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FUZZY MEDIAL FILTERS OF PSEUDO BE-ALGEBRAS

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Abstract. In this paper, the notion of *fuzzy medial filters* of a *pseudo BE-algebra* is defined, and some of the properties are investigated. We show that the set of all fuzzy medial filters of a pseudo BE-algebra is a *complete lattice*. Moreover, we state that in *commutative* pseudo BE-algebras fuzzy filters and fuzzy medial filters coincide. Finally, the notion of a *fuzzy implicative filter* is introduced and proved that every fuzzy implicative filter is a fuzzy medial filter, but the converse is not valid in general.

1. Introduction

Some recent researchers led to generalizations of some types of algebraic structures by pseudo structures. G. Georgescu et al. ([9]), and independently J. Rachunek ([14]), introduced pseudo MV-algebras which are a non-commutative generalization of MV-algebras. After pseudo MV-algebras, the pseudo BL-algebras ([7]) and the pseudo BCK-algebras as an extended notion of BCK-algebras by G. Georgescu et al. ([10]), were introduced and studied. Y.B. Jun et al. introduced the concepts of pseudo-atoms, pseudo BCI-ideals and pseudo BCI-homomorphisms in pseudo BCI-algebras and characterizations of a pseudo BCI-ideal, and provide conditions for a subset to be a pseudo BCI-ideal ([11]). Y.H. Kim et al. ([13]), discussed on minimal elements in pseudo BCI-algebras.

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The notion of a BE-algebra was introduced by H.S. Kim et al. ([12]). A. Borumand Saeid et al. introduced some types of filters in BE-algebras ([1]). Fuzzy subalgebras of BE-algebras were investigated in [15]. Since developing algebraic models for non-commutative multiple-valued logics is a central topic in the study of fuzzy systems, R.A. Borzooei et al. generalized the notion of BE-algebras and introduced the notion of pseudo BE-algebras ([2]). A. Rezaei et al. introduced the notion of distributive pseudo BE-algebra and normal pseudo filters and proved some basic properties ([3]). L.C. Ciungu introduced the notion of commutative pseudo BE-algebras and proved that the class of commutative pseudo BE-algebras is term equivalent to the class of commutative pseudo BCK-algebras ([4]). Also, she defined commutative deductive systems and showed that a pseudo-BCK algebra X is commutative if and only if all the deductive systems of X are commutative ([5, 6]).

Fuzzy ideals of pseudo BCK-algebras were investigated in [8]. Also, A. Walendziak et al. consider the fuzzy ideal theory in pseudo BCH-algebras and provided conditions for a fuzzy set to be a fuzzy ideal ([18]). Recently, A. Rezaei et al. developed the fuzzy filter theory of pseudo-BE algebras. They obtained some characterizations of Noetherian pseudo-BE algebras by fuzzy filters and introduced the notion of the fuzzy commutative filter and investigated some of its properties ([17]).

In this paper, we introduce the notion of a fuzzy medial filter of a pseudo BE-algebra. Also, we show that the set of all fuzzy medial filters of a pseudo BE-algebra is a complete lattice. Several conditions to every fuzzy filter could be a fuzzy medial filter are given. Also, the concept of a fuzzy implicative filter is defined and showed that every fuzzy implicative filter is a fuzzy medial filter.

2. Preliminaries

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [12] An algebra $(X;\to,1)$ of type $(2,0)$ is called a *BE-algebra*, if it satisfies the following axioms:
(BE$_1$) $x\to x=1$,
(BE$_2$) $x\to 1=1$,
(BE$_3$) $1\to x=x$,
(BE$_4$) $x\to(y\to z)=y\to(x\to z)$, for all $x,y,z\in X$.

Definition 2.2. ([2]) An algebra $(X;\to,\rightsquigarrow,1)$ of type $(2,2,0)$ is called a *pseudo BE-algebra,* if it satisfies the following axioms:

(psBE$_1$) $x \to x = x \rightsquigarrow x = 1$,
(psBE$_2$) $x \to 1 = x \rightsquigarrow 1 = 1$,
(psBE$_3$) $1 \to x = 1 \rightsquigarrow x = x$,
(psBE$_4$) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(psBE$_5$) $x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$, for all $x, y, z \in X$.

In a pseudo BE-algebra, one can introduce a binary relation $\leq$ by:

$$x \leq y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1, \text{ for all x, y} \in \text{X}.$$

From now on, we will refer to $(X; \to, \rightsquigarrow, 1)$ by its universe X, unless otherwise is stated.

Proposition 2.3. ([2, 3]) In a pseudo BE-algebra X, the following statements hold:

(p$_1$) $x \to (y \rightsquigarrow x) = 1$, $x \rightsquigarrow (y \to x) = 1$,
(p$_2$) $x \rightsquigarrow (y \rightsquigarrow x) = 1$, $x \to (y \to x) = 1$,
(p$_3$) $x \rightsquigarrow [(x \rightsquigarrow y) \to y] = 1$, $x \to [(x \to y) \rightsquigarrow y] = 1$,
(p$_4$) $x \to [(x \rightsquigarrow y) \to y] = 1$, $x \rightsquigarrow [(x \to y) \rightsquigarrow y] = 1$,
(p$_5$) if $x \leq y \to z$, then $y \leq x \rightsquigarrow z$,
(p$_6$) if $x \leq y \rightsquigarrow z$, then $y \leq x \to z$,
(p$_7$) $1 \leq x$ implies $x = 1$,
(p$_8$) if $x \leq y$, then $x \leq z \to y$ and $x \leq z \rightsquigarrow y$, for all $x, y, z \in X$.

Definition 2.4. (Revisited [3, Definition 4]) A pseudo BE-algebra X is said to be $(\to, \rightsquigarrow)$-distributive (resp. $(\rightsquigarrow, \to)$-distributive) if it satisfies (dis1) (resp. (dis2)).

(dis$_1$) $x \to (y \rightsquigarrow z) = (x \to y) \rightsquigarrow (x \to z)$,
(dis$_2$) $x \rightsquigarrow (y \to z) = (x \rightsquigarrow y) \to (x \rightsquigarrow z)$, for all $x, y, z \in X$.

