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LINKAGE OF IDEALS OVER A MODULE

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Abstract. Inspired by the works in linkage theory of ideals, we define the concept of linkage of ideals over a module. Several known theorems in linkage theory are improved or recovered. Specially, we make some extensions and generalizations of a basic result of Peskine and Szpiro [10, Proposition 1.3], namely if R is a Gorenstein local ring, $\mathfrak{a} \neq 0$ (an ideal of R) and $\mathfrak{b} := 0 :_R \mathfrak{a}$ then $\frac{R}{\mathfrak{a}}$ is Cohen-Macaulay if and only if $\frac{R}{\mathfrak{a}}$ is unmixed and $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

1. Introduction

Classically, linkage theory refers to Halphen (1870) and M. Noether (1882) [9] who worked to classify space curves. In 1974, the significant work of Peskine and Szpiro [10] brought breakthrough to this theory and stated it in the modern algebraic language; two proper ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Cohen-Macaulay local ring R is said to be linked if there is a regular sequence $\underline{x}$ in their intersection such that $\mathfrak{a} = \underline{x} :_R \mathfrak{b}$ and $\mathfrak{b} = \underline{x} :_R \mathfrak{a}$.

A new progress in the linkage theory is the work of Martsinkovsky and Strooker [7] which established the concept of linkage of modules.

Let R be a commutative Noetherian ring with $1 \neq 0$ and M be a finitely generated R-module. In this paper, inspired by the works in the ideal case, we present the concept of linkage of ideals over a module; let $\mathfrak{a}, \mathfrak{b}$ and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be ideals of R such that I is generated by an

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M-regular sequence and $\mathfrak{a}M \neq M \neq \mathfrak{b}M$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by I over M, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. This is a generalization of its classical concept, when $M = R$ ([10]). It is citable that, in general case, linkedness of two ideals over M does not imply linkedness of them over R and vice versa (see Example 2.7). But, in some special cases, it does (see Example 3.1, Theorem 3.2 and Theorem 3.3).

In this paper, we consider the above generalization and study some of its basic facts. The organization of the paper goes as follows.

In Section 2, we present some basic properties of linkage of ideals over a module. Also, we study the koszul homologies of "M-licci" ideals, in a special case (Proposition 2.15).

By the above definition of linkage of ideals over a module it is natural to ask whether linkedness of two ideals over a module implies linkedness of them over the ring and vice versa. Section 3 has considered this question in the case where R is a Cohen-Macaulay local ring with canonical module ω_R. In this section, we study ideals which are linked over ω_R. More precisely, it is shown that if $\mathfrak{a} \sim_{(I;w_R)} \mathfrak{b}$ and $\frac{R}{\mathfrak{a}}$ and $\frac{R}{\mathfrak{b}}$ are unmixed, then $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ (Theorem 3.2). Also, if $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ and $\frac{w_R}{\mathfrak{a}w_R}$ and $\frac{w_R}{\mathfrak{b}w_R}$ are unmixed, then $\mathfrak{a} \sim_{(I;w_R)} \mathfrak{b}$ (Theorem 3.3).

The first main theorem in the theory of linkage is the following.

Theorem A. [10, Theorem 1.3] *If* $(R, \mathfrak{m})$ *is a Gorenstein local ring and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are two linked ideals of* R*, then* $R/\mathfrak{a}$ *is Cohen-Macaulay if and only if* $R/\mathfrak{b}$ *is Cohen-Macaulay.*

Attempts to generalize this theorem lead to several developments in linkage theory, especially the works by C. Huneke [3], B. Ulrich [13] and [6]. A counterexample given by Peskine and Szpiro in [10], shows that *Theorem A* is no longer true if the base ring R is Cohen-Macaulay but not Gorenstein.

In Section 4, we state *Theorem A* for linked ideals over the canonical module ω_R. Namely, it is shown that if the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked over ω_R and the G-dimension of some certain modules are finite then Cohen-Macaulayness of $R/\mathfrak{a}$ and of $R/\mathfrak{b}$ are equivalent (Corollary 4.2).

Moreover, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $\mathfrak{a}, \mathfrak{b}$ be unmixed ideals of R which are in the same evenly linkage class over the maximal Cohen-Macaulay R-module M of finite injective dimension. Then, $\frac{M}{\mathfrak{a}M}$ is Cohen-Macaulay if and only if $\frac{M}{\mathfrak{b}M}$ is Cohen-Macaulay. Also, if $\mathfrak{a}$ is "M-licci" then $\frac{M}{\mathfrak{a}M}$ and $\frac{R}{\mathfrak{a}}$ are Cohen-Macaulay (Corollary 4.3).

Throughout the paper, R denotes a non-trivial commutative Noetherian ring, $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals of R and M will denote a finitely generated R-module.

2. LINKED IDEALS OVER A MODULE

The goal of this section is to introduce the concept of linkage of ideals over a module and study some of its basic properties.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by an M-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M, denoted $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically linked by I over M if $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is linked over M if there exist ideals $\mathfrak{b}$ and I of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. $\mathfrak{a}$ is M-selflinked by I if $\mathfrak{a} \sim_{(I;M)} \mathfrak{a}$.*

$\mathfrak{a}$ and $\mathfrak{b}$ are said to be in the same (evenly) M-linkage class if for some (even) number n, there is a sequence of links $\mathfrak{a} = \mathfrak{a}_0 \sim_{(I_0;M)} \mathfrak{a}_1 \sim_{(I_1;M)} \mathfrak{a}_2 \sim_{(I_2;M)} \dots \sim_{(I_{n-1};M)} \mathfrak{a}_n = \mathfrak{b}$. An ideal $\mathfrak{a}$ is called M-licci if it is in the linkage class of an M-regular sequence.

