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ON REGULAR PRIME INJECTIVITY OF S-POSETS

H. RASOULI*, GH. MOGHADDASI AND N. SARVGHAD

Abstract. In this paper, we define the notion of regular prime monomorphism for S-posets over a pomonoid S and investigate some categorical properties including products, coproducts and pullbacks. We study $\mathcal{M}$-injectivity in the category of S-posets where $\mathcal{M}$ is the class of regular prime monomorphisms and show that the Skornjakov criterion fails for the regular prime injectivity. Considering a weaker form of such kind of injectivity, we obtain some classifications for pomonoids.

1. Introduction and preliminaries

Recall that a monoid (group) S is said to be a *pomonoid* (*pogroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation, it means that $s \leq t, s' \leq t'$ for each $s, t, s', t' \in S$ imply $ss' \leq tt'$. A non-empty subset I of a pomonoid S is said to be a *right ideal* if $IS \subseteq I$. A right ideal I of a pomonoid S is called a *right poideal* whenever $s \leqslant s'$ and $s' \in I, s \in S$ imply $s \in I$. For a subset X of a pomonoid S, the right poideal of S generated by X, denoted as $\downarrow(XS)$, is the set $\{t \in S \mid t \leq xs \text{ for some } x \in X, s \in S\}$. If X is finite, then it is called a *finitely generated right poideal*, and if $X = \{x\}$, then it is called a *principal right poideal* of S which is denoted by $\downarrow(xS)$. For a pomonoid S, a (*right*) *S-poset* is a poset A together with a mapping $A \times S \to A$, $(a, s) \mapsto as$ for $a \in A, s \in S$, called an *action*, satisfying the following conditions:

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*Corresponding author.

(i) $(as)t = a(st)$ for each $a \in A$, $s, t \in S$.
(ii) $a1 = a$ for each $a \in A$.
(iii) $a \leq b, s \leq t$ imply $as \leq bt$ for each $a, b \in A, s, t \in S$.

A non-empty subset B of an S-poset A is called a *sub S-poset* of A, whenever B is closed under the action with the same order as A. An element θ in an S-poset A with $\theta s = \theta$ for all $s \in S$ is called a *zero* element. An *S-poset map* (or *homomorphism*) is an action-preserving as well as order-preserving map between S-posets. Also a *regular monomorphism* (a morphism which is an equalizer) is exactly an *order-embedding*, that is, a homomorphism $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$. We denote the category of all (right) S-posets and homomorphisms between them by **Pos-**S. Recall that the *product* of a family of S-posets is their cartesian product, with componentwise action and order. Also the *coproduct* is their disjoint union, with natural action and componentwise order. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. For more information on acts and S-posets, one may consult [8, 7]. Throughout, S stands for a pomonoid unless otherwise stated.

Recall that a right ideal I of a monoid S is said to be *prime* if for $s, s' \in S$, the inclusion $sSs' \subseteq I$ implies that either $s \in I$ or $s' \in I$ (see [5]). Prime ideals are useful tools in the theory of semigroups. This notion was extended to an arbitrary S-act by Ahsan [1], analogous to the notion of prime module introduced by Dauns [6]. We say that a sub S-poset B of an S-poset A is a *prime sub S-poset* of A, or A is a *regular prime extension* of B, if B is a prime subact of A whenever A is considered as an act over S as a monoid, that is, for each $a \in A$ and $s \in S$, the inclusion $aSs \subseteq B$ implies either $a \in B$ or $As \subseteq B$. So, a right ideal I of a pomonoid S is a prime ideal if and only if it is prime as a sub S-poset of S_S. An S-poset homomorphism $f : A \to B$ is *prime* if $f(A)$ is a prime sub S-poset of B. Clearly, any surjective S-poset homomorphism is prime. By a *regular prime S-poset monomorphism* we mean a regular S-poset monomorphism which is prime. We investigate the products, coproducts, direct sums and the pullback stability property of regular prime S-poset monomorphisms.