Note that if $(X; \to, \rightsquigarrow, 1)$ is a pseudo BE-algebra, then $(X; \rightsquigarrow, \to, 1)$ is a pseudo BE-algebra, too. By [3, Theorem 2], if X satisfies (dis$_1$) and (dis$_2$), then $\to = \rightsquigarrow$. The following we bring another proof of [3, Theorem 2]. Take $x := y$ in (dis$_1$) and (dis$_2$) and applying (psBE$_3$), we get
$x \to (x \rightsquigarrow z) = (x \to x) \rightsquigarrow (x \to z) = 1 \rightsquigarrow (x \to z) = x \to z$ and
$x \rightsquigarrow (x \to z) = (x \rightsquigarrow x) \to (x \rightsquigarrow z) = 1 \to (x \rightsquigarrow z) = x \rightsquigarrow z$.
Now, using (psBE$_4$), we have
$x \to z = x \to (x \rightsquigarrow z) = x \rightsquigarrow (x \to z) = x \rightsquigarrow z$, for all $x, z \in X$. Consequently, $\to = \rightsquigarrow$.

Also, note that if $x \to (z \rightsquigarrow y) = x \rightsquigarrow (z \to y)$, for all $x, y, z \in X$, then $\to = \rightsquigarrow$, since if $z := 1$ and using (psBE$_3$), we get

$$x \to y = x \to (1 \rightsquigarrow y) = x \rightsquigarrow (1 \to y) = x \rightsquigarrow y.$$

Theorem 2.5. ([3]) Let X be a $(\to, \rightsquigarrow)$-distributive pseudo BE-algebra. Then

(i) if $x \leq y$, then $z \to x \leq z \to y$, $z \to x \leq z \rightsquigarrow y$ and
$z \rightsquigarrow x \leq z \to y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(ii) $y \to z \leq (x \to y) \to (x \to z)$ and $y \to z \leq (x \to y) \rightsquigarrow (x \to z)$,
(iii) $y \rightsquigarrow z \leq (x \to y) \to (x \to z)$ and $y \rightsquigarrow z \leq (x \to y) \rightsquigarrow (x \to z)$,

for all $x, y, z \in X$.

Definition 2.6. ([2]) Let X be a pseudo-BE algebra. A subset F of X is called a filter of X if for all $x, y \in X$:
(F_1) $1 \in F$,
(F_2) if $x \to y \in F$ and $x \in F$, then $y \in F$.

Proposition 2.7. ([2]) Let X be a pseudo-BE algebra and F be a subset of X satisfy (F_1). Then F is a filter of X if and only if for all $x, y \in X$,
(F_3) if $x \rightsquigarrow y \in F$ and $x \in F$, then $y \in F$.

We will denote by $\mathsf{F}(\mathrm{X})$ the set of all filters of X.
Obviously, $\{1\}, X \in \mathsf{F}(\mathrm{X})$.

Definition 2.8. ([17]) A fuzzy set $\overline{\mu}$ of X is called a *fuzzy filter*, if it satisfies the following conditions:
(FF_1) $\overline{\mu}(1) \geq \overline{\mu}(x)$,
(FF_2) $\overline{\mu}(y) \geq \min\{\overline{\mu}(x), \overline{\mu}(x \to y)\}$, for all $x, y \in X$.

Let $\mathsf{FF}(\mathrm{X})$ be the set of all fuzzy filters of a pseudo BE-algebra X.

Proposition 2.9. ([17]) A fuzzy set $\overline{\mu}$ in X is a *fuzzy filter* of X if and only if $\overline{\mu}$ verifies (FF_1) and for all $x, y \in X$,
(FF_3) $\overline{\mu}(y) \geqslant \min\{\overline{\mu}(x), \overline{\mu}(x \rightsquigarrow y)\}$.

Definition 2.10. ([4]) A pseudo-BE algebra X is said to be *commutative,* if it satisfies the following conditions:
(C_1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$,
(C_2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$, for all $x, y \in X$.

Proposition 2.11. ([4]) Any commutative pseudo-BE algebra is a pseudo-BCK algebra, therefore commutative pseudo-BE algebras coincide with commutative pseudo-BCK algebras.

Definition 2.12. ([5]) A *filter* F is called *commutative,* if it satisfies the following conditions:
(CF_1) $y \to x \in F$ implies $[(x \to y) \rightsquigarrow y] \to x \in F$,
(CF_2) $y \rightsquigarrow x \in F$ implies $[(x \rightsquigarrow y) \to y] \rightsquigarrow x \in F$, for all $x, y \in X$.

Definition 2.13. ([17]) A fuzzy filter $\overline{\mu}$ is called *fuzzy commutative filter,* if it satisfies the following conditions:
(FCF_1) $\overline{\mu}[((x \to y) \rightsquigarrow y) \to x] \geq \overline{\mu}(y \to x)$,
(FCF_2) $\overline{\mu}[((x \rightsquigarrow y) \to y) \rightsquigarrow x] \geq \overline{\mu}(y \rightsquigarrow x)$, for all $x, y \in X$.

Let $\mathsf{FCF}(X)$ be the set of all fuzzy commutative filters of X.

Definition 2.14. ([16]) A non-empty subset F of X is called a *medial filter,* if it satisfies (F_1) and the following condition:
(MF) $x \to z \in F$ and $z \to y \in F$ imply $x \to y \in F$, for all $x, y, z \in X$.

3. A new fuzzy filter on pseudo BE-algebras

This section aims is to extend the notion of medial filters in BE-algebras ([16]), to the fuzzy medial filters in pseudo BE-algebras, and give a number of it's useful properties. In the following theorems, a necessary and sufficient condition is derived for every fuzzy set to be a fuzzy filter.

