2.2, $E(A) \subseteq \bigoplus_{i\in I} E(A_i)$. So the restriction of f to $E(A)$ is a homomorphism and is fixed on $\{x_j\}_{j\in J}$. Thus by Proposition 1.1, A is LC-injective. □

In [4], it is proved that if all finitely generated weakly injective S-acts are weakly injective, then S is right Noetherian. Now, we have a similar result for C-injective acts.

Proposition 2.4. *Let S be a monoid which contains a zero. If all C-injective S-acts are weakly injective, then S is right Noetherian.*

Proof. Let $\{A_i\}_{i\in I}$ be a family of injective S-acts. By Proposition 2.3, $A = \bigoplus_{i\in I} A_i$ is a C-injective S-act which is weakly injective by assumption. The result now follows by [1, Lemma 1]. $\square$

Proposition 2.5. *If S is a monoid containing a zero, then the following statements are equivalent:*

(i) *All principally weakly injective S-acts are weakly injective.*
(ii) *All C-injective S-acts are weakly injective and every finitely generated right ideal is principal.*
(iii) *All LC-weakly injective S-acts are weakly injective and every locally principal right ideal is principal.*
(iv) *S is a principal right ideal monoid.*

Proof. (i)$\Longleftrightarrow$ (iv) follows by [6, Theorems 4.2.16 and 4.3.6].

(i)$\Longrightarrow$ (ii). Note that every C-injective act is principally weakly injective. So (ii) follows from (i)$\Longleftrightarrow$(iv).

(ii)$\Longrightarrow$ (iv). By the first part of (ii) and Proposition 2.4, S is right Noetherian and so all of its right ideals are finitely generated. So S is a principal right ideal monoid by the second part of (ii).

(iv)$\Longrightarrow$ (iii) is obvious.

(iii)$\Longrightarrow$ (i). Since all locally principal right ideals are principal, the concepts of LC-weak injectivity and principally weak injectivity are equivalent. So by the first part of (iii), all principally weakly injective acts are weakly injective. $\square$

By an adaptation of the technique on the proof of [1, Lemma 1], we have similar results.

Proposition 2.6. *If each direct sum of principally weakly injective (resp. injective) S-acts is LC-weakly injective, then S satisfies the ascending chain condition on locally principal right ideals.*

Note that by [6, Lemma 1.5.9], for a chain $\{I_i\}_{i\in\mathbb{N}}$ of indecomposable right ideals of S, the right ideal $I = \bigcup_{i\in\mathbb{N}} I_i$ is also indecomposable since $\bigcap_{i\in\mathbb{N}} I_i \neq \emptyset$. Using this fact we obtain a similar proposition.

Proposition 2.7. *If all direct sums of principally weakly injective (resp. injective) S-acts are InD-weakly injective, then S satisfies the ascending chain condition on indecomposable right ideals.*

Note that the direct sum of principally weakly injective acts is principally weakly injective, but this is not true for LC-weak injectivity. The following result provides the equivalent conditions for LC-weak injectivity case.

Corollary 2.8. *Suppose that S is a monoid containing a zero. The following statements are equivalent:*

(i) *All principally weakly injective S-acts are LC-weakly injective.*
(ii) *All C-injective S-acts are LC-weakly injective.*
(iii) *All finitely generated weakly injective S-acts are LC-weakly injective.*
(iv) *Every direct sum of LC-weakly injective S-acts is LC-weakly injective.*
(v) *S satisfies the ascending chain condition on its locally principal right ideals.*
(vi) *Every locally principal right ideal is principal.*

Proof. (v)$\Longrightarrow$ (vi). Let I be a locally principal right ideal. If I is not principal, then one may easily see that I contains a strict ascending chain of principal right ideals which is a contradiction.

(vi)$\Longrightarrow$ (iv). By (vi), principally weakly injectivity and LC-weakly injectivity are equivalent. Moreover, it can be easily seen that the direct sum of principally weakly injective acts is principally weakly injective and so is LC-weakly injective.

(iv)$\Longrightarrow$ (v) is clear by Proposition 2.6. Thus (iv), (v) and (vi) are equivalent. (i)$\Longrightarrow$ (ii) is clear.

(ii)$\Longrightarrow$ (v). By Proposition 2.3, each direct sum of C-injective acts is C-injective. So by (ii), each direct sum of injective acts is LC-weakly injective. Hence S satisfies the ascending chain condition on its locally principal right ideals, by Proposition 2.6.