Note that this definition is a generalization of the concept of linkage of ideals in [10]. But, as Example 2.7 shows, these two concepts do not coincide, although, in some cases they do (see Example 2.4).

Example 2.2. Let $x_1, \dots, x_n$ be an M-regular sequence. Then one can see that

$$(x_1, \dots, x_n) \sim_{(((x_1)^2, x_2, \dots, x_n);M)} (x_1, \dots, x_n).$$

In other words, every M-regular sequence is M-selflinked.

Lemma 2.3. Let N be a finitely generated R-module, $x_1, \dots, x_t \in \mathfrak{a} \cap \mathfrak{b}$ and $I := (x_1, \dots, x_t)$. Then $\mathfrak{a} \sim_{(I;M \oplus N)} \mathfrak{b}$ if and only if $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ and $\mathfrak{a} \sim_{(I;N)} \mathfrak{b}$.

The next two items consider the question "Whether linkedness over the ring implies linkedness over a module?". We will come back to this question again in Section 3.

Example 2.4. Let F be a finitely generated free R-module and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$. Then, the above lemma shows that, $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ iff $\mathfrak{a} \sim_{(I;F)} \mathfrak{b}$.

Proposition 2.5. Let M be flat, $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ such that $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$. Then $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$.

Proof. By the assumption, M is projective and every R-regular sequence is an M-regular sequence, too. Now, Via [11, Proposition 3.60] and the above example,

$$\mathfrak{a}M = \mathfrak{a}F \cap M = (IF :_F \mathfrak{b}) \cap M = (IF \cap M) :_M \mathfrak{b} = IM :_M \mathfrak{b}.$$

□

The following lemma, among other things, shows that if $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M then the ideal I has to be generated by a maximal M-regular sequence in $\mathfrak{a} \cap \mathfrak{b}$.

Lemma 2.6. Let I be a proper ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then

(i) $\operatorname{grade}_M \mathfrak{a} = \operatorname{grade}_M \mathfrak{b} = \operatorname{grade}_M I$.
(ii) $\operatorname{Supp} \frac{M}{\mathfrak{a}M} = \operatorname{Supp} \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, \frac{M}{IM})$.
(iii) $\operatorname{Supp} \frac{M}{IM} = \operatorname{Supp} \frac{M}{\mathfrak{a}M} \cup \operatorname{Supp} \frac{M}{\mathfrak{b}M}$.

Proof. (i) The assumption implies that $0 :_{\frac{M}{IM}} \mathfrak{a} \neq 0$. Hence $I \subseteq \mathfrak{a} \subseteq \mathrm{Z}(\frac{M}{IM})$ and $\operatorname{grade}_M I = \operatorname{grade}_M \mathfrak{a}$.

(ii) Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\mathfrak{p} \in \operatorname{Supp} \frac{M}{\mathfrak{a}M}$ if and only if $\mathfrak{p} \in \operatorname{Supp} \frac{\mathfrak{b}M}{IM} = \operatorname{Supp} \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, \frac{M}{IM})$.

(iii) Follows from (ii) and using the following short exact sequence

$$0 \to \frac{\mathfrak{b}M}{IM} \to \frac{M}{IM} \to \frac{M}{\mathfrak{b}M} \to 0.$$

□

The following example shows that the concepts of linkage of ideals over R and over M do not coincide.

Example 2.7. (1) Let $R := \frac{k[x,y]}{(xy)}$ and $M := \frac{k[x,y]}{(x)}$ where k is a field. Via the natural homomorphism $R \to M$, M is a finitely generated R-module. Set $\mathfrak{a} := (x)$ and $\mathfrak{b} := (y)$.

(i) As y is an M-regular sequence, $\mathfrak{b} \sim_{((y^2);M)} \mathfrak{b}$. Assume that $\mathfrak{b} \sim_{(I;R)} \mathfrak{b}$ for some ideals I. Since $\operatorname{grade}_R \mathfrak{b} = 0$, by Lemma 2.6, $I = 0$. But, $0 :_R \mathfrak{b} \neq \mathfrak{b}$.

(ii) $\mathfrak{a} \sim_{(0;R)} \mathfrak{b}$. Assume that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ for some ideals I. As $\operatorname{grade}_M \mathfrak{a} = 0$, by Lemma 2.6, $I = 0$. On the other hand, $0 :_M \mathfrak{a} = M$, which is a contradiction.

(2) Let $R := k[x, y, z]$ and $M := \frac{R}{(x,z)\cap(y)}$ where k is a field. Via the natural homomorphism $R \to M$, M is a finitely generated R-module. Set $\mathfrak{a} := (x)$ and $\mathfrak{b} := (y)$. Then $\mathfrak{a} \sim_{(xy;R)} \mathfrak{b}$ and $x, y \in Z(M)$. One can shows that $0 :_M \mathfrak{a} = \mathfrak{b}M$ but $0 :_M \mathfrak{b} = (x, z)M \neq \mathfrak{a}M$.

Lemma 2.8. Let I be a proper ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then, $\frac{M}{\mathfrak{a}M}$ can be embedded in finite copies of $\frac{M}{IM}$.