Theorem 3.1. *A fuzzy set $\overline{\mu}$ is a fuzzy filter if and only if it satisfies the following conditions:*

(i) $\overline{\mu}(1) \geq \overline{\mu}(x)$,
(ii) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(z), \overline{\mu}[x \to (z \rightsquigarrow y)]\}$, *for all $x, y, z \in X$.*

Proof. Assume that $\overline{\mu}$ is a fuzzy filter of X and $x, y, z \in X$. Applying $(psBE_4)$ and (FF_3), we get

$$\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(z), \overline{\mu}[z \rightsquigarrow (x \to y)]\} = \min\{\overline{\mu}(z), \overline{\mu}[x \to (z \rightsquigarrow y)]\}.$$

Conversely, let $\overline{\mu}$ satisfy (i), (ii) and $x, y, z \in X$. Take $x := 1$ and using $(psBE_3)$, we have

$$\overline{\mu}(1 \to y) = \overline{\mu}(y) \geq \min\{\overline{\mu}(z), \overline{\mu}[1 \to (z \rightsquigarrow y)]\} = \min\{\overline{\mu}(z), \overline{\mu}(z \rightsquigarrow y)\}.$$

□

Theorem 3.2. *A fuzzy set $\overline{\mu}$ is a fuzzy filter if and only if it satisfies the following conditions:*

(i) $\overline{\mu}(1) \geq \overline{\mu}(x)$,
(ii) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(z), \overline{\mu}[x \rightsquigarrow (z \to y)]\}$, *for all $x, y, z \in X$.*

Proof. Similar to the proof of Theorem 3.1. □

Theorem 3.3. *A fuzzy set $\overline{\mu}$ is a fuzzy filter if and only if it satisfies the following conditions:*

(i) $\overline{\mu}(x \to y) \geq \overline{\mu}(y)$,
(ii) $\overline{\mu}[(x \to (y \rightsquigarrow z)) \to z] \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$,

Proof. Assume that $\overline{\mu}$ is a fuzzy filter of X and $x, y \in X$. Applying (p_2) and (FF_2) we deduced that

$$\overline{\mu}(x \to y) \geq \min\{\overline{\mu}[y \to (x \to y)], \overline{\mu}(y)\} = \min\{\overline{\mu}(1), \overline{\mu}(y)\} = \overline{\mu}(y).$$

Using Theorem 3.1 and applying (p_1), we get

$$\begin{aligned}\overline{\mu}[(x \to (y \rightsquigarrow z)) \to z] &\geq \min\{\overline{\mu}[(x \to (y \rightsquigarrow z)) \to (y \rightsquigarrow z)], \overline{\mu}(y)\}\\ &\geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}.\end{aligned}$$

Conversely, assume that $\overline{\mu}$ satisfies (i), (ii), (iii) and $x, y \in X$.
If $x := y$ in (i), then we have $\overline{\mu}(x \to x) = \overline{\mu}(1) \geq \overline{\mu}(x)$.
For (FF_2), using (iii), we have

$$\begin{aligned}\overline{\mu}(y) &= \overline{\mu}(1 \rightsquigarrow y)\\ &= \overline{\mu}[((x \to y) \rightsquigarrow (x \to y)) \rightsquigarrow y]\\ &\geq \min\{\overline{\mu}(x \to y), \overline{\mu}(x)\}.\end{aligned}$$

This means that (FF_2) holds. □

Theorem 3.4. *A fuzzy set $\overline{\mu}$ is a fuzzy filter if and only if it satisfies the following conditions:*

(i) $\overline{\mu}(x \rightsquigarrow y) \geq \overline{\mu}(y)$,
(ii) $\overline{\mu}[(x \to (y \rightsquigarrow z)) \to z] \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$, *for all* $x, y, z \in X$.

Proof. Similar to the proof of Theorem 3.3. □

Proposition 3.5. *A fuzzy filter $\overline{\mu}$ of a pseudo BE-algebra X is order preserving.*

Proof. Assume that $x \leq y$. Then $x \rightsquigarrow y = 1$. Applying Theorem 3.1(ii) and ($psBE_2$), we get

$$\begin{aligned}\overline{\mu}(y) = \overline{\mu}(1 \to y) &\geq \min\{\overline{\mu}[1 \to (x \rightsquigarrow y)], \overline{\mu}(x)\}\\ &= \min\{\overline{\mu}(1 \to 1), \overline{\mu}(x)\} = \overline{\mu}(x).\end{aligned}$$

□

Now, we define a new fuzzy filter, as *fuzzy medial filter* on X.

Definition 3.6. A fuzzy set $\overline{\mu}$ in X is called *fuzzy medial filter,* if it satisfies (FF_1) and the following conditions:
(FMF_1) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}$,
(FMF_2) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$, for all $x, y, z \in X$.

Let $\mathsf{FMF}(X)$ be the set of all fuzzy medial filters of X.

Example 3.7. Consider the pseudo BE-algebra $(X; \to, \rightsquigarrow, 1)$ with the following table:

Define a fuzzy set $\overline{\mu} : X \to [0, 1]$ by $\overline{\mu}(1) = 0.8$, $\overline{\mu}(a) = 0.6$ and $\overline{\mu}(b) = \overline{\mu}(c) = 0.2$. Then $\overline{\mu}$ is a fuzzy medial filter of X.

Theorem 3.8. *A fuzzy filter $\overline{\mu}$ in X is a fuzzy medial filter of X if and only if its nonempty level subset $U(\overline{\mu}, \alpha) = \{x \in X : \overline{\mu}(x) \geq \alpha\}$ is a medial filter of X, for all $\alpha \in [0, 1]$.*

TABLE 1. The Cayley table of the operation $\to$.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

TABLE 2. The Cayley table of the operation $\rightsquigarrow$.

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	a	b	1

Proof. Similar to the proof of [17, Theorem 4.3]. □

Corollary 3.9. *A nonempty subset F is a medial filter of X if and only if χ_F is a fuzzy medial filter of X.*

Proof. The proof is straightforward. □

Proposition 3.10. *Let $\overline{\mu} \in \mathsf{FMF}(X)$. Then*

$$\chi_{\overline{\mu}} := \{x \in X \,|\, \overline{\mu}(x) = \overline{\mu}(1)\}$$

is a filter of X.