Banaschewski [3] indicated the notion of $\mathcal{M}$-injectivity in a category $\mathcal{A}$, when $\mathcal{M}$ is a subclass of monomorphisms the members of which may be called $\mathcal{M}$-morphisms as the following definition. An object A is said to be *$\mathcal{M}$-injective* if for each $\mathcal{M}$-morphism $g : B \to C$, any morphism $f : B \to A$ can be lifted to a morphism $\bar{f} : C \to A$, that is,

$\bar{f}g = f$:

$$\begin{tikzcd} B \arrow[r, "g\in\mathcal{M}"] \arrow[d, "f"'] & C \arrow[dl, dotted, "\bar{f}"] \\ A & \end{tikzcd}$$

Here we study $\mathcal{M}$-injectivity where $\mathcal{M}$ is the class of all regular prime S-poset monomorphisms in the category **Pos-**S which will be called the regular prime injectivity. Analogously to the case of ordinary regular injectivity of S-posets, we show that every regular prime injective S-poset is complete, and the Skornjakov criterion also fails for the regular prime injectivity. Finally, by means of a weaker form of regular prime injectivity, we give some classifications for pomonoids. In particular, for a pomonoid S in which its identity is the bottom element, all principal right poideals of S are principally poideal regular prime injective if and only if S is poregular and principally poideal regular prime injective.

2. Categorical properties of regular prime monomorphisms

In this section, we study some categorical properties of regular prime monomorphisms of S-posets including the products, coproducts, direct sums and the pullback stability.

Proposition 2.1. *Let $f : A \to B$ be an S-poset homomorphism. Then the following assertions are equivalent:*
(i) *f is a regular prime S-poset monomorphism.*
(ii) *The product induced homomorphism $\prod_{i\in I} f : \prod_{i\in I} A \to \prod_{i\in I} B$ is a regular prime S-poset monomorphism.*
(iii) *The coproduct induced homomorphism $\coprod_{i\in I} f : \coprod_{i\in I} A \to \coprod_{i\in I} B$ is a regular prime S-poset monomorphism.*

Proof. (i) $\Rightarrow$ (ii) Assume that $f : A \to B$ is a regular prime S-poset monomorphism. Using [9, Proposition 2], it remains to show that $\prod_{i\in I} f : \prod_{i\in I} A \to \prod_{i\in I} B$ is prime. Note that $(\prod_{i\in I} f)(\prod_{i\in I} A) = \prod_{i\in I} f(A)$. We must prove that $\prod_{i\in I} f(A)$ is a prime sub S-poset of $\prod_{i\in I} B$. Let $\langle b_i\rangle_i Ss \subseteq \prod_{i\in I} f(A)$ for each $s \in S$ and $\langle b_i\rangle_i \in (\prod_{i\in I} B) \setminus (\prod_{i\in I} f(A))$. Then $b_i Ss \subseteq f(A)$ for each $i \in I$ and $b_j \notin f(A)$ for some $j \in I$. Since $f(A)$ is a prime sub S-poset of B, $b_i \in f(A)$ or $Bs \subseteq f(A)$ for each $i \in I$. Particularly, for $j \in I$, $b_j Ss \subseteq f(A)$. As $b_j \notin f(A)$, then $Bs \subseteq f(A)$. Hence, $(\prod_{i\in I} B)s = \prod_{i\in I}(Bs) \subseteq \prod_{i\in I} f(A)$.

(ii) $\Rightarrow$ (i) Let $\prod_{i\in I} f$ be a regular prime S-poset monomorphism. This clearly implies that f is a regular monomorphism. Now we show that $f : A \to B$ is prime. Let $bSs \subseteq f(A)$ for each $b \in B \setminus f(A)$

and $s \in S$. We have $\langle b\rangle \in \prod_{i\in I}(B \setminus f(A)) \subseteq (\prod_{i\in I} B) \setminus (\prod_{i\in I} f(A))$ and $\langle b\rangle Ss \subseteq \prod_{i\in I} f(A)$. Since $\prod_{i\in I} f(A)$ is a prime sub S-poset of $\prod_{i\in I} B$, $(\prod_{i\in I} B)s \subseteq \prod_{i\in I} f(A)$. Consider an arbitrary element $b's \in Bs$. Thus $\langle b's\rangle \in \prod_{i\in I}(Bs) \subseteq \prod_{i\in I} f(A)$ and then $b's \in f(A)$. Hence, $Bs \subseteq f(A)$, as claimed.

