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SOME PROPERTIES ON DERIVATIONS OF LATTICES

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Abstract. In this paper we consider some properties of derivations of lattices and show that (i) for a derivation d of a lattice L with the maximum element 1, it is monotone if and only if $d(x) \leq d(1)$ for all $x \in L$ (ii) a monotone derivation d is characterized by $d(x) = x \wedge d(1)$ and (iii) simple characterization theorems of modular lattices and of distributive lattices are given by derivations. We also show that, for a distributive lattice L and a monotone derivation d of it, the set $\mathrm{Fix}_d(L)$ of all fixed points of d is isomorphic to the lattice $L/\ker(d)$. We provide a counter example to the result (Theorem 4) proved in [3].

1. Introduction

A notion of derivations of algebras with two operations $+$ and $\cdot$ has introduced as an analogy of derivations of analysis and then some properties of derivations are considered. For an algebra $A = (A, +, \cdot)$, a map $f : A \to A$ is called a derivation if it satisfies the conditions, for all $x, y \in A$,

$$f(x+y) = f(x) + f(y)$$
$$f(x \cdot y) = f(x) \cdot y + x \cdot f(y).$$

The notion of derivation is important in the theory of rings ([5]). After that, it is applied to lattices ([4]), where operation $+$ and $\cdot$ are interpreted as lattice operations $\vee$ and $\wedge$, respectively. Following the naive

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interpretation, the derivation d of a lattice L may be defined by

$$(a)\quad d(x \vee y) = d(x) \vee d(y)$$
$$(b)\quad d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)).$$

As proved in [4, 6], the condition (a) represents monotonicity of d and the condition (b) is equivalent to the condition $d(x \wedge y) = d(x) \wedge y$. Hence, as proved later, a monotone derivation $f : L \to L$ is characterized by $f(x \wedge y) = f(x) \wedge y$ for all $x, y \in L$. It follows from the result that a monotone derivation d has the form of $d(x) = x \wedge d(1)$ if L has the maximum element 1 and thus every monotone derivation is determined completely by the value $d(1)$.

In order to obtain more interesting properties of derivations of lattices, we adopt another definition of derivations according to [1, 2, 3, 7] and prove some fundamental properties of them, from which we get new results about derivations of lattices and provide accurate statements described in [1, 2, 3, 6, 7]. Moreover, we consider properties of generalized derivation ([1, 2]) and of f-derivation ([3]) from our view point and give some results which have simpler proofs than those of [3].

Concretely, we prove that

(i). For a derivation d of a lattice L with the maximum element 1, it is monotone if and only if $d(x) \le d(1)$ for all $x \in L$.

(ii). A monotone derivation d is just the form of $d(x) = x \wedge d(1)$.

(iii). For any lattice L and a derivation d, the condition

$$d \text{ is monotone} \Leftrightarrow d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L)$$

is equivalent to that L is a modular lattice.

(iv). For any lattice L and a derivation d, the condition

$$d \text{ is monotone} \Leftrightarrow d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L),$$

is equivalent to that L is a distributive lattice.

We also show that, for a distributive lattice L and a monotone derivation d of it, the set $\mathrm{Fix}_d(L) = \{x \in L \mid d(x) = x\}$ of all fixed points of d is isomorphic to the lattice $L/\ker(d)$.

Lastly, we provide a counter example to the result (Theorem 4) proved in [3].

2. Derivation

According to [6, 7], we give a definition of derivation of a lattice. Let $L = (L, \vee, \wedge)$ be a lattice. A map $d : L \to L$ is called a *derivation* of L if it satisfies the condition

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \quad (\forall x, y \in L)$$

Moreover, a derivation d is called *monotone* if

$$x \le y \Rightarrow d(x) \le d(y) \quad (\forall x, y \in L).$$

We note that the notion of monotone is called isotone in [1, 2, 3, 7])

Example 1. Let L be a lattice and $a \in L$. If we define a map $d_a : L \to L$ by $d_a(x) = x \wedge a$, then d_a is a monotone derivation. Indeed, for all $x, y \in L$, we have $d_a(x \wedge y) = (x \wedge y) \wedge a = ((x \wedge a) \wedge y) \vee (x \wedge (y \wedge a)) = (d_a(x) \wedge y) \vee (x \wedge d_a(y))$.