Proof. Assume that $x, y \in X$ and x, $x \to y \in \chi_{\overline{\mu}}$. Then $\overline{\mu}(x) = \overline{\mu}(x \to y) = \overline{\mu}(1)$. Since $\overline{\mu}$ is a fuzzy medial filter, we have

$$\begin{aligned} \overline{\mu}(1 \to y) = \overline{\mu}(y) &\geq \min\{\overline{\mu}(1 \to x), \overline{\mu}(x \to y)\} \\ &= \min\{\overline{\mu}(x), \overline{\mu}(x \to y)\} \\ &= \min\{\overline{\mu}(1), \overline{\mu}(1)\} \\ &= \overline{\mu}(1). \end{aligned}$$

Therefore, $\overline{\mu}(y) = \overline{\mu}(1)$, and so $y \in \chi_{\overline{\mu}}$. □

Let $\overline{\mu}_i \in \mathsf{FMF}(X)$ for $i \in I$. The meet $\bigwedge_{i \in I} \overline{\mu}_i$ of fuzzy filters $\overline{\mu}_i$ is defined as follows:

$$(\bigwedge_{i \in I} \overline{\mu}_i)(x) = \bigwedge\{\overline{\mu}_i(x) : i \in I\}.$$

Proposition 3.11. *Let $\overline{\mu}_i \in \mathsf{FMF}(X)$ for $i \in I$. Then $\bigwedge_{i \in I} \overline{\mu}_i \in \mathsf{FMF}(X)$.*

Proof. Let $\overline{\mu} := \bigwedge_{i\in I} \overline{\mu}_i$. Then, by ($\mathrm{FF}_1$),

$$\overline{\mu}(1) = \bigwedge\{\overline{\mu}_i(1) : i \in I\} \geqslant \bigwedge\{\overline{\mu}_i(x) : i \in I\} = \overline{\mu}(x)$$

for all $x \in X$. Let $x, y \in X$. Since $\overline{\mu}_i \in \mathsf{FMF}(X)$, we have

$$\overline{\mu}_i(x \to y) \geqslant \min\{\overline{\mu}_i(x \to z), \overline{\mu}_i(z \to y)\}.$$

Hence, by (FMF_2),
$\bigwedge\{\overline{\mu}_i(x \to y) : i \in I\} \geqslant \bigwedge\{\min\{\overline{\mu}_i(x \to z), \overline{\mu}_i(z \to y) : i \in I\}\} =$
$\min\{\bigwedge\{\overline{\mu}_i(x \to z) : i \in I\}, \bigwedge\{\overline{\mu}_i(z \to y) : i \in I\}\}$.
Consequently, $\overline{\mu}(x \to y) \geqslant \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}$.
Similarly, $\overline{\mu}(x \rightsquigarrow y) \geqslant \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$.
Therefore, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Let $\overline{\nu}$ be a fuzzy set in X. A fuzzy medial filter $\overline{\mu}$ of X is said to be *generated by* $\overline{\nu}$ if $\overline{\nu} \leqslant \overline{\mu}$ and for any fuzzy medial filter $\overline{\rho}$ of X, $\overline{\nu} \leqslant \overline{\rho}$ implies $\overline{\mu} \leqslant \overline{\rho}$. The fuzzy medial filter generated by $\overline{\nu}$ will be denoted by $[\overline{\nu})$. The fuzzy medial filter $[\overline{\nu})$ we can define equivalently as follows:

$$[\overline{\nu}) = \bigwedge\{\overline{\rho} : \overline{\rho} \in \mathsf{FMF}(X) \text{ and } \overline{\nu} \leq \overline{\rho}\}.$$

Let $\overline{\mu}, \overline{\nu}$ be two fuzzy medial filters of X. Denote the join of $\overline{\mu}$ and $\overline{\nu}$ by $\overline{\mu} \vee \overline{\nu}$, that is, $\overline{\mu} \vee \overline{\nu} = [\overline{\rho})$, where $\overline{\rho}$ is the fuzzy set of X defined by $\overline{\rho}(x) = \overline{\mu}(x) \vee \overline{\nu}(x)$.

Theorem 3.12. *Let X be a pseudo BE-algebra. Then* $(\mathsf{FMF}(\mathrm{X}); \wedge, \vee)$ *is a complete lattice.*

Proof. The proof is straightforward. □

Proposition 3.13. *Let $\overline{\mu}$ be a fuzzy set on X. Let $\overline{\mu}$ satisfy one of the following conditions:*

(i) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x \to z), \overline{\mu}(y \to z)\}$,
(ii) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(y \rightsquigarrow z)\}$,
(iii) $\min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\} \geq \overline{\mu}(x \to y)$,
(iv) $\min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\} \geq \overline{\mu}(x \rightsquigarrow y)$.

Then $\overline{\mu}(x) = \overline{\mu}(1)$.

Proof. (i) Assume that $\overline{\mu}$ is a fuzzy set satisfying (i). Take $z := 1$. Then $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x \to 1), \overline{\mu}(y \to 1)\} = \min\{\overline{\mu}(1), \overline{\mu}(1)\} = \overline{\mu}(1)$.
If $x = y := 1$, then
$\overline{\mu}(1 \to 1) \geq \min\{\overline{\mu}(1 \to z), \overline{\mu}(1 \to z)\} = \min\{\overline{\mu}(z), \overline{\mu}(z)\} = \overline{\mu}(z)$.
Hence $\overline{\mu}(1) \geq \overline{\mu}(z)$ for all $z \in X$, and so $\overline{\mu}(x \to y) = \overline{\mu}(1)$. Now, take $x := 1$, we have $\overline{\mu}(1 \to y) = \overline{\mu}(y) = \overline{\mu}(1)$, for all $y \in X$.

By a similar argument (ii) holds.