(i) $\Rightarrow$ (iii) Suppose that $f : A \to B$ is a regular prime monomorphism. In view of [9, Proposition 5], it suffices to show that $\coprod_{i\in I} f : \coprod_{i\in I} A \to \coprod_{i\in I} B$ is prime. Note that $(\coprod_{i\in I} f)(\coprod_{i\in I} A) = \coprod_{i\in I} f(A)$. It must be proved that $\coprod_{i\in I} f(A)$ is a prime sub S-poset of $\coprod_{i\in I} B$. Let $(i, b)Ss \subseteq \coprod_{i\in I} f(A)$ for each $s \in S$ and $(i, b) \in (\coprod_{i\in I} B) \setminus (\coprod_{i\in I} f(A)) = \coprod_{i\in I}(B \setminus f(A))$. Since $(i, bSs) \subseteq \coprod_{i\in I} f(A)$, we have $(i, bSs) \subseteq (i, f(A))$. As $f(A)$ is a prime sub S-poset of B, $bSs \subseteq f(A)$ and $b \in B\setminus f(A)$, we get $Bs \subseteq f(A)$ and then $(\coprod_{i\in I} B)s = \coprod_{i\in I}(Bs) \subseteq \coprod_{i\in I} f(A)$.

(iii) $\Rightarrow$ (i) Let $\coprod_{i\in I} f$ be a regular prime S-poset monomorphism. This clearly gives that f is a regular monomorphism. Now we prove that $f : A \to B$ is prime. Let $bSs \subseteq f(A)$ for each $b \in B \setminus f(A)$ and $s \in S$. We have $(i, b)Ss = (i, bSs) \subseteq (i, f(A)) \subseteq \coprod_{i\in I} f(A)$ for each $i \in I$. Since $(i, b) \in \coprod_{i\in I}(B \setminus f(A)) = (\coprod_{i\in I} B) \setminus (\coprod_{i\in I} f(A))$ and $\coprod_{i\in I} f$ is prime, $(\coprod_{i\in I} B)s \subseteq \coprod_{i\in I} f(A)$. Now consider an arbitrary element $b's \in Bs$. Then $(i, b's) \in \coprod_{i\in I}(Bs) = (\coprod_{i\in I} B)s \subseteq \coprod_{i\in I} f(A)$ for each $i \in I$ and so $b's \in f(A)$. Therefore, $Bs \subseteq f(A)$, as required. $\square$

Recall that for a family $\{A_i \mid i \in I\}$ of S-posets with a unique zero element θ, the direct sum $\bigoplus_{i\in I} A_i$ is defined to be the sub S-poset of the product $\prod_{i\in I} A_i$ consisting of all $\langle a_i\rangle_{i\in I}$ such that $a_i = \theta$ for all $i \in I$ except a finite number.

Remark 2.2. Let $f : A \to B$ be an S-poset homomorphism where A and B have a unique zero element and $\bigoplus_{i\in I} f : \bigoplus_{i\in I} A \to \bigoplus_{i\in I} B$ be the homomorphism induced by the product of f. In fact, $\bigoplus_{i\in I} f = (\prod_{i\in I} f)|_{\bigoplus_{i\in I} A}$. It follows directly from Proposition 2.1 that the map f is a regular prime monomorphism if and only if so is $\bigoplus_{i\in I} f$.

The following example shows that the product and coproduct induced homomorphisms of two non-equal (as maps) regular prime homomorphisms on an S-poset are not necessarily regular prime.

Example 2.3. Consider the pomonoid $(\mathbb{N}, \cdot, \leq)$ and the regular monomorphisms $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ given by $f_1(n) = 2n$ and $f_2(n) = 3n$ for each $n \in \mathbb{N}$. Clearly, $f_1(\mathbb{N}) = 2\mathbb{N}$ and $f_2(\mathbb{N}) = 3\mathbb{N}$ are prime sub $\mathbb{N}$-posets of $\mathbb{N}$. But the product and coproduct induced homomorphisms of f_1 and f_2 are not prime. Indeed, $(f_1 \times f_2)(\mathbb{N}\times\mathbb{N}) = 2\mathbb{N}\times 3\mathbb{N}$ is not a prime sub $\mathbb{N}$-poset of $\mathbb{N} \times \mathbb{N}$ because $(2, 2)\mathbb{N}3 = 6\mathbb{N} \times 6\mathbb{N} \subseteq 2\mathbb{N} \times 3\mathbb{N}$ for $(2, 2) \in$

