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ON THE PROJECTIVE DIMENSION OF ARTINIAN MODULES

Y. IRANI, K. BAHMANPOUR* AND G. GHASEMI

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules. In this paper it is shown that R is a Cohen-Macaulay ring if and only if R admits a non-zero Artinian R-module A of finite projective dimension; in addition, for all such Artinian R-modules A, it is shown that $\mathrm{pd}_R A = \dim R$. Furthermore, as an application of these results it is shown that

$$\mathrm{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \mathrm{pd}_R H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N)$$

for each $\mathfrak{p} \in \mathrm{Spec} R$ and each integer $i \geq 0$. This result answers affirmatively a question raised by the present authors in [13].

1. Introduction

Let R denote a commutative Noetherian ring with identity, I be an ideal of R and M, N be two R-modules. Throughout this paper, we denote by $\mathrm{id}_R M$ and $\mathrm{fd}_R M$ the injective dimension and the flat dimension of M respectively. Furthermore, we denote by $\mathrm{pd}_R M$ the projective dimension of M. Also, for technical reasons, we interpret the injective dimension, flat dimension and projective dimension of the zero R-module as -1.

The following celebrated result, primarily known as Bass' conjecture (Bass, 1963), has been achieved by Peskine and Szpiro (1973) in the

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geometric case, by Hochster (1975) for all local rings containing a field, and by Roberts (1987) in mixed characteristic.

Theorem 1.1 (Bass' Theorem)**.** *A Noetherian local ring is Cohen-Macaulay if and only if it admits a non-zero finitely generated module of finite injective dimension.*

In this paper first we prove an analogue result of this theorem. More, precisely we prove that a Noetherian local ring is Cohen-Macaulay if and only if it admits a non-zero Artinian module of finite projective dimension. Also, we show that for any non-zero Artinian module A over a Noetherian local ring R, either $\mathrm{pd}_{\mathrm{R}}\, A = \infty$ or $\mathrm{pd}_{\mathrm{R}}\, A = \dim R$.

In 1970 the notion of generalized local cohomology was introduced by Herzog in [12]. According to his definition, the ith generalized local cohomology modules of M and N with respect to I is defined as

$$H^i_I(M,N) \cong \varinjlim_{n\geq 1} \mathrm{Ext}^i_R(M/I^nM, N).$$

Clearly, this notion is a generalization of the usual local cohomology functor and $H^i_I(R,N)$ is just the ordinary local cohomology module $H^i_I(N)$. There are a lot of articles concerning the generalized local cohomology modules in the literature (see for example [1, 2, 4, 5, 7, 8, 9, 10, 17, 18, 19, 20]).

The authors of the present paper, as a generalization of the main result of [3], in [13] proved the following theorem:

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules. Then for each $\mathfrak{p} \in \mathrm{Spec}R$ and each $i \geq 0$, the following inequalities hold:*

i) $\mathrm{id}_{\mathrm{R}_\mathfrak{p}}\, H^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq \mathrm{id}_{\mathrm{R}}\, H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M,N)$.

ii) $\mathrm{fd}_{\mathrm{R}_\mathfrak{p}}\, H^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq \mathrm{fd}_{\mathrm{R}}\, H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M,N)$.

Then they asked the following question:

Question 1.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules. Whether for each $\mathfrak{p} \in \mathrm{Spec}R$ and each integer $i \geq 0$, the following inequality holds?*

$$\mathrm{pd}_{\mathrm{R}_\mathfrak{p}}\, H^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq \mathrm{pd}_{\mathrm{R}}\, H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M,N).$$

The second aim of this paper is to present an affirmative answer to this question in general.

Throughout this paper, for each Noetherian local ring $(R, \mathfrak{m})$ the symbol $D(-)$ will always denote the Matlis dual functor $\mathrm{Hom}_R(-, E)$, where $E := E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. Also, we will denote the $\mathfrak{m}$-adic completion of R by $\widehat{R}$. For any unexplained notation and terminology, we refer the reader to [6] and [14].