(iii) Take $x := y$ and $z := 1$. For all $x \in X$, we have

$$\min\{\overline{\mu}(x \to 1), \overline{\mu}(1 \to x)\} \geq \overline{\mu}(x \to x) = \overline{\mu}(1).$$

Now, using ($psBE_2$) and ($psBE_3$), we get
$\min\{\overline{\mu}(1), \overline{\mu}(x)\} \geq \overline{\mu}(1)$, and so $\overline{\mu}(x) \geq \overline{\mu}(1)$.
On the other hand, if $x := z = 1$, then

$$\min\{\overline{\mu}(1 \to 1), \overline{\mu}(1 \to y)\} \geq \overline{\mu}(1 \to y).$$

Now, using ($psBE_1$) and ($psBE_3$), we get $\min\{\overline{\mu}(1), \overline{\mu}(y)\} \geq \overline{\mu}(y)$, and so $\overline{\mu}(1) \geq \overline{\mu}(y)$, for all $y \in X$. Then $\overline{\mu}(x) = \overline{\mu}(1)$.

(iv) Similarly to (iii). □

Proposition 3.14. *Let $\overline{\mu} \in \mathsf{FF}(X)$. Let $\overline{\mu}$ satisfy one of the following conditions:*

(i) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(z \to x), \overline{\mu}(y \to z)\}$,
(ii) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(z \rightsquigarrow x), \overline{\mu}(y \rightsquigarrow z)\}$.

Then $\overline{\mu}(x) = \overline{\mu}(1)$.

Proof. (i) Take $x := 1$. Using ($psBE_3$), we have

$$\begin{aligned} \overline{\mu}(1 \to y) = \overline{\mu}(y) &\geq \min\{\overline{\mu}(z \to 1), \overline{\mu}(y \to z)\} \\ &= \min\{\overline{\mu}(1), \overline{\mu}(y \to z)\} \\ &= \overline{\mu}(y \to z). \end{aligned}$$

Hence $\overline{\mu}(y) \geq \overline{\mu}(y \to z)$, for all $y, z \in X$. Now, take $z := 1$, we have $\overline{\mu}(y) \geq \overline{\mu}(1)$, and so $\overline{\mu}(y) = \overline{\mu}(1)$, for all $y \in X$.

Similarly, (ii) holds. □

Definition 3.15. A fuzzy set $\overline{\mu}$ is called a fuzzy subalgebra of X if it satisfies the following conditions:
(FS_1) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$,
(FS_2) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$, for all $x, y \in X$.

Let FS(X) be the set of all fuzzy subalgebras of a X.

Proposition 3.16. *Let $\overline{\mu}$ be a fuzzy set on X. Let $\overline{\mu}$ satisfy the following conditions:*

(i) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(z \to x), \overline{\mu}(z \to y)\}$,
(ii) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(z \rightsquigarrow x), \overline{\mu}(z \rightsquigarrow y)\}$.

Then $\overline{\mu}$ is a fuzzy subalgebra of X.

Proof. Assume that $\overline{\mu}$ is a fuzzy set satisfying (i). Take $z := 1$. By ($psBE_3$), we have

$$\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(1 \to x), \overline{\mu}(1 \to y)\} = \min\{\overline{\mu}(x), \overline{\mu}(y)\}.$$

So,

$$\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}.$$

By a similar argument $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$. □

Proposition 3.17. *If $\overline{\mu} \in \mathsf{FMF}(X)$, then $\overline{\mu} \in \mathsf{FF}(X)$.*

Proof. Suppose that $z, y \in X$ and $x := 1$. Using ($psBE_2$) and ($psBE_3$), we have

$$\overline{\mu}(1 \to y) = \overline{\mu}(y) \geq \min\{\overline{\mu}(1 \to z), \overline{\mu}(z \to y)\} = \min\{\overline{\mu}(z), \overline{\mu}(z \to y)\}.$$

□

The following example shows that the converse of Proposition 3.17, may not be true in general.

Example 3.18. Consider the pseudo BE-algebra $(X; \to, \rightsquigarrow, 1)$ with the following table:

Table 3. The Cayley table of the operation $\to$.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

Table 4. The Cayley table of the operation $\rightsquigarrow$.

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	c	1	1

Define a fuzzy set $\overline{\mu} : X \to [0, 1]$ by $\overline{\mu}(1) = 0.78$, $\overline{\mu}(a) = 0.32$, $\overline{\mu}(b) = 0.7$ and $\overline{\mu}(c) = 0.5$. Then $\overline{\mu}$ is a fuzzy filter, but it is not a fuzzy medial filter, since

$$\begin{aligned} \overline{\mu}(b \to a) &= \overline{\mu}(a) \\ &= 0.32 \\ &\ngeq \min\{\overline{\mu}(b \to c), \overline{\mu}(c \to a)\} \\ &= \min\{0.5, 0.7\} \\ &= 0.5. \end{aligned}$$

Theorem 3.19. *Let $\overline{\mu} \in \mathsf{FF}(X)$. Let $\overline{\mu}$ satisfy the following conditions:*

(i) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$,
(ii) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}$.

Then $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. Assume that $\overline{\mu}$ is a fuzzy filter satisfying (i) and (ii). Using (psBE$_1$) and (FF$_1$), we get

$$\begin{aligned}
\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\} \\
&\geq \min\{\min\{\overline{\mu}(x \to z), \overline{\mu}(z \to z)\}, \min\{\overline{\mu}(z \to z), \overline{\mu}(z \to y)\}\} \\
&= \min\{\min\{\overline{\mu}(x \to z), \overline{\mu}(1)\}, \min\{\overline{\mu}(1), \overline{\mu}(z \to y)\}\} \\
&= \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.
\end{aligned}$$

Also, by a similar argument $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$. Therefore, (FMF$_1$) and (FMF$_2$) hold. □

Theorem 3.20. *Let* $\overline{\mu} \in \mathsf{FF}(X)$. *Let* $\overline{\mu}$ *satisfy the following conditions:*

(i) $\overline{\mu}(z) \geq \min\{\overline{\mu}(x \to y), \overline{\mu}[x \rightsquigarrow (y \to z)]\}$,
(ii) $\overline{\mu}(z) \geq \min\{\overline{\mu}(x \rightsquigarrow y), \overline{\mu}[x \to (y \rightsquigarrow z)]\}$.