$\mathbb{N}\times\mathbb{N}$ and $3\in\mathbb{N}$, whereas $(2,2)\notin 2\mathbb{N}\times 3\mathbb{N}$ and $(\mathbb{N}\times\mathbb{N})3=3\mathbb{N}\times 3\mathbb{N}\nsubseteq 2\mathbb{N}\times 3\mathbb{N}$. Furthermore, $(f_1\sqcup f_2)(\mathbb{N}\sqcup\mathbb{N})=2\mathbb{N}\sqcup 3\mathbb{N}=(1,2\mathbb{N})\cup(2,3\mathbb{N})$ is not a prime sub $\mathbb{N}$-poset of $\mathbb{N}\sqcup\mathbb{N}=(1,\mathbb{N})\cup(2,\mathbb{N})$. This is because $(2,2)\mathbb{N}3=(2,6\mathbb{N})\subseteq 2\mathbb{N}\sqcup 3\mathbb{N}$ for $(2,2)\in(1,\mathbb{N})\cup(2,\mathbb{N}),3\in\mathbb{N}$ while $(2,2)\notin 2\mathbb{N}\sqcup 3\mathbb{N}$ and $(\mathbb{N}\sqcup\mathbb{N})3=3\mathbb{N}\sqcup 3\mathbb{N}\nsubseteq 2\mathbb{N}\sqcup 3\mathbb{N}$.

We say that a subclass $\mathcal{M}$ of monomorphisms of a category is *stable under pullbacks* or *pullbacks transfer* $\mathcal{M}$ if the morphism $p_1\in\mathcal{M}$ whenever $f\in\mathcal{M}$ in the following pullback diagram:

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

In the next result, we study stability of regular prime monomorphisms under pullbacks in **Pos-**S.

Proposition 2.4. *Pullbacks transfer regular prime monomorphisms of S-posets.*

Proof. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

where P is the sub S-poset $\{(a,b):g(a)=f(b)\}$ of $A\times B$, and pullback maps $p_A:P\to A$, $p_B:P\to B$ are restrictions of the projection maps. Assume that f is a regular prime monomorphism. We show that p_A is also a regular prime monomorphism. Using [9, Proposition 4], p_A is a regular monomorphism. We prove that $p_A(P)$ is a prime sub S-poset of A. Let $aSs\subseteq p_A(P)$ for each $a\in A$ and $s\in S$. We have to show that $a\in p_A(P)$ or $As\subseteq p_A(P)$. Suppose that $As\nsubseteq p_A(P)$. For any $x\in aSs\subseteq p_A(P)$, there exists $b\in B$ such that $(x,b)\in P$. Set

$$D:=\{b\in B\mid f(b)=g(x)\text{ for some }x\in aSs\}\subseteq B.$$

We claim that $f(D)=g(aSs)$. Clearly, $f(D)\subseteq g(aSs)$. For the reverse inclusion, let $g(x)\in g(aSs)$ for some $x\in aSs\subseteq p_A(P)$. Then $x=p_A(a,b)=a$ for some $(a,b)\in P$ and so $g(x)=g(a)=f(b)$ which means $b\in D$. Hence, $g(x)\in f(D)$ and then $g(aSs)\subseteq f(D)$. Therefore, $g(a)Ss=f(D)\subseteq f(B)$. Since $f(B)$ is a prime sub S-poset of C, $g(a)\in f(B)$ or $Cs\subseteq f(B)$. Now we show that $Cs\nsubseteq f(B)$

and just we have $g(a) \in f(B)$. It follows from $As \nsubseteq p_A(P)$ that there exists $a' \in A$ such that $a's \notin p_A(P)$. So $g(a')s = g(a's) \neq f(b)$ for all $b \in B$. This means that $Cs \nsubseteq f(B)$. Hence, $g(a) \in f(B)$ and then $g(a) = f(b')$ for some $b' \in B$. This means that $(a, b') \in P$ and so $a \in p_A(P)$. ☐

3. Regular prime injectivity in Pos-S

In this section, we consider regular prime injectivity of S-posets and a weaker form of such kind of injectivity called principal poideal regular prime injectivity and present some homological classifications for pomonoids.