Proof. Let $F \to \frac{R}{I} \to \frac{R}{\mathfrak{b}} \to 0$ be a free resolution of $\frac{R}{\mathfrak{b}}$ as $\frac{R}{I}$-module. Then, applying $(-)^+ := \operatorname{Hom}_{\frac{R}{I}}(-, \frac{M}{IM})$, we get the exact sequence $0 \to (\frac{R}{\mathfrak{b}})^+ \to (\frac{R}{I})^+ \xrightarrow{f} F^+$, where $\frac{M}{\mathfrak{a}M} \cong Im(f) \subseteq F^+ \cong \oplus \frac{M}{IM}$. □

There are some relations between ideals which are linked over a module, as the following lemma shows.

Lemma 2.9. Let I be an ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then the following statements hold.

(i) $\operatorname{Min}\operatorname{Ass} \frac{M}{IM} \subseteq \operatorname{Min}\operatorname{Ass} \frac{M}{\mathfrak{a}M} \cup \operatorname{Min}\operatorname{Ass} \frac{M}{\mathfrak{b}M}$.

(ii) If $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min}\operatorname{Ass} \frac{M}{IM}$, then $\operatorname{Ass} \frac{M}{IM} = \operatorname{Ass} \frac{M}{\mathfrak{a}M} \cup \operatorname{Ass} \frac{M}{\mathfrak{b}M}$ and $\operatorname{Ass} \frac{M}{\mathfrak{a}M} = \operatorname{Ass} \frac{M}{IM} \cap V(\mathfrak{a})$.

(iii) If $\mathfrak{a}M \cap \mathfrak{b}M = IM$, then $\operatorname{Ass} \frac{M}{\mathfrak{a}M} = \operatorname{Ass} \frac{M}{IM} \cap V(\mathfrak{a})$ and $\operatorname{Ass} \frac{M}{\mathfrak{a}M} \cap \operatorname{Ass} \frac{M}{\mathfrak{b}M} = \emptyset$.

Proof. (i) Let $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass} \frac{M}{IM}$ and assume that $\mathfrak{p} \notin \operatorname{Min}\operatorname{Ass} \frac{M}{\mathfrak{a}M}$. Hence, in view of Lemma 2.8, $(\frac{M}{\mathfrak{a}M})_\mathfrak{p} = 0$. Therefore, by Lemma 2.6(iii), $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass} \frac{M}{\mathfrak{b}M}$.

(ii) Follows from (i) and Lemma 2.8.

(iii) $\operatorname{Ass} \frac{M}{\mathfrak{a}M} \subseteq \operatorname{Ass} \frac{M}{IM} \cap V(\mathfrak{a})$, by Lemma 2.8. For the converse, let $\mathfrak{p} \in \operatorname{Ass} \frac{M}{IM} \cap V(\mathfrak{a})$. Hence, $\mathfrak{p} \in \operatorname{Ass}\operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, \frac{M}{IM}) = \operatorname{Ass} \frac{\mathfrak{b}M}{IM}$ and there exists $\alpha \in \mathfrak{b}M$ such that $\mathfrak{p} = 0 :_R \alpha + IM$. Now, it is straight forward to see that $\mathfrak{p} = 0 :_R \alpha + \mathfrak{a}M$.

Let $\mathfrak{q} \in \operatorname{Ass} \frac{M}{\mathfrak{a}M} \cap \operatorname{Ass} \frac{M}{\mathfrak{b}M} = \operatorname{Ass} \frac{M}{IM} \cap V(\mathfrak{a}+\mathfrak{b})$. Then $\operatorname{grade}_M \mathfrak{a}+\mathfrak{b} = \operatorname{grade}_M I$. On the other hand,

$$0 :_{\frac{M}{IM}} \mathfrak{a}+\mathfrak{b} = 0 :_{\frac{M}{IM}} \mathfrak{a} \cap 0 :_{\frac{M}{IM}} \mathfrak{b} = 0,$$

which is a contradiction.

□

The following lemma studies the Artinianness of $\frac{M}{\mathfrak{a}M}$ where $\mathfrak{a}$ is linked over M. It will be used in the next corollary which consider the Cohen-Macaulayness of $M_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.

Lemma 2.10. Let $(R, \mathfrak{m})$ be local, I be an ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ and $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min}\operatorname{Ass} \frac{M}{IM}$. Then the following statements are equivalent.

(i) $\frac{M}{IM}$ is Artinian.

(ii) $\frac{M}{\mathfrak{a}M}$ is Artinian.

(iii) $\frac{M}{\mathfrak{b}M}$ is Artinian.

Moreover, $\mathfrak{a}M \cap \mathfrak{b}M \neq IM$ if one of the above equivalent conditions holds.

Proof. The equivalences follow from Lemma 2.8.

Also, if $\mathfrak{a}M \cap \mathfrak{b}M = IM$ then $IM = IM :_M (\mathfrak{a}+\mathfrak{b})$ and so $(\mathfrak{a}+\mathfrak{b}) \nsubseteq \mathrm{Z}(\frac{M}{IM})$. This implies that $\operatorname{grade}_M(\mathfrak{a}+\mathfrak{b}) > \operatorname{grade}_M I = \operatorname{ht}_M I = \dim M$, which is a contradiction. □

Remark 2.11. Note that if $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M and $\mathfrak{p} \in \operatorname{Supp} \frac{M}{\mathfrak{a}M+\mathfrak{b}M}$, then $\mathfrak{a}R_\mathfrak{p} \sim_{(IR_\mathfrak{p};M_\mathfrak{p})} \mathfrak{b}R_\mathfrak{p}$.