Then $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. Assume that $\overline{\mu}$ is a fuzzy filter satisfying (i) and (ii). Using (p$_1$), we have $y \to z \leq x \rightsquigarrow (y \to z)$. Hence $\overline{\mu}(y \to z) \leq \overline{\mu}(x \rightsquigarrow (y \to z))$, and so

$$\begin{aligned}
\min\{\overline{\mu}(x \to y), \overline{\mu}(y \to z)\} &\leq \min\{\overline{\mu}(x \to y), \overline{\mu}[x \rightsquigarrow (y \to z)]\} \\
&\leq \overline{\mu}(z).
\end{aligned}$$

On the other hand, since $z \leq x \to z$, we get $\overline{\mu}(z) \leq \overline{\mu}(x \to z)$.
Consequently, $\overline{\mu}(x \to z) \geq \min\{\overline{\mu}(x \to y), \overline{\mu}(y \to z))$.

By a similar argument $\overline{\mu}(x \rightsquigarrow z) \geq \min\{\overline{\mu}(x \rightsquigarrow y), \overline{\mu}(y \to z)\}$. Therefore, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Theorem 3.21. *Let* $\overline{\mu} \in \mathsf{FF}(X)$. *Let* $\overline{\mu}$ *satisfy the following conditions:*
$\overline{\mu}[(x \to y) \rightsquigarrow (x \to z)] \geq \min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]\}$,
$\overline{\mu}[(x \rightsquigarrow y) \to (x \rightsquigarrow z)] \geq \min\{\overline{\mu}(x \rightsquigarrow y), \overline{\mu}[(x \rightsquigarrow y) \to (y \rightsquigarrow z)]\}$.
Then $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. Assume that $\overline{\mu}$ is a fuzzy filter satisfying (i) and (ii). Using (p$_1$), we have
$y \to z \leq (x \to y) \rightsquigarrow (y \to z)$ and
$y \rightsquigarrow z \leq (x \rightsquigarrow y) \to (y \rightsquigarrow z)$.
By Propositions 3.17 and 3.5, we get
$\overline{\mu}(y \to z) \leq \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]$ and
$\overline{\mu}(y \rightsquigarrow z) \leq \overline{\mu}[(x \rightsquigarrow y) \to (y \rightsquigarrow z)]$.

Since $\overline{\mu}$ is a fuzzy filter, we have

$$\begin{aligned}\overline{\mu}(x \to z) &\geq \min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (x \to z)]\} \\ &\geq \min\{\overline{\mu}(x \to y), \min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]\} \\ &= \min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]\} \\ &\geq \min\{\overline{\mu}(x \to y), \overline{\mu}(y \to z)\}.\end{aligned}$$

Similarly, $\overline{\mu}(x \rightsquigarrow z) \geq \min\{\overline{\mu}(x \rightsquigarrow y), \overline{\mu}(y \rightsquigarrow z)\}$. Thus, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Theorem 3.22. *Let X be a $(\to, \rightsquigarrow)$-distributive pseudo BE-algebra and $\overline{\mu} \in \mathsf{FF}(X)$. Then $\overline{\mu} \in \mathsf{FMF}(X)$.*

Proof. Assume that X is a $(\to, \rightsquigarrow)$-distributive pseudo BE-algebra, $\overline{\mu} \in \mathsf{FF}(X)$ and $x, y, z \in X$. Applying Theorem 2.5(ii) and (iii), we get $\overline{\mu}[(x \to z) \rightsquigarrow (x \to y)] \geq \overline{\mu}(z \to y)$ and
$\overline{\mu}[(x \rightsquigarrow z) \to (x \rightsquigarrow y)] \geq \overline{\mu}(z \rightsquigarrow y)$.
Since $\overline{\mu}$ is a fuzzy filter, we have

$$\begin{aligned}\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}[(x \to z) \rightsquigarrow (x \to y)]\} \\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.\end{aligned}$$

Similarly, $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$. Thus, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Commutative pseudo BE-algebras were introduced by L.C. Ciungu in [4], and proved that any commutative pseudo BE-algebra is a pseudo BCK-algebra (see [4, Theorem 3.3]) as follows:
(psBCK_1) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1$,
(psBCK_2) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$, for all $x, y, z \in X$.

Proposition 3.23. *Let X be a commutative pseudo BE-algebra and $\overline{\mu} \in FF(X)$. Then*

(i) $\overline{\mu}[(y \to z) \rightsquigarrow (x \to z)] \geq \overline{\mu}(x \to y)$,
(ii) $\overline{\mu}[(y \rightsquigarrow z) \to (x \rightsquigarrow z)] \geq \overline{\mu}(x \rightsquigarrow y)$,
(iii) $\overline{\mu}[(x \to y) \to (x \to z)] \geq \overline{\mu}(y \to z)$,
(iv) $\overline{\mu}[(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)] \geq \overline{\mu}(y \rightsquigarrow z)$.

Proof. The proofs are straightforward. □

The following example shows that any fuzzy medial filter may not be a fuzzy commutative filter, in general.

Example 3.24. Consider the fuzzy medial filter $\overline{\mu}$ given in Example 3.7. It is not a fuzzy commutative filter, since

$$\begin{aligned}\overline{\mu}[((a \to b) \rightsquigarrow b) \to a] &= \overline{\mu}[(a \rightsquigarrow b) \to a] \\ &= \overline{\mu}(c \to a) \\ &= \overline{\mu}(a) \\ &= 0.6 \\ &\not\geq \overline{\mu}(b \to a) \\ &= \overline{\mu}(1) \\ &= 0.8.\end{aligned}$$

Theorem 3.25. *Let X be a commutative pseudo BE-algebra and $\overline{\mu} \in \mathsf{FF}(X)$. Then $\overline{\mu} \in \mathsf{FMF}(X)$.*

Proof. Assume that $\overline{\mu} \in \mathsf{FF}(X)$. Using ($FF_2$) and Proposition 3.23(iii), we get

$$\begin{aligned}\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}[(x \to z) \to (x \to y)]\} \\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.\end{aligned}$$

Also, by (FF_3) and Proposition 3.23(iv), we deduce that

$$\begin{aligned}\overline{\mu}(x \rightsquigarrow y) &\geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}[(x \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow y)]\} \\ &\geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}.\end{aligned}$$