Example 2. ([3]) Let $L = \{0, a, b, 1\}$, $(0 < a < b < c < 1)$. We define $d : L \to L$ by

$$d(x) = \begin{cases} 0 & (x = 0) \\ a & (x = a, b) \\ c & (x = c, 1) \end{cases}$$

It is clear that $d : L \to L$ is the derivation of L.

We have basic results about derivations of lattices.

Proposition 2.1. *Let L be a lattice and d be a derivation of L. For all $x, y \in L$,*

(1) $d(x) \le x$
(2) $d(d(x)) = d(x)$
(3) *If* $1 \in L$, *then* $d(x) = d(x) \vee (x \wedge d(1))$
(4) *If* $1 \in L$, *then* $d(1) = 1 \Leftrightarrow d = id_L$
(5) $d(x) \wedge d(y) \le d(x \wedge y) \le d(x) \vee d(y)$
(6) $d(d(x) \wedge d(y)) = d(x) \wedge d(y)$
(7) *If d is monotone, then* $d(d(x) \vee d(y)) = d(x) \vee d(y)$
(8) *If* $d(d(x) \vee y) = d(x) \vee d(y)$, *then d is monotone.*

Proof. We only prove (7) and (8).

(7) If d is monotone, since $d(x), d(y) \le d(x) \vee d(y)$, then we get $d(d(x)), d(d(y)) \le d(d(x) \vee d(y))$. By $d(d(x)) = d(x)$ and $d(d(y)) = d(y)$, we have $d(x), d(y) \le d(d(x) \vee d(y))$ and $d(x) \vee d(y) \le d(d(x) \vee d(y))$. It is clear from (1) that $d(d(x) \vee d(y)) \le d(x) \vee d(y)$. Hence, $d(d(x) \vee d(y)) = d(x) \vee d(y)$.

(8) Suppose that $x \le y$. Since $d(x) \le x \le y$, we have $d(y) = d(d(x) \vee y) = d(x) \vee d(y)$ and thus $d(x) \le d(y)$. □

We note that the derivation $d_a(x) = x \wedge a$ in Example 1 is monotone. Moreover, any monotone derivation d has just the form of $d(x) = x \wedge a$ for some $a \in L$. In order to prove this fact, we deeply think about properties of monotone derivations.

Theorem 2.2. *For any derivation d, the following conditions are equivalent to each other.*

(1) *d is monotone ;*
(2) $d(x \wedge y) = d(x) \wedge d(y) \quad (\forall x, y \in L)$;
(3) $d(x) \vee d(y) \leq d(x \vee y) \quad (\forall x, y \in L)$.

Proof. We only show the case (1) $\Rightarrow$ (2). The other cases can be proved easily.

Since $x \wedge y \leq x, y$, we have $d(x \wedge y) \leq d(x), d(y)$. On the other hand, since $d(x \wedge y) \leq d(x) \wedge d(y) \leq x \wedge y$, we get $d(x \wedge y) = d(d(x \wedge y)) \leq d(d(x) \wedge d(y)) \leq d(x \wedge y)$. Thus $d(x \wedge y) = d(d(x) \wedge d(y))$. It follows that

$$\begin{aligned} d(x \wedge y) &= d(d(x) \wedge d(y)) \\ &= \{d(d(x)) \wedge d(y)\} \vee \{d(x) \wedge d(d(y))\} \\ &= (d(x) \wedge d(y)) \vee (d(x) \wedge d(y)) \\ &= d(x) \wedge d(y). \end{aligned}$$

□

From the result above, a monotone derivation can be characterized as follows.