2. The results

The main purpose of this section is to prove Theorems 2.12 and 2.14. In order to prove these theorems first we need to prove Proposition 2.9. So, we start this section with some auxiliary lemmas, which are needed in the proof of Proposition 2.9.

Recall that for each commutative ring R, the *finitistic projective dimension of* R, is defined as:

$$\mathrm{FPD}(R) := \sup\{\mathrm{pd}_R\, M \ : \ M \text{ is an } R\text{-module with } \mathrm{pd}_R\, M < \infty\}.$$

The following result shows that for any commutative Noetherian ring R, $\mathrm{FPD}(R)$ is equal to the Krull dimension of R. The inequality $\geq$ was established by Bass in 1962, the other one was completed by Gruson and Raynaud in 1973. Since there are examples of commutative Noetherian rings with an infinite Krull dimension so in general the finitistic projective dimensions of commutative Noetherian rings need not to be finite.

Lemma 2.1. (See [15, Theorem 3.2]) *Let R be a Noetherian ring. Then* $\mathrm{FPD}(R) = \dim R$.

Lemma 2.2. (See [11, Proposition 6]) *Let R be a Noetherian ring and Q be a non-zero flat R-module. If* $\mathrm{FPD(R)} < \infty$ *then* $\mathrm{pd}_R\, Q < \infty$.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero R-module. Then* $\mathrm{fd}_R\, M < \infty$ *if and only if* $\mathrm{pd}_R\, M < \infty$.

Proof. Since R is a local Noetherian ring it follows that $\dim R < \infty$. Hence, by Lemma 2.1 we have $FPD(R) < \infty$. Thus by Lemma 2.2, for each flat R-module Q, we have $\mathrm{pd}_R\, Q < \infty$. Now, assume that $\mathrm{fd}_R\, M := n < \infty$. By induction on n we prove that $\mathrm{pd} M < \infty$. If $n = 0$, then M is a flat R-module, and so the assertion follows from Lemma 2.2. Now suppose, inductively, that $\mathrm{fd}_R\, M = n > 0$ and we have established the result for smaller values of n. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0, \quad (2.3.1)$$

for some free R-module F. Then it is clear that $\mathrm{fd}_R\, K = n - 1 < \infty$ and so by inductive assumption we have $\mathrm{pd}_R\, K < \infty$. Now since F is a projective R-module, from the exact sequence (2.3.1) we deduce that $\mathrm{pd}_{\mathrm{R}} M \leq 1 + \mathrm{pd}_{\mathrm{R}} K < \infty$. This completes the inductive step.

Conversely, assume that $\mathrm{pd}_{\mathrm{R}} M < \infty$. Then, since each projective resolution of M is a flat resolution of M, it follows that $\mathrm{fd}_{\mathrm{R}} M \leq \mathrm{pd}_{\mathrm{R}} M < \infty$. □

Lemma 2.4. (See [12, Theorem 3.1.17]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module. If $\mathrm{id}_{\mathrm{R}} M < \infty$, then $\mathrm{id}_{\mathrm{R}} M = \operatorname{depth} R$.*

Lemma 2.5. (See [16, Theorem 69]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module. If $\mathrm{id}_{\mathrm{R}} M < \infty$, then R is a Cohen-Macaulay ring.*

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module. If $\mathrm{id}_{\mathrm{R}} M < \infty$, then $\mathrm{id}_{\mathrm{R}} M = \dim R$.*

Proof. The assertion follows from Lemmata 2.4 and 2.5. □

Lemma 2.7. (See [6, Theorem 10.2.10]) *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and M be an Artinian R-module. Then $D(M)$ is a finitely generated R-module.*

Lemma 2.8. (See [14, Theorem 2.5]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then each projective module over R is free.*

The following result plays a key role in the proof of our main results.