Therefore, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Corollary 3.26. *If X is a commutative pseudo BE-algebra, then $\mathsf{FF}(X) = \mathsf{FCF}(X) = \mathsf{FMF}(X)$.*

Proof. It follows from Proposition 3.17, Theorem 3.25 and [17, Theorem 4.7]. □

Proposition 3.27. *Let $\overline{\mu}$ be a fuzzy filter of X which satisfies the following conditions:*

(i) $\overline{\mu}[(x \to z) \rightsquigarrow (x \to y)] \geq \overline{\mu}[x \rightsquigarrow (z \to y)]$,
(ii) $\overline{\mu}[(x \rightsquigarrow z) \to (x \rightsquigarrow y)] \geq \overline{\mu}[x \to (z \rightsquigarrow y)]$, *for all $x, y, z \in X$.*

Then $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. Assume that $\overline{\mu} \in \mathsf{FF}(X)$ and $x, y, z \in X$. Applying (p_1) we get $z \to y \leq x \rightsquigarrow (z \to y)$, and so $\overline{\mu}(z \to y) \leq \overline{\mu}[x \rightsquigarrow (z \to y)]$. Thus $\min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\} \leq \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\}$. Now, since $\overline{\mu}$ is fuzzy filter of X and using (i), we get

$$\begin{aligned}\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}[(x \to z) \rightsquigarrow (x \to y)]\} \\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\} \\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.\end{aligned}$$

Thus $\overline{\mu}$ satisfies (FMF$_1$). Similarly, $\overline{\mu}$ also satisfies (FMF$_2$). Consequently, $\overline{\mu} \in \mathsf{FMF}(X)$. □

The following theorem shows that if X is a commutative pseudo BE-algebra, then every fuzzy medial filter $\overline{\mu}$ satisfies (i) and (ii) of Theorem 3.21.

Theorem 3.28. *Let X be a commutative pseudo BE-algebra and $\overline{\mu} \in \mathsf{FMF}(X)$. Then the following conditions hold: for all $x, y, z \in X$,*
(i) $\overline{\mu}[(x \to y) \rightsquigarrow (x \to z)] \geq \min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]\}$,
(ii) $\overline{\mu}[(x \rightsquigarrow y) \to (x \rightsquigarrow z)] \geq \min\{\overline{\mu}(x \rightsquigarrow y), \overline{\mu}[(x \rightsquigarrow y) \to (y \rightsquigarrow z)]\}$.

Proof. Suppose that X is a commutative pseudo BE-algebra and $\overline{\mu} \in \mathsf{FMF}(X)$. Since X is commutative, (psBCK$_1$) and (psBCK$_2$) hold. Thus, $\overline{\mu}[(y \to z) \rightsquigarrow ((x \to y) \rightsquigarrow (x \to z))] = \overline{\mu}(1)$ and $\overline{\mu}[(y \rightsquigarrow z) \to ((x \rightsquigarrow y) \to (x \rightsquigarrow z))] = \overline{\mu}(1)$. Applying (FMF$_2$), we get
$\overline{\mu}[(x \to y) \to ((x \to y) \rightsquigarrow (x \to z))] \geq$
$\min\{\overline{\mu}[(x \to y) \rightsquigarrow (y \to z)], \overline{\mu}[(y \to z) \rightsquigarrow ((x \to y) \rightsquigarrow (x \to z))]\} =$
$\min\{\overline{\mu}[(x \to y) \rightsquigarrow (y \to z)], \overline{\mu}(1)\} =$
$\overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]$.
From this and (FF$_2$), we have
$\overline{\mu}[(x \to y) \rightsquigarrow (x \to z)] \geq$
$\min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \to ((x \to y) \rightsquigarrow (x \to z))] \geq$
$\min\{\overline{\mu}(x \to y), \overline{\mu}[(x \to y) \rightsquigarrow (y \to z)]\}$.
Thus, (i) holds. Similarly, we can get (ii). □

Proposition 3.29. *Let $\overline{\mu}$ be a fuzzy set of X. If*

$$\overline{\mu}(x \to y) = \overline{\mu}(x \rightsquigarrow y) = \min\{\overline{\mu}(x), \overline{\mu}(y)\},$$

for all $x, y \in X$, then $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. For all $x \in X$ using (psBE$_2$), we have

$$\overline{\mu}(1 \to x) = \overline{\mu}(1 \rightsquigarrow x) = \overline{\mu}(x) = \min\{\overline{\mu}(1), \overline{\mu}(x)\}.$$

Hence $\overline{\mu}(x) \leq \overline{\mu}(1)$. Also, we have

$$\begin{aligned} \overline{\mu}(x \to y) &= \min\{\overline{\mu}(x), \overline{\mu}(y)\} \\ &\geq \min\{\overline{\mu}(x), \overline{\mu}(y), \overline{\mu}(z)\} \\ &= \min\{\min\{\overline{\mu}(x), \overline{\mu}(z)\}, \min\{\overline{\mu}(z), \overline{\mu}(y)\}\} \\ &= \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}. \end{aligned}$$

Similarly, $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$. Thus, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Theorem 3.30. *Let $\overline{\mu}$ be a fuzzy set of X, be order-preserving (i. e., if $x \leq y$, then $\overline{\mu}(x) \leq \overline{\mu}(y)$) and let a be a fixed element of X. Define* *a fuzzy set* $\overline{\mu}^a : X \to [0,1]$ *by* $\overline{\mu}^a(x) = \overline{\mu}(a \to x) = \overline{\mu}(a \rightsquigarrow x)$*, for all* $x \in X$*. If* $\overline{\mu}^x$ *is a fuzzy filter, for all* $x \in X$*, then* $\overline{\mu} \in \mathsf{FMF}(X)$*.*

Proof. Assume that $\overline{\mu}$ is a fuzzy set and $x \in X$. Using (psBE$_2$), we have

$$\overline{\mu}^x(1) = \overline{\mu}(x \rightsquigarrow 1) = \overline{\mu}(1).$$