Theorem 2.3. *Let L be a lattice and $f : L \to L$ be a map. Then the following conditions are equivalent.*
(1) *f is a monotone derivation;*
(2) $f(x \wedge y) = f(x) \wedge y \quad (\forall x, y \in L)$;
(3) $f(x) = x \wedge f(1) \quad (\forall x \in L)$.

Proof. Since (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear, we show (1) $\Rightarrow$ (2). Let f be a monotone derivation. Since $x \wedge y \leq x, y$, we get $f(x \wedge y) \leq f(x), f(y)$ and $f(x \wedge y) \leq f(x) \wedge y, x \wedge f(y)$ by $f(x \wedge y) \leq x \wedge y \leq x, y$. On the other hand, since f is the derivation, we have $f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y)) \geq f(x) \wedge y, x \wedge f(y)$. This means that $f(x \wedge y) = f(x) \wedge y = x \wedge f(y)$. □

Corollary 2.4. *If L has a maximum element 1 and d is a derivation, then the following conditions are equivalent.*
(1) *d is monotone;*
(2) $d(x) = x \wedge d(1)$ *for all* $x \in L$;
(3) $d(x) \leq d(1)$ *for all* $x \in L$.

Proof. Since (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are clear, we only show the case (3) $\Rightarrow$ (2). Let $d(x) \leq d(1)$. Since d is the derivation, we have $d(x) \leq x$ and thus $d(x) \leq x \wedge d(1)$. This implies $d(x) = d(x \wedge 1) = (d(x) \wedge 1) \vee (x \wedge d(1)) = d(x) \vee (x \wedge d(1)) = x \wedge d(1)$. □

Corollary 2.5. *If d is a monotone derivation of L, then $d(d(x) \vee d(y)) = d(x) \vee d(y)$ for all $x, y \in L$.*

Proof. The proof follows from $d(x) \vee d(y) = d(d(x)) \vee d(d(y)) \leq d(d(x) \vee d(y)) \leq d(x) \vee d(y)$. □

Unfortunately, the converse of the result above does not hold, namely, d may not be monotone even if $d(d(x) \vee d(y)) = d(x) \vee d(y)$ holds. We have a counter example. Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$. If we define $d : L \to L$ by $d(0) = d(1) = 0, d(a) = d(b) = b$, then it is easy to show that d is a derivation and $d(d(x) \vee d(y)) = d(x) \vee d(y)$, but d is not monotone.

Remark 2.6. A map $f : L \to L$ for a lattice L is called an *interior operator* if

(io1) $x \leq y \Rightarrow f(x) \leq f(y)$
(io2) $f(x) \leq x$
(io3) $f(f(x)) = f(x)$

It follows from our result above that a monotone derivation is an interior operator.

Remark 2.7. Similar results to our Theorem 2.2 are already proved in [7] as Theorem 3.19 and Theorem 3.21.

> Theorem 3.19. Let L be a modular lattice and d be a derivation of L. Then the following conditions are equivalent:
>
> (1) d is monotone;
> (2) $d(x \wedge y) = d(x) \wedge d(y)$;
> (3) If $d(x) = x$, then $d(x \vee y) = d(x) \vee d(y)$,
>
> where a lattice L is called *modular* if
>
> $$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \quad (\text{for all } x, y, z \in L).$$
>
> Theorem 3.21. Let L be a distributive lattice and d be a derivation of it. Then the following conditions are equivalent:
>
> (1) d is monotone.
> (2) $d(x \wedge y) = d(x) \wedge d(y)$.
> (3) $d(x \vee y) = d(x) \vee d(y)$.

Our results are stronger than those of above, because our results say that monotonicity is equivalent to the condition (2) $d(x \wedge y) = d(x) \wedge d(y)$ for all lattices L, namely, we do not assume modularity nor distributivity to get such results.

Moreover, we obtain a following identity condition instead of (3) If $d(x) = x$, then $d(x \vee y) = d(x) \vee d(y)$ in Theorem 3.19 in [7].