Corollary 2.12. Let I be an ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ and $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$. Set $t := \operatorname{grade}_M I$. Then the following statements hold.

(i) $\operatorname{ht}_M \mathfrak{p} = t$ and $M_\mathfrak{p}$ is Cohen-Macaulay for every $\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}$.
(ii) $\operatorname{ht}_M \mathfrak{a} = \operatorname{ht}_M \mathfrak{b} = \operatorname{grade}_M \mathfrak{a} = \operatorname{grade}_M \mathfrak{b} = t$. Moreover, if $\mathfrak{a}M \cap \mathfrak{b}M \neq IM$ then $\operatorname{ht}_M(\mathfrak{a}+\mathfrak{b}) = \operatorname{grade}_M \mathfrak{a}$.
(iii) If $\frac{M}{IM}$ is equidimensional then $\dim M = \operatorname{ht}_M \mathfrak{p} + \dim \frac{R}{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}$. In other words, $\dim M = \operatorname{ht}_M \mathfrak{a} + \dim \frac{M}{\mathfrak{a}M}$.

Proof. (i) Let $\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}$. If $\mathfrak{p} \nsupseteq \mathfrak{b}$, then $\mathfrak{a}M_\mathfrak{p} = IM_\mathfrak{p} :_{M_\mathfrak{p}} \mathfrak{b}R_\mathfrak{p} = IM_\mathfrak{p}$ and so $0 = \dim \frac{M_\mathfrak{p}}{\mathfrak{a}M_\mathfrak{p}} = \dim \frac{M_\mathfrak{p}}{IM_\mathfrak{p}}$. Therefore $M_\mathfrak{p}$ is Cohen-Macaulay of dimension t.

In case $\mathfrak{p} \supseteq \mathfrak{b}$, $\mathfrak{a}R_\mathfrak{p} \sim_{(IR_\mathfrak{p};M_\mathfrak{p})} \mathfrak{b}R_\mathfrak{p}$ and $\operatorname{Ass} \frac{M_\mathfrak{p}}{IM_\mathfrak{p}} = \operatorname{Min} \operatorname{Ass} \frac{M_\mathfrak{p}}{IM_\mathfrak{p}}$. Hence, by Lemma 2.10, $M_\mathfrak{p}$ is Cohen-Macaulay of dimension t.

(ii) The first part follows from (i). Also, assume that $\mathfrak{a}M \cap \mathfrak{b}M \neq IM$ then, $0 :_{\frac{M}{IM}} (\mathfrak{a}+\mathfrak{b}) \neq 0$ and so $(\mathfrak{a}+\mathfrak{b}) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$. Now, by (i), $\operatorname{ht}_M(\mathfrak{a}+\mathfrak{b}) = t$.

(iii) Let $\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}$. Then, by Lemma 2.8, $\dim \frac{R}{\mathfrak{p}} = \dim \frac{M}{IM} = \dim M - t$ and, by (i), the result has desired. □

In the rest of this section we study whether linkedness of ideals over a module transfers via homomorphisms.

Remark 2.13. Let $R' \to R$ be a ring homomorphism and $\mathfrak{a}'$ and $\mathfrak{b}'$ be non-zero proper ideals of R'. Then $\mathfrak{a}' \sim_{(0;M)} \mathfrak{b}'$ if and only if $\mathfrak{a}'R \sim_{(0;M)} \mathfrak{b}'R$.

The following lemma shows that linkage of ideals over a module transfers, in some sense, via faithfully flat homomorphisms.

Lemma 2.14. Let $R \to S$ be a faithfully flat ring homomorphism and $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then, $\mathfrak{a}S \sim_{(IS;M\otimes_R S)} \mathfrak{b}S$. Therefore, if $(R,\mathfrak{m})$ is local then $\mathfrak{a}\hat{R} \sim_{(I\hat{R};M\otimes_R \hat{R})} \mathfrak{b}\hat{R}$.

Let $\mathfrak{a} = (x_l, ..., x_n)$ be an ideal of R and $K_\bullet(\mathfrak{a}; M)$ denote the tensor product of the complexes $0 \to M \xrightarrow{x_i} M \to 0$, for all $1 \leq i \leq n$.

More explicitly,

$$K_p(\mathfrak{a}; M) \cong (\bigwedge^p R^n) \bigotimes_R M$$

and the differentials are given by exterior multiplication with $\sum_{i=1}^n x_i e_i$, where $e_l, ..., e_n \in R^n$ denotes the canonical basis. The cohomology of this complex is denoted by $H_\bullet(\mathfrak{a}; M)$ and called the Koszul cohomology of M with respect to $\mathfrak{a}$. Clearly, $K_\bullet(\mathfrak{a}; M) \cong K_\bullet(\mathfrak{a}; R) \otimes M$.

The following proposition can be considered as a generalization of [3, Corollary 1.12]. Note that [3, Corollary 1.12] holds when the ring is only Cohen-Macaulay, too.

Proposition 2.15. Let M be a finitely generated Cohen-Macaulay flat R-module and $\mathfrak{a}$ and $\mathfrak{b}$ be in the same evenly M-linkage class. If $H_i(\mathfrak{a}; M)$ is Cohen-Macaulay for all i, then $H_i(\mathfrak{b}; M)$ is also Cohen-Macaulay for all i.

Proof. It is enough to consider the case where $\mathfrak{a} \sim_{(I_0;M)} \mathfrak{c} \sim_{(I_1;M)} \mathfrak{b}$.