Definition 3.1. Let A be an S-poset. Then A is called *regular prime injective* if for each regular prime monomorphism $g : B \to C$, any homomorphism $f : B \to A$ can be lifted to a homomorphism $\bar{f} : C \to A$, that is, $\bar{f}g = f$.

We call a homomorphism $f : A \to B$ a *prime embedding* of A into B if f is a regular prime monomorphism. In this case, A is regularly prime embedded into B or B contains an isomorphic copy of A. Obviously, every S-poset may be replaced by an isomorphic S-poset in the definition of regular prime injectivity. Hence, we may assume that a regular prime monomorphism $f : A \to B$ is a prime embedding so that A can be considered as a prime sub S-poset of B, and use the definition of regular prime injectivity in a slightly different form. More precisely, an S-poset A is regular prime injective if and only if for every S-poset C, any prime sub S-poset B of C and a homomorphism $f : B \to A$, there exists a homomorphism $\bar{f} : C \to A$ such that $\bar{f}|_B = f$.

Remark 3.2. (i) Let S be a pogroup. Then each sub S-poset B of an S-poset A is prime. Indeed, $aSs \subseteq B$ for $a \in A, s \in S$ implies that $a = as^{-1}s \in aSs$ and so $a \in B$. This implies that regular prime injectivity of S-posets over a pogroup S coincides to regular injectivity.

(ii) It is easily seen that if an S-poset A is regular prime injective, then it is a retract of each ot its prime extensions, that is, for each regular prime extension B of A, there exists a homomorphism $f : B \to A$ (a retraction) which maps A identically.

(iii) Every regular prime injective prime right poideal I of a pomonoid S is a principal right ideal which is generated by an idempotent element. Indeed, consider the prime embedding $j : I \hookrightarrow S$ and extend the identity map $id : I \to I$ to $\overline{id} : S \to I$ by regular prime injectivity. We show that I is generated by $\overline{id}(1)$ which is an idempotent element of I. This is because, $\overline{id}(1) \in I$ and so $\overline{id}(1)\overline{id}(1) = \overline{id}(1\overline{id}(1)) = \overline{id}(\overline{id}(1)) = \overline{id}(1)$. Also $s = \overline{id}(s) = \overline{id}(1s) = \overline{id}(1)s$ for each $s \in I$.

Recall from [7] that a regular injective S-poset is bounded by two zero elements. For regular prime injectivity of S-posets, we have the same result as follows.

Lemma 3.3. *Any non-singleton regular prime injective S-poset is bounded by two zero elements.*

Proof. Let A be a non-singleton regular prime injective S-poset. Consider the S-poset $B = A \sqcup \{\theta_1\} \sqcup \{\theta_2\}$ obtained by adjoining two zero elements to A such that $\theta_1 \leq a \leq \theta_2$ for every $a \in A$. We show that B is a prime extension of A. Let $bSs \subseteq A$ for $b \in B$ and $s \in S$. Clearly, $b \neq \theta_1, \theta_2$. Thus $b \in A$ which gives that A is a prime sub S-poset of B. Since A is regular prime injective, there exists a retraction $f : B \to A$. Note that $f(\theta_1) \leq f(a) = a \leq f(\theta_2)$ for every $a \in A$ and so $f(\theta_1) \neq f(\theta_2)$ because otherwise $|A| = 1$ which is a contradiction. Hence, the zero elements $f(\theta_1)$ and $f(\theta_2)$ are the bottom and top elements of A, respectively. $\square$

Clearly, if $1 \in S$ is a zero element, then $S = \{1\}$. Regarding this fact and Lemma 3.3 we get:

Corollary 3.4. *Let the identity of a pomonoid S be its top or bottom element. If S as an S-poset is regular prime injective, then $S = \{1\}$.*

Remark 3.5. (i) Let $\{A_i : i \in I\}$ be a non-empty family of S-posets. As in the case of regular injectivity, using Lemma 3.3, one can show that $\prod_{i\in I} A_i$ is regular prime injective if and only if so is each A_i (cf. [10, Remark 2.12]). As for the coproduct, if $|I| > 1$, then $\coprod_{i\in I} A_i$ is not regular prime injective; because otherwise, using Theorem 3.3, it would be bounded which is a contradiction. As a conclusion, there exists no pomonoid S over which all S-posets are regular prime injective. This is because, for a pomonoid S, if A is an S-poset, then $A \sqcup A$ is not regular prime injective.

