Journal of Algebraic Systems
Vol. 9, No 1, (2021), pp 35-43

DEFICIENCY ZERO GROUPS IN WHICH PRIME POWER OF GENERATORS ARE CENTRAL

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Abstract. The infinite family of groups defined by the presentation $G_p = \langle x, y | x^p = y^p,\ xyx^m y^n = 1\rangle$, in which p is a prime in $\{2, 3, 5\}$ and $m, n \in \mathbb{N}_0$, will be considered and finite and infinite groups in the family will be determined. For the primes $p = 2, 3$ the group G_p is finite and for $p = 5$, the group is finite if and only if $m \equiv n \equiv 1 \pmod 5$ is not the case.

1. Introduction

Deficiency zero groups are those, presented by an equal number of generators and relations, that is a finitely presented group $G = \langle X \mid R\rangle$ in which X is the set of generators of G and R is the set of relations, is called deficiency zero if $\mid X \mid = \mid R \mid$. Finite deficiency zero groups are of much interest in group theory, see for example [1, 3, 5]. For a general introduction to group presentations and deficiency zero groups see [4].

In this article we consider the groups $G_p = \langle x, y | x^p = y^p,\ xyx^m y^n = 1\rangle$, of zero deficiency, where $m, n \in \mathbb{N}_0$ and $p = 2, 3$ and 5. In some states, we use the modified Todd-Coxeter coset enumeration algorithm in the form given in [2]. Also we use the Tietz transformations (see [4]), to find out that the group G_p is finite or infinite. Using `GAP` ([6]), we checked finiteness of G_p with small m and n by examining its quotients

DOI: 10.22044/jas.2020.9361.1456
MSC(2010): Primary: 20F05; Secondary: 20D15.
Keywords: deficiency zero group, finitely presented group, coset enumeration alghorithm.
Received: 3 February 2020, Accepted: 28 July 2020.
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and subgroups and then tried to generalize the results. The notations we use here are standard.

2. Preliminaries

Let p be a prime number and let m, n be non-negative integers. Let G_p be the group defined by the presentation $G_p = \langle x, y | x^p = y^p,\ xyx^m y^n = 1\rangle$. The easiest case to think about is the case that the prime p divides one of m or n. In that case the second relation of G_p simplifies to $x = y^r$ in which r is an integer, therefore the group G_p is generated by y. The following lemma shows the detail.

Lemma 2.1. *Let p be a prime number. If $m \equiv 0 \pmod p$ or $n \equiv 0 \pmod p$ then the group G_p is a finite cyclic group of order $p(m+n+2)$.*

Proof. By the first relation of G_p, the elements x^p and y^p are central in G_p. Let $m = kp$. By the second relation of G_p it follows that $xyx^m y^n = xyx^{kp} y^n = xyy^{kp+n} = xy^{m+n+1} = 1$. Therefore the relation $x = y^{-(m+n+1)}$ holds in G_p. Using this relation to remove the generator x by a Tietz transformation, we get the presentation $G_p = \langle y | (y^{-m-n-1})^p = y^p\rangle = \langle y | y^{p(m+n+2)} = 1\rangle$ for the group G_p. Hence G_p is cyclic with $|G_p| = p(m+n+2)$. A similar argument works if $n \equiv 0 \pmod p$. □

Lemma 2.2. *Let $p \geq 3$ be a prime number. If $m \equiv 1 \pmod p$ and $n \equiv r \pmod p$ with $1 < r < p$ then the group G_p is a finite abelian group of order $p(m+n+2)$.*

Proof. Let $m = pk_1 + 1$ and $n = pk_2 + r$. By the second relation of G_p we have $1 = xyx^m y^n = xyxy^{pk+r}$ where $k = k_1 + k_2$. Therefore $xyxy^r (= y^{-pk})$ is a central element of G_p, that is $xyxy^r = xy^r xy$. Hence $xy^{r-1} = y^{r-1}x$. Consequently y^{r-1} commutes with x and hence is a central element of G_p. As $y^p, y^{r-1} \in Z(G_p)$ and $\gcd(p, r-1) = 1$ we see that $[y, x] = 1$. Therefore G_p is abelian. Now it is easy to see that $\mid G_p \mid = p(m+n+2)$. □

Lemma 2.3. *Let $p \geq 3$ be a prime number. If $m \equiv p-1 \pmod p$ then the subgroup $H = \langle y\rangle$ of the group G_p has a presentation of the form $H = \langle a \mid R_i, i = 1, \cdots, p\rangle$ where the relation R_i is $a^{p(1+(-1)^{i-1}(m+n+1)^i)} = 1$ for $i = 1, \cdots, p-1$ and R_p is $a^{(m+n+1)^p+1} = 1$.*