Theorem 2.8. *Let L be a modular lattice and d be a derivation. Then d is monotone $\Leftrightarrow d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L)$*

Proof. Suppose that d is monotone. Then we have

$$\begin{aligned} d(y) &= d(y \wedge (d(x) \vee y)) \\ &= \{d(y) \wedge (d(x) \vee y))\} \vee \{y \wedge d(d(x) \vee y)\} \\ &= d(y) \vee \{y \wedge d(d(x) \vee y)\} \quad (d(y) \leq y \leq d(x) \vee y) \\ &= (d(y) \vee d(d(x) \vee y)) \wedge y \quad \text{(modularity)} \\ &= y \wedge d(d(x) \vee y) \end{aligned}$$

and thus

$$\begin{aligned} d(x) \vee d(y) &= d(x) \vee \{y \wedge d(d(x) \vee y)\} \\ &= (d(x) \vee y) \wedge d(d(x) \vee y) \quad \text{(modularity)} \\ &= d(d(x) \vee y). \end{aligned}$$

Conversely, suppose $d(d(x) \vee y) = d(x) \vee d(y)$. If $x \leq y$, since $d(x) \leq x \leq y$, then we have $d(x) \vee y = y$ and $d(y) = d(d(x) \vee y) = d(x) \vee d(y)$. Therefore $d(x) \leq d(y)$ and d is monotone. □

Moreover we prove the converse.

Theorem 2.9. *A lattice L in which any derivation d satisfies the identity*

$$d(d(x) \vee y) = d(x) \vee d(y) \quad (\forall x, y \in L)$$

is a modular lattice.

Proof. For every $z \in L$, if we consider a map $d_z(x) = x \wedge z$ then it is a monotone derivation. By assumption, the map d_z satisfies

$$d_z(d_z(x) \vee y) = d_z(x) \vee d_z(y) \quad (\forall x, y \in L)$$

and hence $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. This implies that if $x \leq z$ then $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore L is the modular lattice. □

We also have a similar result about distributive lattices.

Theorem 2.10. *Let L be a distributive lattice and d be a derivation. Then we have*

$$d \text{ is monotone} \Leftrightarrow d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L).$$

Conversely,

Theorem 2.11. *A lattice L in which any derivation d satisfies the identity*

$$d(x \vee y) = d(x) \vee d(y) \ (\forall x, y \in L),$$

is a distributive lattice.

The above results provide characterization theorems of modular lattices and of distributive lattices in terms of derivations.

Remark 2.12. If d is a monotone derivation then a subset

$$\mathrm{Fix}_d(L) = \{x \in L \mid d(x) = x\}$$

of L is an *ideal* of L, that is, $\mathrm{Fix}_d(L)$ satisfies the conditions

(I1) $0 \in \mathrm{Fix}_d(L)$
(I2) $x \in \mathrm{Fix}_d(L), y \leq x \;\Rightarrow\; y \in \mathrm{Fix}_d(L)$
(I3) $x, y \in \mathrm{Fix}_d(L) \;\Rightarrow\; x \vee y \in \mathrm{Fix}_d(L)$.

In the case of d being monotone, we have a following result.

Theorem 2.13. *If d is a monotone derivation of a lattice L, then* $\mathrm{Fix}_d(L)$ *is a lattice.*

Proof. For all $x, y \in \mathrm{Fix}_d(L)$, since d is monotone, we have $d(x \wedge y) = d(x) \wedge d(y) = x \wedge y$ and hence $x \wedge y \in \mathrm{Fix}_d(L)$. □

Remark 2.14. We note that $(\mathrm{Fix}_d(L), \wedge, \vee)$ is a lattice for a monotone derivation d, but it is not always a sublattice of L if L has the maximum element 1. Because, if $1 \in L$, then $(\mathrm{Fix}_d(L), \wedge, \vee, 0, d(1))$ is also a lattice, however $d(1) = 1$ does not hold in general.