Let $\mathfrak{p} \in \operatorname{Spec} R$. We may assume that $\mathfrak{b} \subseteq \mathfrak{p}$. If $\mathfrak{c} \nsubseteq \mathfrak{p}$ then $\mathfrak{b}R_\mathfrak{p} = I_1 R_\mathfrak{p}$ and so $H_i(\mathfrak{b}R_\mathfrak{p}; M_\mathfrak{p})$ is Cohen-Macaulay for all i. Therefore, assume that $\mathfrak{p} \in V(\mathfrak{b} + \mathfrak{c})$. If $\mathfrak{p} \nsupseteq \mathfrak{a}$ then $\mathfrak{c}R_\mathfrak{p} = I_0 R_\mathfrak{p}$. This, in conjunction with Example 2.2, implies that $\mathfrak{c}R_\mathfrak{p} \sim_{(IR_\mathfrak{p};M_\mathfrak{p})} \mathfrak{c}R_\mathfrak{p} \sim_{(I_1R_\mathfrak{p};M_\mathfrak{p})} \mathfrak{b}R_\mathfrak{p}$, for some $IR_\mathfrak{p}$ which is generated by an $M_\mathfrak{p}$-regular sequence, and that, $H_i(\mathfrak{c}R_\mathfrak{p}; M_\mathfrak{p})$ is Cohen-Macaulay for all i. Hence, one can also assume that $\mathfrak{p} \in V(\mathfrak{a})$.

By Remark 2.11 and Example 2.4, $\mathfrak{a}R_\mathfrak{p}$ and $\mathfrak{b}R_\mathfrak{p}$ are in a same evenly $R_\mathfrak{p}$-linkage class and, that, $H_i(\mathfrak{a}R_\mathfrak{p}; R_\mathfrak{p})$ is Cohen-Macaulay for all i. Therefore, using [3, Corollary 1.12], $H_i(\mathfrak{b}R_\mathfrak{p}; R_\mathfrak{p})$ is Cohen-Macaulay for all i. This implies that $H_i(\mathfrak{b}R_\mathfrak{p}; M_\mathfrak{p}) \cong (H_i(\mathfrak{b}; M))_\mathfrak{p}$ is Cohen-Macaulay, too. □

The following lemma consider a case where linkedness of ideals over M passes over $\frac{M}{xM}$, where x is an M-regular element.

Lemma 2.16. Let $\mathfrak{a} \sim_{(0;M)} \mathfrak{b}$ and $\operatorname{Ext}_R^1(\frac{R}{\mathfrak{a}}, M) = \operatorname{Ext}_R^1(\frac{R}{\mathfrak{b}}, M) = 0$. Also, assume that $x \notin \mathrm{Z}(M)$ and $(\mathfrak{a}, x)M \neq M \neq (\mathfrak{b}, x)M$. Then $(\mathfrak{a}, x) \sim_{(0;\frac{M}{xM})} (\mathfrak{b}, x)$, in other words, $\mathfrak{a} \sim_{(0;\frac{M}{xM})} \mathfrak{b}$.

Proof. Set $(-)^* := \mathrm{Hom}\,_R(-, M)$. Using the assumptions, we get the following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\frac{R}{\mathfrak{a}})^* & \xrightarrow{.x} & (\frac{R}{\mathfrak{a}})^* & \longrightarrow & \mathrm{Hom}\,_R(\frac{R}{\mathfrak{a}}, \frac{M}{xM}) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow = & & \downarrow \exists\phi & & \\ 0 & \longrightarrow & \mathfrak{b}M & \xrightarrow{.x} & \mathfrak{b}M & \longrightarrow & \frac{\mathfrak{b}M}{x\mathfrak{b}M} & \longrightarrow & 0. \end{array}$$

This implies that $\mathrm{Hom}\,_R(\frac{R}{\mathfrak{a}}, \frac{M}{xM}) \cong \frac{\mathfrak{b}M}{x\mathfrak{b}M}$. Considering the composition of natural isomorphisms

$$0 :_{\frac{M}{xM}} \mathfrak{a} \to \mathrm{Hom}\,_R(\frac{R}{\mathfrak{a}}, \frac{M}{xM}) \xrightarrow{\phi} \frac{\mathfrak{b}M}{x\mathfrak{b}M} \to \frac{\mathfrak{b}M + xM}{xM}$$

and using the fact that this composition is actually the identity map, we get $0 :_{\frac{M}{xM}} \mathfrak{a} = \mathfrak{b}(\frac{M}{xM})$. Now, the result follows using Remark 2.13. □

3. LINKED IDEALS OVER A RING AND ITS CANONICAL MODULE

Throughout this section, we assume that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with canonical module w_R. The main goal of this section is to study the ideals which are linked over the canonical module.

More precisely, we study whether linkedness of two ideals over w_R implies linkedness over R and vice versa.

In spite of Example 2.7, there are some cases where linkage of ideals over an R-module ends to linkedness of them over R, as the following example shows.

Example 3.1. (1) Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are radical, M is faithful and $\mathfrak{a} \sim_{(0;M)} \mathfrak{b}$. Then $\mathfrak{a} \sim_{(0;R)} \mathfrak{b}$. Indeed, by the assumption, $\mathfrak{b} \subseteq 0 :_R \mathfrak{a}$. On the other hand, if $r \in 0 :_R \mathfrak{a}$ then, in view of [8, Theorem 2.1], there exist $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathfrak{b}$ such that $(r^n + r^{n-1}b_1 + \cdots + b_n)M = 0$. This implies that $r \in \mathfrak{b}$. Therefore, $0 :_R \mathfrak{a} = \mathfrak{b}$ and $\mathfrak{a} \sim_{(0;R)} \mathfrak{b}$.