Proof. By the first relation of $G = G_p$ the elements x^p and y^p are central elements in G. Hence the second relation of G could be written in the form $xyx^{-1}y^{m+n+1} = 1$, that is $xyx^{-1} = y^{-(m+n+1)}$. A p power of the latter relation gives us the relation $y^{p(m+n+2)} = 1$. Consider the subgroup $H = \langle a = y\rangle$ of the group $G = G_p = \langle x, y | x^p =$

$y^p,\ xyx^{-1}y^{m+n+1} = 1\rangle$. We find a presentation for the subgroup H. The subgroup relation table gives us the bonus $1.y = a.1$ and by defining $1.x = 2$, the first row of the table of the second relation of G deduces $2.y = a^{-(m+n+1)}.2$. Now for $i = 2, \cdots, p-1$ define $i.x = i+1$ and the i-th row of the table of the second relation of G completes to deduce the bonus $(i+1).y = a^{(-1)^i(m+n+1)^i}.(i+1)$. Now the first row of the table of the relation $x^p y^{-p} = 1$ completes and we deduce $p.x = a^p.1$. All the tables are now complete and we have the presentation $H = \langle a \mid R_i, i = 1, \cdots, p\rangle$ for the subgroup H in which the relations $R_i, i = 1, \cdots, p-1$ is $a^{p(1+(-1)^{i-1}(m+n+1)^i)} = 1$ and correspond to the rows $2, \cdots, p$ of the table of the relation $x^p y^{-p} = 1$ and the relation R_p is $a^{(m+n+1)^p+1} = 1$ and corresponds to the last row of the table of the second relation of G_p. □

Lemma 2.4. *Let $p \geq 5$ be a prime number and let $m \equiv p-1 \pmod p$. Then the following hold*

(i) *If $n \equiv r \pmod p$ with $2 < r < p-1$ then the group G_p is a finite group of order $p(m+n+2)$.*
(ii) *If $n \equiv p-1 \pmod p$ then the group G_p is a finite group of order $p^2(m+n+2)$.*

Proof. By the previous lemma, the index of the subgroup H in G_p is p and the order of H is

$$h = \gcd((p(1+(-1)^{i-1}(m+n+1)^i), i = 1, \cdots, p-1), (m+n+1)^p+1).$$

On the other hand if $n \equiv r \pmod p$ with $2 < r < p-1$ then the number $(m+n+1)^p + 1$ is not divisible by p and therefore the number h is $(m+n+2)$ and if $n \equiv p-1 \pmod p$ then the number $(m+n+1)^p+1$ is divisible by p and therefore the number h is $p(m+n+2)$. □

Lemma 2.5. *Let $p \geq 5$ be a prime number and let $m \equiv 1 \pmod p$. If $n \equiv 1 \pmod p$ then the group G_p is an infinite group.*

Proof. Consider the quotient group $H = \langle x, y | x^p = y^p,\ xyx^m y^n = 1,\ x^p = 1\rangle$ of the group G_p. As $m, n \equiv 1 \pmod p$, the second relation of the group H is $(xy)^2 = 1$. Hence $H = \langle x, y | x^p = y^p = 1,\ (xy)^2 = 1\rangle$ is isomorphic to the triangle group $D(2, p, p)$. As $p \geq 5$ the group $D(2, p, p)$ is an infinite group and hence G_p is infinite. □

3. Main Results

For the prime $p = 2$, using Lemma 2.1, the only case which remains to consider is the case where m, n are both odd numbers.

Lemma 3.1. *Let m and n are odd numbers. Then the group G_2 is a finite group of order $4(m+n+2)$.*

Proof. Similar to the argument in the proof of Lemma 2.3, the subgroup $H = \langle y \rangle$ is of index 2 in G_2 and has the presentation $H = \langle a \mid a^{2(m+n+2)} = 1, a^{(m+n)^2-4} = 1\rangle$. As $2(m+n+2)$ divides $(m+n)^2-4$, H is cyclic of order $2(m+n+2)$ and hence the order of G_2 is $|G_2| = 2|H| = 4(m+n+2)$ where m and n are both odd numbers. □

The next theorem shows that the group G_p is finite for $p = 2$ and $m, n \in \mathbb{N}_0$.

Theorem 3.2. *Let $m, n \in \mathbb{N}_0$ and $p = 2$. Then the group*

$$G_p = \langle x, y | x^p = y^p,\ xyx^m y^n = 1\rangle$$

is a finite group.