(v)$\Longrightarrow$ (i). By the above proof, (v) implies (vi), and clearly (vi) implies (i). (i)$\Longrightarrow$ (iii) is obvious.

(iii)$\Longrightarrow$ (v). By the same proof as Proposition 2.3, every direct sum of finitely generated weakly injective S-acts is finitely generated weakly injective. So by (iii), every direct sum of finitely generated weakly injective S-acts is LC-weakly injective. Thus S satisfies the ascending chain condition on locally principal right ideals, by Proposition 2.6. $\square$

Recall that for a (semi-) hereditary monoid the quotient of every injective act is (finitly generated) weakly injective (see, [2]). Motivated by this, here we will characterize monoids over which every quotient of injective acts is principally (LC-, InD-) weakly injective.

First we present an extension of X-injectivity, where X stands for C, LC or InD, in order to characterize monoids over which each factor of an X-weakly injective act has the same property.

Definition 2.9. Let A and B be S-acts. A is called C-*injective (resp. LC-injective, InD-injective) relative to* B or B-C-*injective* (resp. B-LC-*injective*, B-InD-*injective)*, if all homomorphisms from cyclic (resp. locally cyclic, indecomposable) subacts of B into A can be extended to B. Also an S-act A is called C-*quasi injective* (CQ-*injective* for short) if it is A-C-injective.

Clearly A is C-injective (resp. LC-injective, InD-injective) if it is B-C-injective (resp. B-LC-injective, B-InD-injective) for all S-acts B.

Since every S-act is contained in an injective (a principally weakly injective) S-act, it is easy to see that an S-act A is C-injective if and only if it is B-C-injective for each injective (principally weakly injective) S-act B. Similar assertions are true for LC-injectivity and InD-injectivity.

Recall that an S-act A is regular if and only if every cyclic subact of A is projective (see [6, Corollary 3.19.3]). The following practical proposition is a characterization for regular acts using the concept of C-injectivity.

Proposition 2.10. *If B is a regular S-act, then every factor act of any B-C-injective (injective) S-act is B-C-injective. The converse is also true if B is projective. In particular every factor of a regular CQ-injective S-act A is A-C-injective.*

Proof. First assume that A is a B-C-injective (injective) S-act and C is a cyclic subact of B and ρ is a right congruence on A. Let $f : C \longrightarrow A/\rho$ be a homomorphism and $\pi : A \longrightarrow A/\rho$ be the natural projection. By projectivity of C, there exists a homomorphism $h : C \longrightarrow A$, such that $f = \pi h$. Now since A is B-C-injective (injective), there exists a homomorphism $\overline{h} : B \longrightarrow A$, which extends h. Now $\pi\overline{h}$ is an extension of f, that is, A/ρ is B-C-injective.

Conversely, assume that B is projective and any factor act of any B-C-injective (injective) S-act is B-C-injective. Let D be a cyclic subact of B, $h : M \longrightarrow N$ an epimorphism of S-acts and $g : D \longrightarrow N$ a homomorphism. To show that D is projective, it suffices to show that g can be lifted to M. We have $N \cong M/kerh \subseteq E(M)/kerh$. So we may consider g as a homomorphism into $E(M)/kerh$. Also by assumption, $E(M)/kerh$ is B-C-injective. Thus there exists an extension $\overline{g} : B \longrightarrow E(M)/kerh$ of g. Now since B is projective, there exists k which lifts $\overline{g}$ to $E(M)$, that is, $\overline{g} = \pi k$, where $\pi : E(M) \longrightarrow E(M)/kerh$ is the natural epimorphism. If $j : M/kerh \longrightarrow E(M)/kerh$ is the inclusion map, then for each $d \in D$,

$$\pi k(d) = \overline{g}(d) = jg(d).$$

So $kkerh = mkerh$, that is, $(k(d), m) \in kerh$, for some $m \in M$. Since $kerh \subseteq M \times M$, $k(D) \subseteq M$. Hence $k|_D$ lifts g.

To prove the last part, it suffices to put $B = A$ in the above proof. $\square$

Using similar proofs as in Proposition 2.10 , one can get the following results.