(2) Assume that $\mathfrak{a}$ is a linked radical ideal over M by zero and $\mathrm{Ass}\ M = \mathrm{Ass}\ R$. Then, by Lemma 2.8 , $\mathfrak{a} = \sqrt{\mathfrak{a} + \mathrm{Ann}\ M} = \bigcap_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for some $\Lambda \subseteq \mathrm{Ass}\ M$. Now, using [4, Corollary 2.8], $\mathfrak{a}$ is linked over R.

In the following theorem, we consider a case where linkedness over canonical module implies linkedness over R.

Theorem 3.2. Let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal of R which is generated by an w_R-regular sequence. Also, assume that $Iw_R :_{w_R} \mathfrak{b} = \mathfrak{a}w_R \neq Iw_R$ and $\frac{R}{\mathfrak{b}}$ is unmixed. Then $\mathfrak{b} = I :_R \mathfrak{a}$. In particular, if $\mathfrak{a} \sim_{(I;w_R)} \mathfrak{b}$ and $\frac{R}{\mathfrak{a}}$ and $\frac{R}{\mathfrak{b}}$ are unmixed then $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$.

Proof. First note the that using [2, Theorem 3.3.5(a)] and the fact that every w_R-regular sequence is an R-regular sequence, one can assume that $I = 0$.

As w_R is faithful, $\mathfrak{b} \subseteq 0 :_R \mathfrak{a}$. Assume to the contrary that $\mathfrak{b} \subsetneq 0 :_R \mathfrak{a}$. Then $\frac{0:_{R_\mathfrak{p}} \mathfrak{a}R_\mathfrak{p}}{\mathfrak{b}R_\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \in \operatorname{Ass} \frac{R}{\mathfrak{b}}$. By the assumption and [2, Theorem 3.3.5(a)], $R_\mathfrak{p}$ is an Artinian ring with canonical module $w_{R_\mathfrak{p}} \cong E_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}})$. Set $D(-) = \operatorname{Hom}_{R_\mathfrak{p}}(-, w_{R_\mathfrak{p}})$. Then, by [1, Lemma 10.2.3(iv) and Lemma 10.2.2], $\frac{R_\mathfrak{p}}{\mathfrak{b}R_\mathfrak{p}} \cong D(\mathfrak{a}w_{R_\mathfrak{p}})$ and $\mathfrak{b}R_\mathfrak{p} = 0 :_{R_\mathfrak{p}} \mathfrak{a}w_{R_\mathfrak{p}} = 0 :_{R_\mathfrak{p}} \mathfrak{a}$, which is a contradiction.

□

In the following theorem, we consider a case which linking over R implies linking over the canonical module w_R.

Theorem 3.3. Let I be an ideal of R where is generated by an R-regular sequence in $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} \neq I \neq \mathfrak{b}$. Also, assume that $I :_R \mathfrak{b} = \mathfrak{a}$ and $\frac{w_R}{\mathfrak{b}w_R}$ is unmixed. Then $\mathfrak{b}w_R = Iw_R :_{w_R} \mathfrak{a}$. In particular, if $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ and $\frac{w_R}{\mathfrak{a}w_R}$ and $\frac{w_R}{\mathfrak{b}w_R}$ are unmixed, then $\mathfrak{a} \sim_{(I;w_R)} \mathfrak{b}$.

Proof. First, note that replacing R with $\frac{R}{I}$, we may assume that $I = 0$. By the assumption, $\mathfrak{b}w_R \subseteq 0 :_{w_R} \mathfrak{a}$. Assume to the contrary that $0 :_{w_R} \mathfrak{a} \neq \mathfrak{b}w_R$. Then, there exists $\mathfrak{p} \in \operatorname{Ass}_R \frac{0:_{w_R}\mathfrak{a}}{\mathfrak{b}w_R}$.

Let $E := E_R(\frac{R}{\mathfrak{p}})$ and $\mathrm{D}(-) := \operatorname{Hom}_R(-, E)$. Then $\mathfrak{b}E \subseteq 0 :_E \mathfrak{a}$ and there is a natural monomorphism $h : \mathfrak{b}E \to \mathrm{D}(\frac{R}{\mathfrak{a}})$. Also, using [1, Lemma 10.2.16], there are the following natural isomorphisms

$$\begin{aligned} \frac{E}{\mathfrak{b}E} \cong \frac{R}{\mathfrak{b}} \otimes_R \operatorname{Hom}_R(R, E) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\frac{R}{\mathfrak{b}}, R), E) \\ &\cong \operatorname{Hom}_R(\mathfrak{a}, E) \\ &= \mathrm{D}(\mathfrak{a}). \end{aligned}$$

Therefore, we get the following commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{b}E & \longrightarrow & E & \longrightarrow & E/\mathfrak{b}E & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{D}(\frac{R}{\mathfrak{a}}) & \xrightarrow{\mathrm{D}(\pi)} & \mathrm{D}(R) & \xrightarrow{\mathrm{D}(i)} & \mathrm{D}(\mathfrak{a}) & \longrightarrow & 0. \end{array}$$

This implies that h is an isomorphism and so the combination $\mathfrak{b}E \xrightarrow{h} \mathrm{D}(\frac{R}{\mathfrak{a}}) \to 0 :_E \mathfrak{a}$, which is the inclusion map, is an isomorphism. Hence $\mathfrak{b}E = 0 :_E \mathfrak{a}$.