Corollary 2.15. *If L is a bounded distributive lattice and d is a monotone derivation of L, then the quotient lattice $L/\ker(d)$ is isomorphic to the lattice* $\mathrm{Fix}_d(L)$*, that is,*

$$L/\ker(d) \cong \mathrm{Fix}_d(L).$$

Proof. Let L be a bounded distributive lattice and d be a monotone derivation. Since d is monotone, $d(z) = z \wedge d(1)$ for all $z \in L$. It follows that $d(x \vee y) = (x \vee y) \wedge d(1) = (x \wedge d(1)) \vee (y \wedge d(1)) = d(x) \vee d(y)$. This means that a map $f : L \to \mathrm{Fix}_d(L)$ defined by $f(x) = d(x)$ for all $x \in L$ is a surjective homomorphism. It follows from the homomorphism theorem of lattices that $L/\ker(f) \cong \mathrm{Fix}_d(L)$ and $\ker(f) = \ker(d)$, where $x/\ker(d) = y/\ker(d)$ is defined by $d(x) = d(y)$ for all $x, y \in L$. Therefore, we have $L/\ker(d) \cong \mathrm{Fix}_d(L)$. □

3. Other derivations

Some types of derivations, such as *generalized derivation, generalized (f, g)-derivation* and *f-derivation*, are defined and properties of them are considered in [1, 2, 3]. For instance, a map $D : L \to L$ is called a *generalized derivation* in [1] if it satisfies the condition: For a derivation d,

$$D(x \wedge y) = (D(x) \wedge y) \vee (x \wedge d(y))$$

We get basic results about a generalized derivation D without difficulty.

Proposition 3.1 (cf. Proposition 3.4, 3.9 [1]). *Let d be a derivation and D be a generalized derivation. Then we have*

(1) $d(x) \leq D(x) \leq x$;
(1) $D(D(x)) = D(x)$;
(1) $D(x) \wedge D(y) \leq D(x \wedge y)$;
(1) $D(x) \wedge D(y) = D(D(x) \wedge D(y))$;
(1) $D(x) = d(x) \vee (x \wedge D(1))$.

We also have a new result about a generalized derivation D.

Proposition 3.2. *Let d be a derivation and D be a generalized derivation. Then we have $D \circ d = d \leq d \circ D$*

Proof. Since

$$\begin{aligned}(D \circ d)(x) &= D(d(x)) \\ &= D(x \wedge d(x)) \\ &= (D(x) \wedge d(x)) \vee (x \wedge d(d(x))) \\ &= d(x) \vee (x \wedge d(x)) \\ &= d(x),\end{aligned}$$

we get $D \circ d = d$.

For $d \circ D$, we have $d(D(x)) = d(x \wedge D(x)) = (d(x) \wedge D(x)) \vee (x \wedge d(D(x)) = d(x) \vee d(D(x)) \geq d(x)$ and hence $d \leq d \circ D$. □

It follows from our result that a characterization theorem about monotone generalized derivations can be proved similarly.

Proposition 3.3 (Proposition 3.12 [1]). *For a generalized derivation D, the following conditions are equivalent to each other:*

(1) *D is monotone.*
(1) $D(x \wedge y) = D(x) \wedge D(y)$.
(1) $D(x) \vee D(y) \leq D(x \vee y)$.
(1) $D(x) = x \wedge D(1)$ *if L has a maximum element* 1.

Proposition 3.4. *If L has a maximum element 1, then any generalized derivation D has a following form*

$$D(x) = (D(1) \wedge x) \vee d(x).$$

Corollary 3.5. $D(1) = 1 \;\Leftrightarrow\; D = id_L$

Lemma 3.6. *If L has a maximum element* 1 *and $d(x) \leq D(1)$ for all $x \in L$, then*

$$D(x) = x \wedge D(1).$$

In this case, the generalized derivation D is monotone. Conversely, if D is monotone then $d(x) \leq D(1)$ for all $x \in L$. Therefore, we have another characterization of monotone generalized derivations.