Proposition 2.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero Artinian R-module. If $\mathrm{pd}_{\mathrm{R}} M < \infty$ then $\mathrm{pd}_{\mathrm{R}} M = \dim R$ and R is a Cohen-Macaulay ring.*

Proof. Assume that $\mathrm{pd}_{\mathrm{R}} M = n < \infty$. Then by Lemma 2.8, M has a finite R-free resolution of length n. Now, using the fact that $\widehat{R}$ is a flat R-algebra it follows that the $\widehat{R}$-module $M \otimes_R \widehat{R}$ has a finite $\widehat{R}$-free resolution of length n, and so $\mathrm{pd}_{\widehat{\mathrm{R}}} M \otimes_R \widehat{R} \leq n$. Since by the hypothesis M is an Artinian R-module it follows that M has an $\widehat{R}$-module structure and there is an isomorphism of $\widehat{R}$-modules as $M \simeq M \otimes_R \widehat{R}$, whence we get

$$\mathrm{pd}_{\widehat{\mathrm{R}}} M = \mathrm{pd}_{\widehat{\mathrm{R}}} M \otimes_R \widehat{R} \leq \mathrm{pd}_{\mathrm{R}} M < \infty.$$

Now by Lemma 2.1 we can deduce that $\mathrm{pd}_{\widehat{\mathrm{R}}} M \leq \mathrm{pd}_{\mathrm{R}} M \leq \dim R$.

Now, let $\mathrm{pd}_{\widehat{\mathrm{R}}} M = t$ and assume that

$$0 \longrightarrow Q_t \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

is a $\widehat{R}$-projective resolution of the $\widehat{R}$-module M. Then by effecting the exact functor $D_{\widehat{R}}(-)$ to this exact sequence we get an exact sequence

$$0 \longrightarrow D_{\widehat{R}}(M) \longrightarrow D_{\widehat{R}}(Q_0) \longrightarrow \cdots \longrightarrow D_{\widehat{R}}(Q_t) \longrightarrow 0,$$

which is an $\widehat{R}$-injective resolution of the $\widehat{R}$-module $D_{\widehat{R}}(M)$. Therefore, $\operatorname{id}_{\widehat{R}} D_{\widehat{R}}(M) \leq t < \infty$. Now, since $\operatorname{id}_{\widehat{R}} D_{\widehat{R}}(M) < \infty$ and by lemma 2.7, $D_{\widehat{R}}(M)$ is a finitely generated $\widehat{R}$-module, it follows from Lemma 2.5 that $\widehat{R}$ is a Cohen-Macaulay ring. But this means that R is a Cohen-Macaulay ring. Also, by Corollary 2.6 we get

$$\operatorname{id}_{\widehat{R}} D_{\widehat{R}}(M) = \dim \widehat{R} = \dim R,$$

and so

$$\dim R = \operatorname{id}_{\widehat{R}} D_{\widehat{R}}(M) \leq t = \operatorname{pd}_{\widehat{R}} M \leq \dim R.$$

Thus $t = \dim R$ and hence $\dim R = t = \operatorname{pd}_{\widehat{R}} M \leq \operatorname{pd}_R M \leq \dim R$, which means that $\operatorname{pd}_R M = \dim R$. □

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero Artinian R-module. Then either* $\operatorname{pd}_R M = \infty$ *or* $\operatorname{pd}_R M = \dim R$.

Proof. The assertion follows from Proposition 2.9. □

Lemma 2.11. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring. Then there exists a non-zero Artinian R-module M such that* $\operatorname{pd}_R M < \infty$.