On the other hand, by the assumption, $0 = \dim R_{\mathfrak{p}} = \dim (w_R)_{\mathfrak{p}}$. This implies that, $(w_R)_{\mathfrak{p}} \cong w_{R_{\mathfrak{p}}} \cong E_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}})$. Therefore, $\mathfrak{b}(w_R)_{\mathfrak{p}} = 0 :_{(w_R)_{\mathfrak{p}}} \mathfrak{a}$ and so, $(\frac{0 :_{w_R} \mathfrak{a}}{\mathfrak{b}w_R})_{\mathfrak{p}} = 0$ which is a contradiction. Hence $\mathfrak{b}w_R = 0 : w_R\mathfrak{a}$.

Now, the result follows from the fact that every R-regular sequence is a w_R-regular sequence too. □

4. Some generalizations of a theorem of Peskine and Szpiro

In this section, we present some generalizations of a basic result of Peskine and Szpiro [10, Proposition 1.3], namely if R is Gorenstein, $\mathfrak{a} \neq 0$ and $\mathfrak{b} := 0 :_R \mathfrak{a}$ then $\frac{R}{\mathfrak{a}}$ is Cohen-Macaulay if and only if $\frac{R}{\mathfrak{a}}$ is unmixed and $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

Theorem 4.1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module w_R, $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal of R generated by an R-regular sequence such that $\mathfrak{a} \neq I \neq \mathfrak{b}$ and $I :_R \mathfrak{a} = \mathfrak{b}$. Then the following are equivalent:

(i) $\frac{w_R}{\mathfrak{a}w_R}$ is Cohen-Macaulay.
(ii) $\frac{w_R}{\mathfrak{a}w_R}$ is unmixed and $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

Proof. Considering the fact that $w_{\frac{R}{I}} \cong \frac{w_R}{Iw_R}$, we may replace R by $\frac{R}{I}$ and assume that $I = 0$.

"$(i) \to (ii)$" Using the exact sequence $0 \to \mathfrak{a}w_R \to w_R \to \frac{w_R}{\mathfrak{a}w_R} \to 0$,

$$\text{depth } \mathfrak{a}w_R \geq \text{Min}\,\{\text{depth } w_R, \text{depth } \frac{w_R}{\mathfrak{a}w_R} + 1\} = \text{depth } R.$$

Therefore, $\mathfrak{a}w_R$ is maximal Cohen-Macaulay, and so is $\text{Hom }_R(\mathfrak{a}w_R, w_R)$. Now, Set $(-)^* := \text{Hom }_R(-, w_R)$. Considering the isomorphisms

$$\begin{aligned} \text{Hom }_R(\frac{w_R}{\mathfrak{a}w_R}, w_R) &\cong \text{Hom }_R(\frac{R}{\mathfrak{a}}, \text{Hom }_R(w_R, w_R)) \\ &\cong \text{Hom }_R(\frac{R}{\mathfrak{a}}, R) \cong \mathfrak{b}, \end{aligned}$$

and [2, Theorem 3.3.10], we get the following commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\frac{w_R}{\mathfrak{a}w_R})^* & \longrightarrow & (w_R)^* & \longrightarrow & (\mathfrak{a}w_R)^* & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \exists & & \\ 0 & \longrightarrow & \mathfrak{b} & \longrightarrow & R & \longrightarrow & \frac{R}{\mathfrak{b}} & \longrightarrow & 0. \end{array}$$

This implies that $\frac{R}{\mathfrak{b}} \cong \mathrm{Hom}_R(\mathfrak{a}w_R, w_R)$ is Cohen-Macaulay.

"$(ii) \to (i)$" Follows using similar argument as used above and Theorem 3.3. □

Recall that

$$\text{G-dim}_R \frac{R}{\mathfrak{a}} := \text{Max}\,\{i \mid \text{Ext}^i_R(\frac{R}{\mathfrak{a}}, R) \neq 0\}.$$

Corollary 4.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with the canonical module w_R and I be an ideal of R such that $\mathfrak{a} \sim_{(I;w_R)} \mathfrak{b}$ and $\frac{R}{\mathfrak{a}}, \frac{R}{\mathfrak{b}}$ are unmixed. Then the following statements hold:

(i) $\frac{w_R}{\mathfrak{a}w_R}$ is Cohen-Macaulay if and only if $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

(ii) $\text{G-dim}_R \frac{R}{\mathfrak{a}} < \infty$ and $\frac{R}{\mathfrak{a}}$ is Cohen-Macaulay iff $\text{G-dim}_R \frac{R}{\mathfrak{b}} < \infty$ and $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

Proof. Note that, by Theorem 3.2, $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$.

(i) Follows from Theorem 4.1.

(ii) In view of,

$$\text{Ext}^i_R(\frac{R}{\mathfrak{a}}, R) \cong \text{Ext}^{i-t}_{\frac{R}{I}}(\frac{R}{\mathfrak{a}}, \frac{R}{I}),$$

where $t := \text{grade}_R \mathfrak{a}$, $\text{G-dim}_R \frac{R}{\mathfrak{a}} < \infty$ if and only if $\text{G-dim}_{\frac{R}{I}} \frac{R}{\mathfrak{a}} < \infty$. Hence, we may assume that $I = 0$. By [5, Theorem 1.11], $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay. On the other hand, $\text{G-dim}\, \frac{R}{\mathfrak{a}} = \text{depth}\, R - \text{depth}\, \frac{R}{\mathfrak{a}} = 0$ and, by [7, Theorem 1], $\text{G-dim}\, \frac{R}{\mathfrak{b}} = 0$.