(ii) Skornjakov criterion states that the injectivity of acts with a zero is equivalent to being injective relative to all inclusions into cyclic acts (see [8, Theorem III.1.8]). Example 2.14 in [10] shows that this criterion fails for regular injectivity of S-posets. Using this example and Lemma 3.3, the Skornjakov criterion also fails for regular prime injectivity of S-posets. Indeed, the pomonoid $S = \{0, 1\}$ with equality relation as an S-poset is not regular prime injective by Lemma 3.3 since it is not bounded. But, as it was shown in the example, S is regular injective with respect to embeddings into cyclic S-posets which clearly gives that it is regular prime injective relative to prime embeddings into cyclic S-posets.

In what follows, we investigate the relation between regular prime injectivity in **Pos-**S and regular injectivity in the category **Pos** of all posets and order-preserving maps between them.

Recall from [4] that the free functor F : **Pos** $\to$ **Pos-**S is given by $F(P) = P \times S$ with componentwise order and the action $(x, s)t = (x, st)$ for $x \in P, s, t \in S$. It is a left adjoint to the forgetful functor U : **Pos-**$S \to$ **Pos**.

The following will be used in the sequel.

Lemma 3.6 ([4])**.** *Let $\boldsymbol{F} : \boldsymbol{C} \to \boldsymbol{D}$ and $\boldsymbol{G} : \boldsymbol{D} \to \boldsymbol{C}$ be two functors such that $\boldsymbol{F}$ be a left adjoint to $\boldsymbol{G}$. Also let $\mathcal{M}, \mathcal{M}'$ be certain subclasses of $\boldsymbol{C}, \boldsymbol{D}$, respectively. If for all $f \in \mathcal{M}, \boldsymbol{F}f \in \mathcal{M}'$, then for each $\mathcal{M}'$-injective object $D \in \boldsymbol{D}$, $\boldsymbol{G}D$ is an $\mathcal{M}$-injective object of $\boldsymbol{C}$.*

Now consider the following lemma:

Lemma 3.7. *The free functor F :* **Pos** $\to$ **Pos-**S *sends any regular monomorphism in* **Pos** *to a regular prime monomorphism.*

Proof. Consider a regular monomorphism $f : A \to B$ in **Pos**. Using [7, Lemma 2.3], $F(f) : A \times S \to B \times S$ is a regular monomorphism in **Pos-**S. It suffices to prove that $F(f)$ is prime. Let $(b, t)Ss \subseteq F(f)(A \times S)$ for $(b, t) \in B \times S$ and $s \in S$. Then $(b, tSs) \subseteq F(f)(A \times S) = f(A) \times S$. Therefore, $b \in f(A)$ and so $(b, t) \in F(f)(A \times S)$. This means that $F(f)(A \times S)$ is a prime sub S-poset of $B \times S$. $\square$

An S-poset A is called *complete* if it is complete as a poset. It is known that regular injective posets are exactly complete posets (see [2]). So, in light of Lemmas 3.6 and 3.7, the following is immediate:

Corollary 3.8. *Any regular prime injective S-poset is complete.*

Definition 3.9. An S-poset A is said to be *principally poideal regular prime injective* if every S-poset homomorphism $f : I \to A$ from a principal prime right poideal I of a pomonoid S can be extended to an S-poset homomorphism $\bar{f} : S \to A$.

Note that any S-poset homomorphism $f : S \to A$ is of the form λ_a, where $\lambda_a : S \to A$ is the right translation mapping, for $a = f(1)$ since $f(s) = f(1s) = f(1)s = as$ for each $s \in S$. Thus, the fact that an S-poset map $f : I \to A$ from a right poideal I of S to an S-poset A can be extended to an S-poset map $\bar{f} : S \to A$ is equivalent to f being of the form λ_a for some $a \in A$. This means that:

An S-poset A is principally poideal regular prime injective if and only if for each S-poset map $f : I \to A$, where $I \subseteq S$ is a principal right poideal, there exists an element $a \in A$ such that $f = \lambda_a$.