Proof. The result follows from Lemmas 2.1 and 3.1. □

We continue with the case $p = 3$. We need the following lemmas to complete the case $p = 3$.

Lemma 3.3. *Let $m \equiv 1 \pmod 3$, then the followings hold*

(i) *If $n \equiv 1 \pmod 3$, then the group G_3 is a finite group of order $24(m+n+2)$.*

(ii) *If $n \equiv 2 \pmod 3$, then the group G_3 is a finite group of order $3(m+n+2)$.*

Proof.

(i) Let $m = 3k_1 + 1$ and $n = 3k_2 + 1$. By the second relation of G_3 it follows that $xyx^m y^n = xyx^{3k_1+1}y^{3k_2+1} = 1$ and therefore the following relation holds in G_3

$$(xy)^2 y^{3k} = 1,$$

where $k = k_1 + k_2$. Consider the subgroup $N = \langle a = x\rangle$ of the group $G_3 = \langle x, y \mid x^3y^{-3} = 1, (xy)^2y^{3k} = 1\rangle$. We use the modified Todd-Coxeter coset enumeration algorithm to find a presentation for N. By the table of the generator a we obtain $1 \cdot x = a \cdot 1$. Defining $1.y = 2$ and $2.y = 3$ completes the first row of the table of the relation $x^3y^{-3} = 1$ to deduce $3.y = a^3 \cdot 1$. Now the first row of the table of the second relation of G_3 also completes to get $2.x = a^{-3k-4} \cdot 3$. Also by defining $3.x = 4$ the second row of the table of the first relation of G_3 completes and we deduce that $4.x = a^{3k+7} \cdot 2$. Now the third row of the table of the second relation of G_3 completes and we find $4.y = a^{-6k-7} \cdot 4$. All the tables are complete and we obtain the following presentation for N

$$N \cong \langle a | a^{18k+24} = 1\rangle.$$

On the other hand we have $|G_3 : N| = 4$. Hence $|G_3| = 4(18k + 24) = 24(m + n + 2)$.

(ii) Lemma 2.2. $\square$

Lemma 3.4. *Let* $m \equiv 2 \pmod 3$ *and* $n \equiv 2 \pmod 3$*. Then the group* G_3 *is a finite group of order* $9(m + n + 2)$*.*

Proof. By Lemma 2.3 the subgroup H of the group G_3 has the presentation $H = \langle a | a^{3(m+n+2)} = a^{3(1-(m+n+1)^2)} = a^{(m+n+1)^3+1} = 1\rangle$ which simplifies to $H = \langle a | a^{3(m+n+2)} = 1\rangle$, as the numbers $(m + n + 1)^3 + 1$ and $3(1 - (m + n + 1)^2)$ are divisible by $3(m + n + 2)$. Therefore the group G_3 is a finite group of order $9(m + n + 2)$ in this case. $\square$

Theorem 3.5. *Let* $m, n \in \mathbb{N}_0$ *and let* $p = 3$*. Then the group*

$$G_p = \langle x, y | x^p = y^p,\ xyx^m y^n = 1\rangle,$$

is a finite group.

Proof. The result follows from Lemmas 2.1, 3.3 and 3.4. $\square$

Lemma 3.6. *Let* $m \equiv 2 \pmod 5$*. Then the followings hold*

(i) *If* $n \equiv 2 \pmod 5$*, then the group* G_5 *is a finite group of order* $55(m + n + 2)$*.*

(ii) *If* $n \equiv 3 \pmod 5$*, then the group* G_5 *is a finite group of order* $55(m + n + 2)$*.*

(iii) *If* $n \equiv 4 \pmod 5$*, then the group* G_5 *is a finite group of order* $5(m + n + 2)$*.*

Proof.

(i) Let 5 divides both $m-2$ and $n-2$. The second relation of the group G_5 is in the form $1 = xyx^2y^2x^{m+n-4}$ as x^{m-2} and y^{n-2} are central elements of G_5. Therefore the element xyx^2y^2 is also a central element of G_5. Hence the relation $xyx^2y^2 = x^2y^2xy$ holds in the group G_5 and thus $(yx)(xy) = (xy)(yx)$, or equivalently xy commutes with yx. In other words the relation $xy^2xy^{-1}x^{-2}y^{-1} = 1$ holds in G_5. Therefore we have $G_5 = \langle x, y | x^5y^{-5} = 1, xyx^2y^2x^{m+n-4} = 1, xy^2xy^{-1}x^{-2}y^{-1} = 1\rangle$ and we call the relations of G_5 in this order, that is the first relation is $x^5y^{-5} = 1$, the second is $xyx^2y^2x^{m+n-4} = 1$ and the third is $xy^2xy^{-1}x^{-2}y^{-1} = 1$.

