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$C^{\#}$-IDEALS OF LIE ALGEBRAS

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Abstract. Let L be a finite dimensional Lie algebra. A subalgebra H of L is called a $c^{\#}$-ideal of L, if there is an ideal K of L with $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L. This is analogous to the concept of a $c^{\#}$-normal subgroup of a finite group. Now, we consider the influence of this concept on the structure of finite dimentional Lie algebras.

1. Introduction

In this paper, L will denote a finite dimensional Lie algebra over a field F. We denote the largest ideal of L contained in all the maximal subalgebras of L, the Frattini ideal of L, by $\phi(L)$. For a subalgebra H of L, the core of H with respect to L, H_L, is the largest ideal of L contained in H. Also vector space direct sums will be denoted by $\dot{+}$. We say the factor algebra A/B is a chief factor of L if B is an ideal of L and A/B is a minimal ideal of L/B. Also, a Lie algebra L is called supersolvable, if there is a chain of ideals $\{0\} \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n = L$ such that $\dim L_i = i$.

In 1996, Wang [7] introduced the concept of c-normal subgroups. This concept has been studied by many mathematicians. Analogously, Towers [4] introduced the notion of a c-ideal of a Lie algebra as follows:

A subalgebra H of L is a c-ideal of L, if there is an ideal K of L such that $L = H + K$ and $H \cap K \leq H_L$. He obtained some properties of c-ideals and used them to give some characterizations of solvable and

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supersolvable Lie algebras. Also, similarly to the case of finite groups, Towers [5] defined the notion of CAP-subalgebras of Lie algebras, as follows:

Let L be a Lie algebra and H be a subalgebra of L and A/B be a chief factor of L. We say that

(i) H covers A/B, if $H + A = H + B$; and

(ii) H avoids A/B, if $H \cap A = H \cap B$.

A subalgebra H of L is called a CAP-subalgebra of L, if H either covers or avoids every chief factor of L. It can be easily seen that each ideal of L is a c-ideal as well as a CAP-subalgebra of L.

In this paper, we define the notion of a $c^{\#}$-ideal of a Lie algebra and give some conditions for solvability and supersolvability of a Lie algebra.

Definition. A subalgebra H of L is called a $c^{\#}$-ideal of L, if there is an ideal K of L with $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L.

This is analogous to the concept of $c^{\#}$-normal subgroups of finite groups as introduced by Wang and Wei [6].

Remark. If H is a CAP-subalgebra of L, then we have $L = H + L$ and $H \cap L = H$ is a CAP-subalgebra of L. Therefore H is a $c^{\#}$-ideal of L. Also, if H is a c-ideal of L, then by [2, Lemma 2.3(i)], there is an ideal K of L with $L = H + K$ and $H \cap K = H_L$ and H_L is a CAP-subalgebra of L, thanks to [5, Lemma 2.1(iii)]. Therefore, CAP-subalgebras and c-ideals of L are $c^{\#}$-ideals of L

Now, in the following example, we show that a $c^{\#}$-ideal of L is not necessarily a c-ideal of L.

Example. Let $L = \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ be a complex Lie algebra with non-zero multiplications $[x, y] = y$ and $[x, z] = 2z$. If we put $H = \mathbb{F}(y + z)$, then H is not a c-ideal of L, but since H either covers or avoids each chief factor of L, so H is a CAP-subalgebra of L and therefore it is a $c^{\#}$-ideal of L.

2. PRELIMINARY RESULTS

This section is devoted to some basic results which are needed in our investigation. In the following lemma, we provide a condition under which in a Lie algebra L, a $c^{\#}$-ideal of L becomes a CAP-subalgebra of L.

Lemma 2.1. *Let L be a Lie algebra and N be an ideal of L. Then*
(i) If $N \leq H$, then H is a $c^{\#}$-ideal of L if and only if H/N is a $c^{\#}$-ideal of L/N.