Proposition 2.11. *If S is a monoid and B is a projective S-act, then the following statements are equivalent:*

(i) *Every cyclic subact of B is projective (that is, B is a regular S-act).*
(ii) *Every factor act of every B-C-injective S-act is B-C-injective.*
(iii) *Every factor act of every B-LC-injective S-act is B-C-injective.*
(iv) *Every factor act of every B-InD-injective S-act is B-C-injective.*
(v) *Every factor act of every injective S-act is B-C-injective.*

Proposition 2.12. *If S is a monoid and B is a projective S-act, then the following statements are equivalent:*

(i) *Every locally cyclic subact of B is projective.*
(ii) *Every factor act of every B-LC-injective S-act is B-LC-injective.*
(iii) *Every factor act of every B-InD-injective S-act is B-LC-injective.*
(iv) *Every factor act of every injective S-act is B-LC-injective.*

Proposition 2.13. *If S is a monoid and B is a projective S-act, then the following statements are equivalent:*

(i) *Every indecomposable subact of B is projective.*
(ii) *Every factor act of every B-InD-injective S-act is B-InD-injective.*
(iii) *Every factor act of every injective S-act is B-InD-injective.*

Recall that a monoid S is called a *right PP-monoid* if all principal right ideals of S are projective. Also we call a monoid a *right* LCP-*monoid* (resp. *right* IndP-*monoid*) if all locally principal (resp. indecomposable) right ideals of S are projective.

Note that S as an S-act is projective. As consequences of the above three propositions, putting $B = S$, we have the following three corollaries.

Corollary 2.14. *The following are equivalent for a monoid S:*

(i) *S is a right PP-monoid.*
(ii) *Every factor of every principally weakly injective S-act is principally weakly injective.*
(iii) *Every factor of every LC-weakly injective S-act is principally weakly injective.*
(iv) *Every factor of every InD-weakly injective S-act is principally weakly injective.*

(v) *Every factor of every weakly injective S-act is principally weakly injective.*
(vi) *Every factor of every injective S-act is principally weakly injective.*

Corollary 2.15. *The following are equivalent for a monoid S:*

(i) *S is a right LCP-monoid.*
(ii) *Every factor of every LC-weakly injective S-act is LC-weakly injective.*
(iii) *Every factor of every InD-weakly injective S-act is LC-weakly injective.*
(iv) *Every factor of every weakly injective S-act is LC-weakly injective.*
(v) *Every factor of every injective S-act is LC-weakly injective.*

Corollary 2.16. *The following are equivalent for a monoid S:*

(i) *S is a right IndP-monoid.*
(ii) *S is a right hereditary monoid.*
(iii) *Every factor of every InD-weakly injective S-act is InD-weakly injective.*
(iv) *Every factor of every weakly injective S-act is InD-weakly injective.*
(v) *Every factor of every injective S-act is InD-weakly injective.*

Proof. Note that every right ideal I of S has a decomposition into indecomposable right ideals. So if (i) is satisfied, then I is projective as the coproduct of projective right ideals. Thus (i) and (ii) are equivalent. The rest of the proof is easy and it is omitted as the proofs of the above two corollaries. □

3. Quasi Injectivity

In this section, we shall study the relation between quasi injectivity and other kinds of injectivity of acts. The following proposition has a key role in the rest of this section.

Proposition 3.1. *Let A be an S-act with a zero. Then $A \oplus E(A)$ is quasi injective if and only if A is injective. The result is also true when A is a cyclic act with a zero and $A \oplus E(A)$ is CQ-injective.*

Proof. Let $A \oplus E(A)$ be quasi injective and consider the following diagram of S-acts and S-morphisms,

$$\begin{tikzcd}
A \arrow[r, "\imath"] \arrow[d, "id"'] & E(A) \arrow[r, "\jmath_2"] & A \oplus E(A) \\
A \arrow[d, "\jmath_1"'] & & \\
A \oplus E(A) & &
\end{tikzcd}$$

where $\imath, \jmath_1$ and $\jmath_2$ are inclusions. Let $p \in Hom(A \oplus E(A), A)$ be the canonical projection into A. Then $p\jmath_1 = id$. Since $\jmath_1 id$ is a monomorphism and $A \oplus E(A)$ is quasi injective, there exists $f \in End(A \oplus E(A))$, such that, $f\jmath_2\imath = \jmath_1 id$. Thus $(pf\jmath_2)\imath = pf\jmath_2\imath = p\jmath_1 id = id$, that is, A is a retract of $E(A)$. So A is injective.