We find a presentation for the subgroup $N = \langle a = x\rangle$ of the group G_5. Let $k = m + n - 4$. The subgroup table gives us $1.x = a.1$. Define $i.y = i + 1$ for $i = 1, \cdots, 4$ and the first row of the table of the first relation of G_5 completes to obtain $5.y = a^5.1$. Now by defining $2.x = 6$ the first rows of the tables of the second and the third relations complete and we get $6.x = a^{-k-6}.4$ and $3.x = a^{-k-7}.5$ respectively. Defining

$4.x = 7$ and then $7.x = 8$ completes the second row of the table of the first relation of G_5, the $4-th$ and the $7-th$ rows of the table of the second relation and we obtain $8.x = a^{k+11}.2$, $7.y = a^{-2k-11}.7$ and $8.y = a^{4k+28}.6$ respectively. Finally defining $5.x = 9$, $9.x = 10$ and $6.y = 11$ complete all the tables and we deduce $10.x = a^{k+10}.11$ from the third row of the table of the first relation. Also by the second, $6-th$ and $9-th$ rows of the table of the third relation we obtain $11.y = a^{-3k-19}.9$, $11.x = a^2.3$ and $9.y = a^{-2k-12}.10$ respectively. From the $6-th$ row of the table of the second relation we deduce $10.y = a^{-4k-22}.8$ and now all the entries of the monitor table are complete. Therefore the index of N in G_5 is 11 and we have the following presentation for N

$$N = \langle a \mid a^{5(k+6)} = 1 \rangle,$$

that is N is a cyclic subgroup with order $\mid N \mid = 5(m+n+2)$ and therefore $\mid G_5 \mid = 55(m+n+2)$ in this case.

(ii) Similar to the previous case the second relation of G_5 could be written in the form $1 = xyx^2y^3x^{m+n-5}$ as x^{m-2} and y^{n-3} are central elements of G_5. Therefore the element xyx^2y^3 is also a central element of G_5. Hence $xyx^2y^3 = x^2y^3xy$ in the group G_5 and thus $(yx)(xy^2) = (xy^2)(yx)$, or equivalently xy^2 commutes with yx. In other words the relation $xy^3xy^{-2}x^{-2}y^{-1} = 1$ holds in G_5. Therefore we have $G_5 = \langle x, y | x^5y^{-5} = 1, xyx^2y^3x^{m+n-5} = 1, xy^3xy^{-2}x^{-2}y^{-1} = 1\rangle$ and we call the relations of G_5 in this order, that is the first relation is $x^5y^{-5} = 1$, the second is $xyx^2y^3x^{m+n-5} = 1$ and the third is $xy^3xy^{-2}x^{-2}y^{-1} = 1$.

We find again a presentation for the subgroup $N = \langle b = x\rangle$ of the group G_5 and show that its index is 11. Let $d = m+n-5$. The subgroup table gives us $1.x = b.1$. Define $i.y = i+1$ for $i = 1, \cdots, 4$ and the first row of the table of the first relation of G_5 completes to obtain $5.y = b^5.1$. Now by defining $2.x = 6$ the first rows of the tables of the second and the third relations complete and we got $6.x = b^{-d-6}.3$ and $4.x = b^{-d-7}.5$ respectively. Defining $3.x = 7$ and then $7.x = 8$ completes the second row of the table of the first relation of G_5 and we obtain $8.x = b^{d+11}.2$. Now define $7.y = 9$ to completing the third row of the table of the second relation to get the bonus $9.x = b^2.4$ and define $6.y = 10$ to complete and get the bonus $10.x = b^{(-d-7)}.9$ from the second row of that table. Finally defining $5.x = 11$ completes all the tables and we deduce $11.x = b^{2d+17}.10$ from the $5-th$ row of the table of the first relation. Also by the $8-th$ and $11-th$ rows of the table of the third relation we obtain $8.y = b^{d+8}.8$ and $10.y = b^{-d-8}.11$ respectively. From the $5-th$ and $6-th$ rows of the table of the second relation we deduce $11.y = b^{-2d-11}.7$ and $9.y = b^{3d+24}.6$ respectively.

Now all the entries of the monitor table are complete. Therefore the index of N in G_5 is 11 and we have the presentation

$$N = \langle b \mid b^{5(d+7)} \rangle,$$

that is N is a cyclic subgroup with order $\mid N \mid= 5(m+n+2)$ and therefore $\mid G_5 \mid= 55(m+n+2)$ in this case.