(*ii*) *If K is a subalgebra of L with $H \leq \phi(K)$ and H is a $c^{\#}$-ideal of L, then H is a CAP-subalgebra of L.*

Proof. (i) We suppose that H is a $c^{\#}$-ideal of L. Then there is an ideal K of L with $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L. So $L/N = H/N + (K+N)/N$ and $H/N \cap (K+N)/N = ((H \cap K)+N)/N$. Now, since $(H \cap K) + N$ is a CAP-subalgebra of L, by [5, Lemma 2.5], so $((H \cap K)+N)/N$ is a CAP-subalgerbra of L/N, thanks to [5, Lemma 2.1(v)]. Therefore H/N is a $c^{\#}$-ideal of L/N.
Conversely, if H/N is a $c^{\#}$-ideal of L/N, then there is an ideal K/N of L/N with $L/N = H/N + K/N = (H + K)/N$ and $H/N \cap K/N = (H \cap K)/N$ is a CAP-subalgebra of L/N. Therefore $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L, by [5, Lemma 2.1(v)].
(ii) Since H is a $c^{\#}$-ideal of L, there exists an ideal N of L such that $L = H + N$ and $H \cap N$ is a CAP-subalgebra of L. Also, $K = H + (K \cap N)$. Now, by using [3, Lemma 2.1], we conclude that $K = K \cap N$ and so $H \subseteq K \subseteq N$. Hence $L = N$ and $H = H \cap N$ is a CAP-subalgebra of L. □

In the following example, we show that the relation 'to be a $c^{\#}$-ideal' is not transitive.

Example 2.2. Let L be a real Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and with non-zero multiplications $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$ and $[e_2, e_4] = e_1$. (See Example 1.1 of [5])
If we put $H = \mathbb{R}e_1 + \mathbb{R}e_3$ and $K = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$, then K is an ideal of L and so K is a $c^{\#}$-ideal of L. Also, we can easily show that H is a $c^{\#}$-ideal of K. But H is not a $c^{\#}$-ideal of L, because for every non-zero ideal A of L that $L = H + A$, we have $H \cap A = \mathbb{R}e_1$ or $H \cap A = H$. But neither $\mathbb{R}e_1$, nor H is a CAP-subalgebra of L, thanks to Example 1.1 of [5].
A non-zero Lie algebra L is called $c^{\#}$-simple, if for each $c^{\#}$-ideal H of L, either $H = 0$ or $H = L$.

Lemma 2.3. *A Lie algebra L is $c^{\#}$-simple if and only if L is a simple Lie algebra.*

Proof. Suppose that L is $c^{\#}$-simple and is non-simple. Then there is a non-zero proper ideal N of L. But N is a $c^{\#}$-ideal of L, so we have $N = L$ or $N = 0$, a contradiction.
Conversely, we suppose that L is not $c^{\#}$-simple and H is a non-zero proper subalgebra of L that is $c^{\#}$-ideal of L. Then there is an ideal K of L such that $L = H + K$ and $H \cap K$ is a CAP-subalgebra of L. Since L is simple, so either $K = L$ or $K = 0$. If $K = 0$, then $H = L$ that is

contradiction. But if $K = L$, then $H \cap K = H$ and so $H + L = H + 0$ or $H \cap L = H \cap 0$, that is a contradiction again. □

Lemma 2.4. *Let L be a Lie algebra and N be a minimal ideal of L and M be a maximal subalgebra of N. If M is a $c^{\#}$-ideal of L, then $dimN = 1$.*

Proof. Since M is a $c^{\#}$-ideal of L, there is an ideal K of L such that $L = M + K$ and $M \cap K$ is a CAP-subalgebra of L. Also $N = M + (N \cap K)$ and $N \cap K$ is an ideal of L. Hence $N \cap K = 0$ or $N \cap K = N$. Because the former case is impossible, we have $N \cap K = N$. In this case, $M = M \cap K$ is a CAP-subalgebra of L and so covers or avoids $N/\{0\}$. But M can not cover N. Therefore $M \cap N = M \cap 0$ which concludes that $dimN = 1$. □

Also, we will use the following lemma where proved in [4].