For the converse note that if A is injective, then $A \oplus E(A)$ is injective and so is quasi injective. The proof for CQ-injectivity is similar. $\square$

Corollary 3.2. *Suppose that S is a monoid with a zero. Then the direct sum of two quasi injective S-acts is quasi injective if and only if every quasi injective S-act is injective.*

Proof. Let A be a quasi injective S-act. Then by assumption, $A \oplus E(A)$ is quasi injective. So A is injective by Proposition 3.1.

The converse is obvious because the direct sum of two injective acts is their product, which is injective by assumption. $\square$

Theorem 3.3. *The following statements are equivalent for a monoid S with a zero:*

(i) *Every C-injective S-act is injective.*
(ii) *Every C-injective S-act is quasi injective.*
(iii) *Every direct sum of C-injective S-acts is injective.*
(iv) *Every direct sum of C-injective S-acts is quasi injective.*

In particular, these equivalent statements hold, provided that all right ideals of S are principal.

Proof. (i)$\Longrightarrow$ (iii) is clear because the direct sum of C-injective right S-acts is C-injective by Proposition 2.3. The implications (iii)$\Longrightarrow$ (iv) and (iv)$\Longrightarrow$ (ii) are clear. (ii)$\Longrightarrow$ (i). Let A be a C-injective S-act. By Proposition 2.3, $A \oplus E(A)$ is C-injective and so is quasi injective by assumption. So by Proposition 3.1, A is injective.

The proof of the last part follows from the fact that all subacts of cyclic S-acts are cyclic when all right ideals of S are principal. $\square$

Proposition 3.4. *If the equivalent statements of Theorem 3.3 are satisfied, then every right ideal of S is finitely generated and indecomposable. In particular, every right ideal of S is principal if and only if* *every C-injective S-act is injective and every two right ideals of S are comparable with respect to inclusion.*

Proof. By Proposition 2.3, every direct sum of C-injective acts is C-injective. So by assumption every direct sum of injective S-acts is injective. Thus by [12, Theorem 2], S is right Noetherian. So every right ideal of S is finitely generated. In a similar way, using [14, Proposition 8] and [6, Theorem 3.1.13] S is left reversible. So by [10, Proposition 2.2], every right ideal of S is indecomposable. To prove the last part note that if every two ideals are comparable, then every finitely generated right ideal of S is principal. □

Recall that an idempotent $e \in S$ is called *right special* if for any right congruence ρ of S, there exists an element $k \in eS$ such that, $(ke)\rho\, e$ and $s\rho\, t$, $s, t \in S$, implies $(ks)\rho(kt)$ (see, [6, Definition 4.4.2]).

Theorem 3.5. *For a monoid S with a zero the following statements are equivalent:*

(i) *All S-acts are injective.*
(ii) *All S-acts are quasi injective.*
(iii) *Every right ideal of S is generated by a right special idempotent.*

Moreover, if the idempotents of S are central, then the above statements are equivalent to:

(iv) *All right ideals of S are injective.*
(v) *All S-acts are CC-injective and S is a principal right ideal monoid.*
(vi) *For each S-act A, $A \oplus S$ is quasi injective.*
(vii) *For each right ideal I of S, $I \oplus S$ is quasi injective.*
(viii) *For each S-act A, $A \oplus S$ is injective.*
(ix) *For each right ideal I of S, $I \oplus S$ is injective.*
(x) *For each indecomposable right ideal I of S, $I \oplus S$ is injective.*
(xi) *Every right ideal of S is generated by an idempotent.*

Proof. It is easy to see that by Proposition 3.1 and [13, Theorem 2], (i), (ii) and (iii) are equivalent. (i) $\Longrightarrow$ (iv) is clear.

(iv) $\Longrightarrow$ (i). Since every right ideal of S is weakly injective, it is generated by an idempotent. The result now follows by [3, Theorem 2.6].

(iv) $\Longleftrightarrow$ (v). Again since each right ideal of S is weakly injective, it is generated by an idempotent. Also since all idempotents are central, all idempotents are right special. Thus by [15, Corollary 10], all S-acts are CC-injective. Conversely, by [15, Corollary 10], every right ideal of S is generated by an idempotent. So by [3, Theorem 2.6], all right ideals of S are injective. The implications (vii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(ix) follow by [1, Theorem 3].

(ix) $\Longrightarrow$ (x) is clear.