Theorem 3.7. *For any generalized derivation D,*

$$D \text{ is monotone} \Leftrightarrow d(x) \leq D(1) \quad (\forall x \in L).$$

Corollary 3.8. *If d is monotone, then so D is.*

Proof. We assume that d is monotone. Since $d(x) = x \wedge d(1)\;\; (\forall x \in L)$, we have $d(x) = x \wedge d(1) \leq x \wedge D(1) \leq D(1)$ and thus D is monotone. □

We may ask whether the converse holds, that is, if a generalized derivation D is monotone then so d is ?

Unfortunately, this does not hold by the following example.

Example 3 Let $L = \{0, a, b, 1\},\ (0 < a < b < 1)$ and $d, D : L \to L$ be maps defined by

$$d(x) = \begin{cases} 0 & (x = 0, 1) \\ a & (x = a, b) \end{cases}$$

$$D(x) = \begin{cases} x & (x = 0, a, b) \\ b & (x = 1) \end{cases}$$

It is easy to show that d is a derivation and D is a generalized derivation. Moreover D is monotone. However, it is obvious that d is not monotone.

In the previous section, we provide characterization theorems of modular lattices and of distributive lattices in terms of derivations. We also have similar results about generalized derivations.

Theorem 3.9. *Let L be a modular lattice and D be a generalized derivation. Then, D is monotone if and only if $D(D(x) \vee y) = D(x) \vee D(y)$.*

Proof. Suppose that D is monotone. Since

$$\begin{aligned} D(y) &= D((D(x) \vee y) \wedge y) \\ &= \{D(D(x) \vee y) \wedge y) \vee ((D(x) \vee y) \wedge d(y))\} \\ &= (D(D(x) \vee y) \wedge y) \vee d(y) \\ &= y \wedge D(D(x) \vee y), \end{aligned}$$

we have

$$\begin{aligned} D(x) \vee D(y) &= D(x) \vee (y \wedge D(D(x) \vee y)) \\ &= D(D(x)) \vee (y \wedge D(D(x) \vee y)) \\ &= (D(x) \vee y) \wedge D(D(x) \vee y) \quad \text{(modularity)} \\ &= D(D(x) \vee y). \end{aligned}$$

Conversely, suppose $D(D(x) \vee y) = D(x) \vee D(y)$ for all $x, y \in L$. If $x \leq y$, since $D(x) \leq x \leq y$, then we have $D(x) \vee y = y$ and $D(y) = D(D(x) \vee y) = D(x) \vee D(y)$. Therefore $D(x) \leq D(y)$ and D is monotone. □

Theorem 3.10. *A lattice L in which any generalized derivation D satisfies the identity*

$$D(D(x) \vee y) = D(x) \vee D(y) \quad (\forall x, y \in L)$$

is a modular lattice.

Proof. For every $z \in L$, if we define maps d_z and D_z by $d_z(x) = x \wedge z = D_z(x)$ for all $x \in L$. It is clear that d_z is a derivation and D_z is also a generalized derivation. Since D_z is monotone, it follows from assumption that $D_z(D_z(x) \vee y) = D_z(x) \vee D_z(y)$ and thus $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. This implies that if $x \leq z$ then $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore L is the modular lattice. □

We also have a similar result about distributive lattices.

Theorem 3.11 (Theorem 3.14 [1])**.** *Let L be a distributive lattice and D be a generalized derivation. Then we have*

$$D \text{ is monotone} \Leftrightarrow D(x \vee y) = D(x) \vee D(y) \ (\forall x, y \in L).$$

Conversely,

Theorem 3.12. *A lattice L in which any generalized derivation D satisfies the identity*

$$D(x \vee y) = D(x) \vee D(y) \ (\forall x, y \in L)$$

is a distributive lattice.

The above results provide characterization theorems of modular lattices and of distributive lattices in terms of generalized derivations.