Lemma 2.5. [4, Lemma 4.1] *Let L be a Lie algebra over any field F, let N be an ideal of L, and let U/N be a maximal nilpotent subalgebra of L/N. Then $U = A + N$, where A is a maximal nilpotent subalgebra of L.*

3. Main results

In this section, we will first give a condition to imply Lie algebras to be solvable.

Theorem 3.1. *Let L be a Lie algebra over a field of characteristic zero. Then L is solvable if and only if every maximal subalgebra of L is a $c^{\#}$-ideal of L.*

Proof. First, we suppose that L is a non-solvable Lie algebra of the smallest dimension satisfying the hypothesis. We can easily show that L is non-simple. Now, if N is a minimal ideal of L and M/N is a maximal subalgebra of L/N, then M is a maximal subalgebra of L and it is a $c^{\#}$-ideal of L, by the assumption. By using Lemma 2.1(i), we conclude that M/N is a $c^{\#}$-ideal of L/N and so L/N is solvable. Since the class of all solvable Lie algebras is a saturated formation, we can assume that N is a unique minimal ideal of L. If $N \leq \phi(L)$, then L is solvable. But if $N \nleq \phi(L)$, then there is a maximal subalgebra M of L such that $N \nleq M$ and $L = M + N$. Also, M is a $c^{\#}$-ideal of L and there is an ideal K of L such that $L = M + K$ and $M \cap K$ is a CAP-subalgebra of L. Therefore $M \cap K$ covers or avoids $N/\{0\}$. Hence either $(M \cap K) + N = M \cap K$ and so $N \subseteq M \cap K \subseteq M$, a contradiction, or $M \cap K \cap N = M \cap K \cap 0$. Since $N \subseteq K$, $M \cap N = 0$. It follows that $L = M \dotplus N$ and so M is a solvable maximal subalgebra

that is a c-ideal of L. Therefore L is solvable, by [4, Theorem 3.2].
Conversely, If L is solvable, then it follows from [4, Theorem 3.1], all maximal subalgebras of L are c-ideals of L and so are $c^{\#}$-ideals of L. □

Theorem 3.2. *Let L be a Lie algebra over a field of characteristic zero. Then L is solvable if and only if L has a solvable maximal subalgebra which is $c^{\#}$-ideal of L.*

Proof. Let L be a minimal counterexample and let M be a solvable maximal subalgebra of L which is a $c^{\#}$-ideal of L. clearly, $M_L \leq R(L)$. Now, if $R(L) \nleq M$, then $L = R(L) + M$ and so $L/R(L)$ is solvable, that is contradiction.
If $R(L) \leq M$, then $M/R(L)$ is a $c^{\#}$-ideal of $L/R(L)$, by Lemma 2.1(i). Therefore $L/R(L)$ satisfies the hypothesis of this theorem and so $L/R(L)$ is solvable, a contradiction.
The converse follows from the previous theorem. □

Proposition 3.3. *Let L be a Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of L are $c^{\#}$-ideals of L. If N is a minimal ideal of L, then all maximal subalgebras of each maximal nilpotent subalgebra of L/N are $c^{\#}$-ideals of L/N.*