Recall from [11] that a pomonoid S is called *poregular* if every $s \in S$ for which sS is a poideal of S is a *regular element*, that is, there exists $t \in S$ such that $s = sts$. Also we say that S is *weakly regular* if every $s \in S$ is a *weakly regular element*, that is, there exists $t \in S$ such that $s \leq sts$.

Lemma 3.10. *Let the identity of a pomonoid S be its bottom element and $s \in S$. Then the principal right poideal $\downarrow(sS)$ is a prime right ideal of S.*

Proof. Assume that $s_1Ss_2 \subseteq sS = \downarrow(sS)$ for each $s_1, s_2 \in S$. We show that $s_1 \in \downarrow(sS)$ or $s_2 \in \downarrow(sS)$. Then for each $t \in S$, there exist $l, r \in S$ such that $s_1ts_2 \leq sr$. As 1 is the bottom element, we have $1 \leq ts_2$, and hence $s_1 = s_1 1 \leq s_1ts_2$. This implies that $s_1 \leq s_1ts_2 \leq sr$ and then $s_1 \in \downarrow(sS)$. Therefore, S is a regular prime extension of $\downarrow(sS)$. □

Theorem 3.11. *Let S be a pomonoid whose identity is the bottom element. Then the following statements are equivalent:*

(i) *All principal right poideals of S which are principal as ideals are principally poideal regular prime injective.*

(ii) *S is poregular and principally poideal regular prime injective.*

Proof. (i) $\Rightarrow$ (ii) Since S is itself a principal right poideal, it is principally poideal regular prime injective by (i). We show that S is poregular. Let $s \in S$ and sS is a poideal. It is easily seen that $sS = \downarrow(sS)$. Consider the natural embedding $\iota : \downarrow(sS) \hookrightarrow S$. Using Lemma 3.10, ι is a prime embedding and so $\downarrow(sS)$ is a retract of S because $\downarrow(sS)$ is principally poideal regular prime injective by the assumption. Let $f : S \to \downarrow(sS)$ denote such a retraction. Now, taking $f(1) = u$, we have $u = st$ for some $t \in S$, since $sS = \downarrow(sS)$, and then $s = f(s) = f(1s) = f(1)s = us = sts$, which means that s is a regular element. Therefore, S is poregular.

(ii) $\Rightarrow$ (i) Consider a principal right poideal of the form $\downarrow(sS) = sS$ of S. Let $\iota : I \hookrightarrow S$ be a prime embedding from a principal right poideal I and $f : I \to sS$ be a homomorphism. Since S is poregular, there exists $t \in S$ such that $sts = s$. This gives that sS is a retract of S with the retraction $\phi : S \to sS$ given by $\phi(r) = str$ for each $r \in S$.

Consider the following diagram:

$$\begin{tikzcd}
I \arrow[r, "\iota"] \arrow[d, "f"'] & S \\
sS \arrow[d, hook, shift left, "i"] & \\
S \arrow[u, shift left, "\phi"] &
\end{tikzcd}$$

Since S is principally poideal regular prime injective, there exists an S-poset map $h : B \to S$ such that $hg = if$. Considering the map $\bar{f} := \phi h$, we get $\bar{f}g = \phi hg = \phi if = id_{sS}f = f$ and then sS is principally poideal regular prime injective. □

Recall from [11] that a pomonoid S is called *right po-PP* if for every $s \in S$ there exists an idempotent $e \in S$ such that $s = se$, and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$.

Proposition 3.12. *Let S be a pomonoid whose identity is the bottom element. Then we have the following assertions:*

(i) *If any principal right poideal of S is principally poideal regular prime injective, then S is weakly regular.*

(ii) *If S is a principally poideal regular prime injective as well as a right po-PP pomonoid, then S is poregular.*

Proof. (i) Let $s \in S$. Using the assumption, the principal right poideal $\downarrow (sS)$ is principally poideal regular prime injective. Consider the natural embedding $\iota :\downarrow (sS) \hookrightarrow S$. By Lemma 3.10, ι is a prime embedding. Then there is a retraction $f : S \to \downarrow (sS)$, because $\downarrow (sS)$ is principally poideal regular prime injective. Now, taking $f(1) = u$ we have $u \leq st$ for some $t \in S$, and $s = f(s) = f(1s) = f(1)s = us \leq sts$. Therefore, s is a weakly regular elemnt and then S is weakly regular.