(iii) Lemma 2.4. □

Lemma 3.7. *Let* $m \equiv 3 \pmod 5$. *Then the followings hold*

(i) *If* $n \equiv 3 \pmod 5$, *then the group* G_5 *is a finite group of order* $55(m+n+2)$.

(ii) *If* $n \equiv 4 \pmod 5$, *then the group* G_5 *is a finite group of order* $5(m+n+2)$.

Proof.

(i) Let 5 divides both $m-3$ and $n-3$. The second relation of the group G_5 is in the form $1 = xyx^3y^3x^m + n - 6$ as x^{m-3} and y^{n-3} are central elements of G_5. Therefore the element xyx^3y^3 is also a central element of G_5. Hence the relation $xyx^3y^3 = x^3y^3xy$ holds in the group G_5 and thus $(yx)(x^2y^2) = (x^2y^2)(yx)$, or equivalently x^2y^2 commutes with yx. In other words the relation $x^2y^3xy^{-2}x^{-3}y^{-1} = 1$ holds in G_5. Therefore we have $G_5 = \langle x, y | x^5y^{-5} = 1, xyx^3y^3x^{m+n-6} = 1, x^2y^3xy^{-2}x^{-3}y^{-1} = 1\rangle$ and we call the relations of G_5 in this order, that is the first relation is $x^5y^{-5} = 1$, the second is $xyx^3y^3x^{m+n-6} = 1$ and the third is $x^2y^3xy^{-2}x^{-3}y^{-1} = 1$.

Suppose $a = yx$, $b = x^2y^2$, $c = x^5$, $u = xy$ and $w = x^3y^3$. Consider the subgroup $N = \langle a, b, c, u, w\rangle$ of the group G_5. We find a presentation for N. Defining $1.y = 2$ completes the table of the generator a and gives us the bonus $2.x = a.1$ and defining $1.x = 3$ completes the table of u with bonus $3.y = u.1$. By defining $3.x = 4$ the table of b completes with the result $4.y = bu^{-1}.3$ and finally by defining $4.x = 5$ all the tables became complete and from the table of the generator c we get $5.x = ca^{-1}.2$ and from the table of w we conclude $5.y = wb^{-1}.4$ and from the first row of the table of the first relation of G_5 we deduce $2.y = cw^{-1}.5$. Now the relations of N are as follows, from the rows of the table of the first relation we get the relations $[c, a] = [c, u] = [c, b] = [c, w] = 1$, that is the generator c is central in N. From the table of the third relation of G_5 we deduce the relations $[b, a] = auw^{-1}a^{-1}u^{-1}w = [w, u] = a^{-1}u^{-1}baub^{-1} = [b, w] = 1$ and from the table of the second relation the following relations for N,

$$R_1 : uwc^k = 1,$$
$$R_2 : a^2bc^k = 1,$$
$$R_3 : bu^{-1}a^{-1}u^{-1}c^{k+2} = 1,$$
$$R_4 : wb^{-2}c^{k+2} = 1,$$
$$R_5 : a^{-1}w^{-1}uw^{-1}c^{k+4} = 1,$$

where $k = (m+n-6)/5$. It is easy to show that N is abelian and after some straightforward calculations we get the following presentation for N

$$N \cong \langle a, c | [a, c] = 1, a^{11}c^{4k+2} = 1, c^{5k+8} = 1\rangle.$$

The subgroup N is cyclic if and only if $\gcd(11, 4k + 2) = 1$ and the order of N is $|N| = 11(5k + 8)$. As the index of N in G_5 is 5, we see that G_5 is finite with order $|G_5| = 55(5k + 8) = 55(m + n + 2)$.

(ii) Lemma 2.3. □

Lemma 3.8. *Let* $m \equiv 4 \pmod 5$ *and* $n \equiv 4 \pmod 5$. *Then the group* G_5 *is a finite group of order* $25(m + n + 2)$.

Proof. Lemma 2.4. □

Theorem 3.9. *Let* $m, n \in \mathbb{N}_0$ *and* $p = 5$. *Then the group*

$$G_p = \langle x, y | x^p = y^p,\ xyx^my^n = 1\rangle,$$

is a finite group except in the case that $m \equiv 1 \pmod 5$ *and* $n \equiv 1 \pmod 5$.

Proof. The result follows from Lemmas 2.1, 2.2, 2.5, 3.6, 3.7 and 3.8. □

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