Proof. We suppose that U/N is a maximal nilpotent subalgebra of L/N. Then $U = A + N$, where A is a maximal nilpotent subalgebra of L, by Lemma 2.5. If B/N is a maximal subalgebra of U/N, then $B = B \cap (A+N) = (B \cap A) + N = D + N$, where D is a maximal subalgebra of A and $B \cap A \leq D$. Since D is a $c^{\#}$-ideal of L, there exists an ideal K of L with $L = D + K$ and $D \cap K$ is a CAP-subalgebra of L. Therefore $D \cap K$ covers or avoids $N/\{0\}$. If $D \cap K + N = D \cap K$, then $N \subseteq D \cap K \subseteq D$ and so $B = D$. It follows from Lemma 2.1(i) that B/N is a $c^{\#}$-ideal of L/N and so the result holds. If $D \cap K \cap N = 0$, then we consider two cases:
1. $N \leq K$: In this case, $L/N = (D + N)/N + K/N = B/N + K/N$ and $(D + N)/N \cap K/N = ((D \cap K) + N)/N$. Since $D \cap K$ is a CAP-subalgebra of L and N is an ideal of L, then by [5, Lemma 2.5], $(D \cap K) + N$ is a CAP-subalgebra of L and so $((D \cap K) + N)/N$ is a CAP-subalgebra of L/N, thanks to [5, Lemma 2.1]. Thus B/N is a $c^{\#}$-ideal of L/N.
2. $N \nleq K$: In this case, $N \cap K = 0$ and $(N + K)/K$ is a minimal ideal of L/K and so $(N + K)/K \subseteq Z(L/K)$. This concludes that $[N + K, L] \subseteq K$ and so $[N, L] \subseteq N \cap K = 0$ and $N \subseteq Z(L)$. Consequently, $U = A + N$ is a nilpotent subalgebra of L and so we must have $A = A + N$. Therefore $N \leq A$ and so $N \leq B \cap A$. Hence $N \leq D$ and therefore B/N is a $c^{\#}$-ideal of L/N. □

Finally, we obtain a condition implying a Lie algebra L to be supersolvable.

Theorem 3.4. *Let L be a solvable Lie algebra, in which all maximal subalgebras of each maximal nilpotent subalgebra of L are $c^{\#}$-ideals of L. Then L is supersolvable.*

Proof. Let L be a minimal counterexample and N be a minimal ideal of L. Then by the previous proposition, L/N satisfies the hypothesis of this theorem and so L/N is supersolvable. It is enough to show that $dimN = 1$. If there is another ideal N' of L, then $N \cong (N + N')/N' \leq L/N'$ and so $dimN = 1$ and L is supersolvable, a contradiction.
Therefore, we suppose that N is a unique minimal ideal of L. If $N \leq \phi(L)$, then $L/\phi(L)$ is supersolvable and so L is supersolvable by [1, Theorem 7], a contradiction. If $N \nleq \phi(L)$, then there is a maximal subalgebra of L such that $L = N \dotplus M$. Now, if C is a maximal nilpotent subalgebra of L with $N \leq C$, then we consider two cases:
1. $C = N$: In this case, N is a maximal nilpotent subalgebra of L and so by the assumption, every maximal subalgebra of N is a $c^{\#}$-ideal of L. Hence $dimN = 1$, thanks to Lemma 2.4.
2. $N < C$: In this case, we have $C = N + (C \cap M)$. Now, let B be a maximal subalgebra of C that contains $C \cap M$. Then B is a $c^{\#}$-ideal of L and so there is an ideal K of L such that $L = B + K$ and $B \cap K$ is a CAP-subalgebra of L. Therefore $B \cap K$ covers or avoids $N/\{0\}$. If $(B \cap K) + N = B \cap K$, then $N \leq B \cap K \leq B$ and so $C \leq B$, a contradiction.
If $B \cap K \cap N = 0$, then $B \cap N = 0$. It follows that $C = B \dotplus N$. Thus $C/B \cong N$ and consequently $dimN = 1$ and therefore L is supersolvable, a contradiction. □

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$c^{\#}$- ایدآل‌های جبرهای لی

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فرض کنیم L یک جبرلی متناهی بعد باشد. زیرجبر H از L را یک $c^{\#}$-ایدآل از L می گوییم، هرگاه ایدآل K از L موجود باشد به طوری که $L = H + K$ و $H \cap K$ یک CAP-زیرجبر از L باشد. این مفهوم مشابه با مفهوم یک زیرگروه $c^{\#}$-نرمال از یک گروه متناهی است. اکنون ما تأثیر این مفهوم را روی ساختار جبرهای لی متناهی بعد مورد بررسی قرار می دهیم.

کلمات کلیدی: $c^{\#}$-ایدآل، جبرلی، CAP-زیرجبر، حل پذیر، ابرحل پذیر