(ii) Consider any pomonoid S satisfying the assumption. Let $s \in S$ such that sS is a poideal. By Lemma 3.10, the natural embedding $\iota :\downarrow (sS) \hookrightarrow S$ is prime. Using the hypothesis, there exists an idempotent element $e \in S$ such that $s = se$ and $su \leq sv$ implies $eu \leq ev$ for all $u, v \in S$. Define $f :\downarrow (sS) = sS \to S$ by $f(st) = et$ for each $t \in S$. Let $st \leq st'$ for some $t, t' \in S$. Then $et \leq et'$ and so $f(st) \leq f(st')$. This means that f is an order-preserving and then well-defined. Hence, f is an S-poset map. Since S is principally poideal regular prime injective, $f = \lambda_x$ for some $x \in S$. Thus $e = f(s) = xs$. Then $s = se = sxs$, so s is regular and hence S is poregular. □

Theorem 3.13. *Let $\{A_i \mid i \in I\}$ be a non-empty family of S-posets with a zero $\theta_i \in A_i$ for each $i \in I$. Then the product $\prod_{i\in I} A_i$ is poideal regular prime injective if and only if so is each A_i.*

Proof. The fact that the product of poideal regular prime injective S-posets is poideal regular prime injective is proved analogously to the case of general injectivity in a category using the universal property of products. To prove the converse, let $\prod_{i\in I} A_i$ be poideal regular prime injective and take any $j \in I$. Let $f : I \to A_j$ be an S-poset map from a prime right poideal I of S to A_j. Consider the S-poset map $\bar{f} : I \to \prod_{i\in I} A_i$ defined by $\bar{f}(s) = (\dots, \theta_{j-1}, f(s), \theta_{j+1}, \dots)$ for each $s \in S$. It follows from the assumption that there exists an element $(a_i)_i \in \prod_{i\in I} A_i$ such that $\bar{f} = \lambda_{(a_i)_i}$. Now it is easily seen that $f = \lambda_{a_j}$ which shows that A_j is poideal regular prime injective. □

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Hamid Rasouli
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Email: hrasouli@srbiau.ac.ir; hrasouli5@yahoo.com

Gholam Reza Moghaddasi
Department of Mathematics, Hakim Sabzevary University, Sabzevar, Iran.
Email: r.moghadasi@hsu.ac.ir

Nasrin Sarvghad
Department of Mathematics, Hakim Sabzevary University, Sabzevar, Iran.
Email: n.sarvghad@hsu.ac.ir

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ON REGULAR PRIME INJECTIVITY OF S-POSETS

H. RASOULI, GH. MOGHADDASI AND N. SARVGHAD

انژکتیوی اول منظم S-مجموعه های مرتب

حمید رسولی[۱]، غلام‌رضا مقدسی[۲] و نسرین سروقد[۳]

[۱] دانشگاه آزاد اسلامی، واحد علوم و تحقیقات، گروه ریاضی، تهران، ایران

[۲,۳] گروه رياضي محض، دانشگاه حکیم سبزواري، سبزوار، ایران

در این مقاله، مفهوم تکریختی اول منظم را برای S-مجموعه‌های مرتب روی یک تکواره مرتب S تعریف و برخی خواص رسته‌ای از جمله ضرب‌ها، هم‌ضرب‌ها و عقب‌برها را بررسی می‌کنیم. ما $\mathcal{M}$-انژکتیوی را در رسته‌ی S-مجموعه‌های مرتب مورد مطالعه قرار می‌دهیم جایی‌که $\mathcal{M}$ کلاس تکریختی‌های اول منظم است و نشان می‌دهیم محک اسکورناخوف برای انژکتیوی اول منظم برقرار نیست. با در نظر گرفتن فرم ضعیف‌تری از این نوع انژکتیوی، برخی رده‌بندی‌ها را برای تکواره‌های مرتب به‌دست می‌آوریم.

کلمات کلیدی: زیر S-مجموعه مرتب اول، تکریختی اول منظم، انژکتیو اول منظم.