Proof. Set $d := \dim R$. If $d = 0$, then $M = R$ satisfies the condition of the lemma. Now assume that $d > 0$ and let $x_1, ...x_d$ be a system of parameters for R. Then by [14, p. 151], the Koszul complex

$$0 \longrightarrow R \longrightarrow R^d \longrightarrow \cdots \longrightarrow R^d \longrightarrow R \longrightarrow R/(x_1, ..., x_d) \longrightarrow 0,$$

with respect to $x_1, ..., x_d$ provides an R-free resolution of the Artinian R-module $R/(x_1, ..., x_d)$. Therefore, $\operatorname{pd}_R R/(x_1, ..., x_d) < \infty$ and hence $M = R/(x_1, ..., x_d)$ satisfies the condition of the lemma. □

The following theorem is the first main result of this paper.

Theorem 2.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then R is a Cohen-Macaulay ring if and only if and only if R admits a non-zero Artinian R-module of finite projective dimension.*

Proof. The assertion holds by Proposition 2.9 and Lemma 2.11. □

Lemma 2.13. (See [10, Theorem 2.2]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M, N be two finitely generated R-modules. Then the R-module $H^i_{\mathfrak{m}}(M, N)$ is Artinian, for all $i \geq 0$.*

Now we are ready to state and prove the second main result of this paper, which answers affirmatively a question raised by the present authors in [13].

Theorem 2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M, N be two finitely generated R-modules. Then for each $\mathfrak{p} \in \operatorname{Spec} R$ and each integer $i \geq 0$, we have:*

$$\operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N).$$

Proof. If $H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ then the assertion is clear. So, we may assume that $H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$. Then by [13, Theorem 2.11] we have $H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N) \neq 0$.

First, suppose that $\operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \infty$. Then by Lemma 2.3, $\operatorname{fd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \infty$ and so by [13, Theorem 2.11], we have

$$\begin{aligned} \infty &= \operatorname{fd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \\ &\leq \operatorname{fd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N) \\ &\leq \operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N), \end{aligned}$$

which means that $\operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N) = \infty$ and so the assertion is clear in this case. So, in order to prove the assertion we can make the additional assumption that $\operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) < \infty$. Now if

$$\operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N) = \infty,$$

then there is nothing to prove. So we may assume that

$$\operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N) < \infty \text{ and } \operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) < \infty.$$

Then by Proposition 2.9 and Lemma 2.13, we get

$$\begin{aligned} \operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) &= \dim R_{\mathfrak{p}} \\ &= \operatorname{height} \mathfrak{p} \\ &\leq \dim R \\ &= \operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N). \end{aligned}$$

□

Corollary 2.15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and N be a finitely generated R-module. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $i \geq 0$ be an integer. Then*

$$\operatorname{pd}_{R_{\mathfrak{p}}} H^i_{\mathfrak{p} R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \leq \operatorname{pd}_{R} H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(N).$$

Proof. Replacing M by R, the assertion follows immediately from Theorem 2.14. □

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فرض کنید $(R, \mathfrak{m})$ یک حلقه‌ی موضعی و نوتری و M، N دو R-مدول با تولید متناهی باشند. در این مقاله نشان داده شده است که حلقه‌ی R کوهن-مکالی است اگر و فقط اگر یک R-مدول آرتینی غیر صفر مانند A با بعد تصویری متناهی موجود باشد. بعلاوه نشان داده شده است که برای هر چنین R-مدول آرتینی مانند A، داریم $\mathrm{pd}_R A = \dim R$. علاوه بر آن به عنوان کاربردی از این نتیجه نشان داده شده است که

$$\mathrm{pd}_{R_\mathfrak{p}} H^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \leq \mathrm{pd}_R H^{i+\dim R/\mathfrak{p}}_{\mathfrak{m}}(M, N).$$

که در آن $\mathfrak{p} \in \mathrm{Spec} R$ و $i \geq 0$.
این نتیجه، یک جواب مثبت به سوال مطرح شده توسط نویسندگان این مقاله در مرجع [۱۳] است.

کلمات کلیدی: مدول کوهمولوژی موضعی، مدول کوهمولوژی موضعی تعمیم یافته، بعد انژکتیو، بعد تصویری و بعد یکدست.