We also consider another type of derivation, *f-derivation*, according to [3]. A map $d : L \to L$ is called an *f-derivation* if there exists a map $f : L \to L$ such that

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \quad (\forall x, y \in L).$$

It is clear that if $f = id_L$ then an f-derivation is the same as the derivation defined in the previous section.

As basic results about f-derivations, we have

Proposition 3.13 (Proposition 1,2 [3]). *Let $d : L \to L$ be an f-derivation. Then, for all $x, y \in L$,*
(1) $d(x) \leq f(x)$;
(2) $d(x) \wedge d(y) \leq d(x \wedge y) \leq d(x) \vee d(y)$;
(3) $f(x) \leq d(1), f(1) = 1 \;\Rightarrow\; d(x) = f(x)$;
(4) $d(1) = 1 \;\Rightarrow\; d = f$, *hence d is monotone.*

We also have similar results about monotone f-derivations.

Proposition 3.14 (cf. Theorem 1 [3]). *For an f-derivation d, the following conditions are equivalent:*
(1) *d is monotone;*
(2) $d(x) \vee d(y) \leq d(x \vee y) \quad (\forall x, y \in L)$;
(3) $d(x \wedge y) = d(x) \wedge d(y) \quad (\forall x, y \in L)$;
(4) $d(x) = f(x) \wedge d(x \vee y) \quad (\forall x, y \in L)$.

Proof. We only show that (1) is equivalent to (4). Suppose that d is monotone. Since $d(x) \leq f(x)$ and $d(x) \leq d(x \vee y)$, we get $d(x) \leq f(x) \wedge d(x \vee y)$. On the other hand, since $d(x) \leq d(x \vee y) \leq f(x \vee y)$, we have

$$\begin{aligned} d(x) &= d(x \wedge (x \vee y)) \\ &= (d(x) \wedge f(x \vee y)) \vee (f(x) \wedge d(x \vee y)) \quad (\because\ d \text{ is an } f\text{-derivation}) \\ &= d(x) \vee (f(x) \wedge d(x \vee y)) \quad (\because\ d(x) \leq f(x \vee y)). \end{aligned}$$

This means that $f(x) \wedge d(x \vee y) \leq d(x)$. Therefore, we get $d(x) = f(x) \wedge d(x \vee y)$.

Conversely, we assume that $d(x) = f(x) \wedge d(x \vee y) \quad (\forall x, y \in L)$. If $x \leq y$, then $d(x) = f(x) \wedge d(x \vee y) = f(x) \wedge d(y) \leq d(y)$. Hence d is monotone.

□

We note that the result above was already proved in [3] as Theorem 1 under the conditions $f(1) = 1$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$. Our result shows that the conditions are redundant. Moreover, our result implies that the modularity condition is also redundant in the Theorem 2 (a) in [3], where it said that

> Theorem 2. Let L be a modular lattice and d be an f-derivation on L.
> (a) d is a monotone f-derivation if and only if $d(x \wedge y) = dx \wedge dy$.

In cases of modular lattices and of distributive lattices, we have following results.

Theorem 3.15. *Let L be a modular lattice and $d : L \to L$ be an f-derivation. Then, d is monotone if and only if $d(x) \vee d(y) = (d(x) \vee f(y)) \wedge d(x \vee y)$ for all $x, y \in L$.*

Proof. Suppose that d is monotone. Since $d(y) = f(y) \wedge d(x \vee y)$, we have $d(x) \vee d(y) = d(x) \vee (f(y) \wedge d(x \vee y)) = (d(x) \vee f(y)) \wedge d(x \vee y)$ by modularity.