(x) $\Longrightarrow$ (ix). Note that every right ideal I has the decomposition $I = \coprod_{i\in I} I_i$, into indecomposable right ideals I_i of S and $I \oplus S = \coprod_{i\in I}(I_i \oplus S)$. Then each $I_i \oplus S$ is injective by (x) and so $I \oplus S$ is injective, since S is left reversible. The implications (i) $\Longrightarrow$ (vi) $\Longrightarrow$ (vii) and (i) $\Longrightarrow$ (viii) $\Longrightarrow$ (ix) are clear.

Now we show that (ix) $\Longrightarrow$ (i). Let I be a right ideal of S, $I \oplus S$ be injective and consider the following diagram of S-acts and S-morphisms, where $\imath$ and $\jmath$ are inclusions.

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[d, "id"'] & S \\ I \arrow[d, "\jmath"'] & \\ I \oplus S & \end{tikzcd}$$

Let $p \in Hom(I \oplus S, I)$ be the canonical projection onto I. Then $p\jmath = id$. Since $\jmath\ id$ is a monomorphism and $I \oplus S$ is injective, there exists $f \in Hom(S, I \oplus S)$, such that, $f\imath = \jmath\ id$. Now put $g := pf \in Hom(S, I)$. Then $g\imath = pf\imath = p\jmath\ id = id$. Thus $I = g(S) = g(1)S$. Moreover $g(1)g(1) = g(1g(1)) = g(g(1)) = g(1)$. So I is generated by an idempotent. Thus (i) follows by [3, Theorem 2.6].

Finally, since all idempotents are central, (xi) is equivalent to (v) by [15, Corollary 10]. Thus the proof is complete. □

A monoid over which all right acts are quasi injective (resp. CQ-injective) is called a *completely right quasi injective monoid* (resp. *completely right CQ-injective monoid*). In the above theorem we have shown that completely right quasi injective monoids are exactly completely right injective monoids. The following theorem is a similar result for monoids over which all cyclic acts are injective (that is, completely cyclic injective monoids) and is a characterization for completely right CQ-injective monoids. Let K be a right ideal of S, μ a right congruence on S and s an element of S. As in Section 2 of [14], we use $K(s,\mu)$ to show the set $\{a \in S | [sa]_\mu \in \overline{K}_\mu\}$, where $\overline{K}_\mu = \{[k]_\mu \in S/\mu | k \in K\}$.

Theorem 3.6. *The following statements are equivalent for a monoid S with a left zero:*

(i) *All S-acts are CQ-injective.*

(ii) *All cyclic S-acts are injective.*
(iii) *All S-acts are C-injective.*
(iv) *For every right ideal K of S, right congruences μ and λ on S, and every homomorphism $f \in Hom(\overline{K}_\mu, S/\lambda)$, there exists an element $q \in S$ such that $f([m]_\mu) = [q]_\lambda m$ for each $[m]_\mu \in \overline{K}_\mu$, and $K(s,\mu) = K(t,\mu)$, $s,t \in S$, and $(qsa)\lambda(qta)$, for all $a \in K(s,\mu)$ implies that $(qs)\lambda(qt)$.*

Proof. In light of [14, Theorem 14], it suffices to prove the non-trivial implication (i)$\Longrightarrow$ (ii). Let C be a cyclic S-act. By assumption $C \oplus E(C)$ is CQ-injective. So C is injective by Proposition 3.1. ☐

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SOME CLASSIFICATIONS OF MONOIDS BY VARIOUS NOTIONS OF INJECTIVITY OF ACTS

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برخی از دسته‌بندی‌های تکواره‌ها بر اساس مفهوم تزریقی بودن سیستم‌ها

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این مقاله ادامه‌ی مباحث اخیر در مورد تعمیم‌هایی از مفهوم سیستم‌های تزریقی روی نیم‌گروه‌هاست. از جمله‌ی این مفاهیم سیستم‌های تزریقی-دوری و تزریقی-تجزیه‌ناپذیر هستند که در مراجع [۱۵، ۱۴، ۱۰، ۷] به آن‌ها پرداخته شده است. در این مقاله سیستم‌های تزریقی-موضعاً دوری (ضعیف) به عنوان تعمیم دیگری از سیستم‌های تزریقی معرفی و مطالعه شده‌اند و دسته‌بندی‌هایی از نیم گروه‌ها برای حالات ضعیف این مفهوم انجام گرفته است. همچنین دسته‌بندی جدیدی برای تکواره‌های ارثی ارائه گردیده است.

کلمات کلیدی: سیستم‌های تزریقی، سیستم‌های تزریقی-موضعاً دوری، سیستم‌های تزریقی-تجزیه‌ناپذیر.