□

One can generalize the above corollary for the ideals which are in a same evenly M-linkage class, as follows.

Corollary 4.3. Let $(R, \mathfrak{m})$ be local and M be maximal Cohen-Macaulay of finite injective dimension. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are in the evenly M-linkage class $\mathfrak{a} = \mathfrak{a}_0 \sim_{(I_0;M)} \mathfrak{a}_1 \sim_{(I_1;M)} \dots \sim_{(I_{n-1};M)} \mathfrak{a}_n = \mathfrak{b}$, such that $\frac{R}{\mathfrak{a}_i}$ is unmixed for all i. Then,

(i) $\frac{M}{\mathfrak{a}M}$ is Cohen-Macaulay if and only if $\frac{M}{\mathfrak{b}M}$ is Cohen-Macaulay.

(ii) If $\mathfrak{a}$ is M-licci then $\frac{M}{\mathfrak{a}M}$ and $\frac{R}{\mathfrak{a}}$ are Cohen-Macaulay.

(iii) $\text{G-dim}_R \frac{R}{\mathfrak{a}} < \infty$ and $\frac{R}{\mathfrak{a}}$ is Cohen-Macaulay iff $\text{G-dim}_R \frac{R}{\mathfrak{b}} < \infty$ and $\frac{R}{\mathfrak{b}}$ is Cohen-Macaulay.

Proof. By Lemma 2.14, we may assume that R has canonical module w_R. In view of [2, Exercise 3.3.28], $M \cong \oplus^l w_R$ for some $l \in \mathbb{N}_0$ and, using Lemma 2.3, $\mathfrak{a}$ and $\mathfrak{b}$ are in the evenly w_R-linkage class $\mathfrak{a} = \mathfrak{a}_0 \sim_{(I_0;w_R)} \mathfrak{a}_1 \sim_{(I_1;w_R)} \cdots \sim_{(I_{n-1};w_R)} \mathfrak{a}_n = \mathfrak{b}$. Now, the results follow from Corollary 4.2. □

Theorem 4.4. Let $(R, \mathfrak{m})$ be a local ring, M be a Cohen-Macaulay R-module of finite injective dimension and $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \sim_{(0;M)} \mathfrak{b}$. Also, assume that depth $R =$ depth $\frac{R}{\mathfrak{a}} =$ depth $\frac{R}{\mathfrak{b}}$. Then, $\frac{M}{\mathfrak{a}M}$ is Cohen-Macaulay if and only if $\frac{M}{\mathfrak{b}M}$ is Cohen-Macaulay.

Proof. We prove the claim by induction on $n :=$ depth R. If $n = 0$ then, by [8, Theorem 18.9] and [2, Exercise 3.1.23], M is injective and dim $R = 0$. So, the result is clear. Now, assume that $n > 0$. In case dim $M = 0$, there is nothing to prove. Therefore, we may assume that dim $M > 0$ and there exists $x \in \mathfrak{m} - \mathrm{Z}(M) \cup \mathrm{Z}(R) \cup \mathrm{Z}(\frac{R}{\mathfrak{a}}) \cup \mathrm{Z}(\frac{R}{\mathfrak{b}})$. In view of [2, Exercise 3.1.24], $\mathrm{Ext}\,^i_R(\frac{R}{\mathfrak{a}}, M) = \mathrm{Ext}\,^i_R(\frac{R}{\mathfrak{b}}, M) = 0$ for all $i > 0$. Hence, by Lemma 2.16, $(\mathfrak{a}, x) \sim_{(0;\frac{M}{xM})} (\mathfrak{b}, x)$. Therefore, by inductive hypothesis, $\frac{M}{(\mathfrak{a},x)M}$ is Cohen-Macaulay if and only if $\frac{M}{(\mathfrak{b},x)M}$ is Cohen-Macaulay. Now, the result follows from Lemma 2.8 and the fact that $x \notin \mathrm{Z}(\frac{M}{\mathfrak{a}M}) \cup \mathrm{Z}(\frac{M}{\mathfrak{b}M})$. □

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LINKAGE OF IDEALS OVER A MODULE

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پیوند ایده‌آل‌ها روی یک مدول

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به عنوان تعمیمی از مفهوم کلاسیک پیوند ایده‌آل‌ها، در این مقاله مفهوم پیوند ایده‌آل‌ها روی یک مدول را معرفی می‌کنیم. برخی قضایای مهم و شناخته شده در نظریه پیوند ایده‌آل‌ها، در این حالت بیان و ثابت می‌شوند. به ویژه تعمیم‌هایی از قضیه اساسی پسکین (Peskine) و اسپیرو (Szpiro)، که بیان می‌کند اگر R حلقه‌ای موضعی گرنشتاین، a ایده آلی ناصفر از R و $b = 0 :_R a$، آنگاه $\frac{R}{a}$ یک حلقه کوهن-مکالی است اگر و تنها اگر $\frac{R}{a}$ مخلوط نشدنی و $\frac{R}{b}$ کوهن-مکالی باشد، را در این مفهوم جدید ثابت می‌کنیم.

کلمات کلیدی: پیوند ایده‌آل‌ها، مدول‌های کوهن-مکالی، مدول‌های کانونی.