Conversely, assume that $d(x) \vee d(y) = (d(x) \vee f(y)) \wedge d(x \vee y)$ for all $x, y \in L$. If $x \leq y$ then $d(x) \leq d(x) \vee d(y) = (d(x) \vee f(y)) \wedge d(y) \leq d(y)$ and thus d is monotone. □

Corollary 3.16. *Let L be a modular lattice and $d : L \to L$ be a derivation. Then, d is monotone if and only if $d(d(x) \vee y) = d(x) \vee d(y)$.*

Proof. For an f-derivation d of a modular lattice L, if we take $f = id_L$, then d is a derivation of L and thus $d \circ d = d$ and $d(x) \leq x$ for all $x \in L$. It follows from the above that $d(x) \vee d(y) = (d(x) \vee y) \wedge d(x \vee y)$ for all $x, y \in L$. By use of these facts, if d is monotone, then we have $d(x) \vee d(y) = d(d(x)) \vee d(y) = (d(d(x)) \vee y) \wedge d(d(x) \vee y) = (d(x) \vee y) \wedge d(d(x) \vee y) = d(d(x) \vee y)$. The converse is obvious. □

For the case of distributive lattices, we also have a following result.

Theorem 3.17. *Let L be a distributive lattice and d be an f-derivation. Then, d is monotone if and only if $d(x) \vee d(y) = (f(x) \vee f(y)) \wedge d(x \vee y)$ for all $x, y \in L$.*

Proof. Let d be a monotone f-derivation. Since $d(x) = f(x) \wedge d(x \vee y)$ and L is the distributive lattice, we have $d(x) \vee d(y) = (f(x) \wedge d(x \vee y)) \vee (f(y) \wedge d(x \vee y)) = (f(x) \vee f(y)) \wedge d(x \vee y)$.

Conversely, suppose that $d(x) \vee d(y) = (f(x) \vee f(y)) \wedge d(x \vee y)$ for all $x, y \in L$. If $x \leq y$, then $d(x) \leq d(x) \vee d(y) = (f(x) \vee f(y)) \wedge d(y) \leq d(y)$. Therefore d is monotone. □

Corollary 3.18. *Let L be a distributive lattice and d be a derivation. Then, d is monotone if and only if $d(x) \vee d(y) = d(x \vee y)$ for all $x, y \in L$.*

Remark 3.19. The following result was proved as theorem 4 which was one of the main results of [3].

> Theorem 4. Let L be a lattice. If there exists an f-derivation d on L such that $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in L$ and f is an epimorphism, then L is a distributive lattice.

Unfortunately, this is not true, because we have a following counter example. Let $L = N_5 = \{0, a, b, c, 1\}$, $(0 < a < 1, 0 < b < c < 1)$ and $f = d = id_L$. Then it is trivial that d and f satisfy the assumption of the theorem, but the lattice L is neither distributive nor modular.

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SOME PROPERTIES ON DERIVATIONS OF LATTICES

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برخی خواص مشتقات مشبکه‌ها

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در این مقاله برخی خواص مشتقات مشبکه‌ها را بررسی کرده و نشان می‌دهیم (i) مشتق d از مشبکه L با عنصر ماکسیمم ۱، یکنوا است اگر و تنها اگر $d(x) \leq d(۱)$ برای هر $x \in L$، (ii) مشتق یکنوای d توسط $d(x) = x \wedge d(۱)$ مشخص (توصیف) می‌شود و (iii) برخی قضایای رده بندی ساده برای مشبکه‌های پیمانه‌ای و مشبکه‌های توزیع‌پذیر (پخشی)، با استفاده از مشتقات مشبکه‌ها بیان خواهد شد. همچنین نشان خواهیم داد که برای مشبکه توزیع‌پذیر L و مشتق یکنوای d از این مشبکه، مجموعه $\mathrm{Fix}_d(L)$ متشکل از تمام نقاط ثابت d با مشبکه $L/ker(d)$ یکریخت می‌باشد. به‌علاوه، مثال نقضی برای قضیه ۴ که از مرجع [۳]، ارائه خواهیم کرد.

کلمات کلیدی: مشتق، حافظ ترتیب، مشبکه پیمانه‌ای، مشبکه توزیع‌پذیر.