Now, from $x \to y \leq 1$ and assumption, we have

$$\overline{\mu}^x(y) = \overline{\mu}(x \rightsquigarrow y) \leq \overline{\mu}(1) = \overline{\mu}^x(1).$$

Also, let $x, y, z \in X$. Since $z \to y \leq x \rightsquigarrow (z \to y)$ and $\overline{\mu}$ is order-preserving we obtain

$$\overline{\mu}(z \to y) \leq \overline{\mu}[x \rightsquigarrow (z \to y)].$$

Now, since $\overline{\mu}^x$ is a fuzzy filter, we get

$$\begin{aligned} \overline{\mu}(x \to y) = \overline{\mu}^x(y) &\geq \min\{\overline{\mu}^x(z), \overline{\mu}^x(z \to y)\} \\ &= \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\} \\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}. \end{aligned}$$

Similarly, $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$. Thus, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Now, we will investigate several theorems for these fuzzy medial filters of a pseudo BE-algebra.

Theorem 3.31. *Let* $\overline{\mu}, \overline{\nu} \in \mathsf{FMF}(X)$*. Then* $\lambda\overline{\mu} + (1-\lambda)\overline{\nu} \in \mathsf{FMF}(X)$*, for all* $\lambda \in [0,1]$*.*

Proof. Assume that $\overline{\mu}, \overline{\nu} \in \mathsf{FMF}(X)$ and $\lambda \in [0,1]$. Then, for all $x \in X$ we have

$$\begin{aligned} (\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x) &= \lambda\overline{\mu}(x) + (1-\lambda)\overline{\nu}(x) \\ &\leq \lambda\overline{\mu}(1) + (1-\lambda)\overline{\nu}(1) \\ &= (\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(1). \end{aligned}$$

For (FMF$_2$), assume that $x, y, z \in X$. Then
$(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \to y) =$
$\lambda\overline{\mu}(x \to y) + (1-\lambda)\overline{\nu}(x \to y) \geq$
$\lambda \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\} + (1-\lambda)\min\{\overline{\nu}(x \to z), \overline{\nu}(z \to y)\} =$
$\min\{\lambda\overline{\mu}(x \to z), \lambda\overline{\mu}(z \to y)\} + \min\{(1-\lambda)\overline{\nu}(x \to z), (1-\lambda)\overline{\nu}(z \to y)\} =$
$\min\{\lambda\overline{\mu}(x \to z) + (1-\lambda)\overline{\nu}(x \to z), \lambda\overline{\mu}(z \to y) + (1-\lambda)\overline{\nu}(z \to y)\} =$
$\min\{(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \to z), (\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(z \to y)\}$, and so
$(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \to y) \geq$
$\min\{(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \to z), (\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(z \to y)\}$.
By a similar argument we have
$(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \rightsquigarrow y)$

$\geq \min\{(\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(x \rightsquigarrow z), (\lambda\overline{\mu} + (1-\lambda)\overline{\nu})(z \rightsquigarrow y)\}$.
Therefore, $\lambda\overline{\mu} + (1-\lambda)\overline{\nu} \in \mathsf{FMF}(X)$. □

Proposition 3.32. *Let $\overline{\mu} \in \mathsf{FMF}(X)$ and $\alpha \in [0, \overline{\mu}(1)]$. Then $\overline{\mu} \vee \alpha \in \mathsf{FMF}(X)$, where $(\overline{\mu} \vee \alpha)(x) = \overline{\mu}(x) \vee \alpha$, for all $x \in X$.*

Proof. Since $\overline{\mu}$ is a fuzzy medial filter, we get

$$(\overline{\mu} \vee \alpha)(x) = \overline{\mu}(x) \vee \alpha \leq \overline{\mu}(1) \vee \alpha = (\overline{\mu} \vee \alpha)(1).$$

Now, let $x, y, z \in X$. Then

$$\begin{aligned}
(\overline{\mu} \vee \alpha)(x \to y) &= \overline{\mu}(x \to y) \vee \alpha \\
&\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\} \vee \alpha \\
&= \min\{\overline{\mu}(x \to z) \vee \alpha, \overline{\mu}(z \to y) \vee \alpha\} \\
&= \min\{(\overline{\mu} \vee \alpha)(x \to z), (\overline{\mu} \vee \alpha)(z \to y)\}.
\end{aligned}$$

Thus, $\overline{\mu}$ satisfies (FMF_1). Similarly, $\overline{\mu}$ also satisfies (FMF_2).
Consequently, $\overline{\mu} \in \mathsf{FMF}(X)$. □

Theorem 3.33. *Let $f : X \to X$ be a homomorphism and $\overline{\mu} \in \mathsf{FMF}(X)$. Then $(\overline{\mu})^f \in \mathsf{FMF}(X)$, where $(\overline{\mu})^f(x) = \overline{\mu}(f(x))$.*

Proof. Assume that $\overline{\mu} \in \mathsf{FMF}(X)$ and $x, y, z \in X$. Then

$$\begin{aligned}
(\overline{\mu})^f(x \to y) &= \overline{\mu}(f(x \to y)) \\
&= \overline{\mu}(f(x) \to f(y)) \\
&\geq \min\{\overline{\mu}(f(x) \to f(z)), \overline{\mu}(f(z) \to f(y))\} \\
&= \min\{(\overline{\mu})^f(x \to z), (\overline{\mu})^f(z \to y)\}.
\end{aligned}$$

By a similar argument we have
$(\overline{\mu})^f(x \rightsquigarrow y) \geq \min\{(\overline{\mu})^f(x \rightsquigarrow z), (\overline{\mu})^f(z \rightsquigarrow y)\}$. Also, since f is a homomorphism, we have $f(1) = 1$, and so

$$(\overline{\mu})^f(1) = \overline{\mu}(f(1)) \geq \overline{\mu}(f(x)) = (\overline{\mu})^f(x).$$

Therefore, $(\overline{\mu})^f \in \mathsf{FMF}(X)$. □

Theorem 3.34. *Let $f : X \to Y$ be a homomorphism from pseudo BE-algebras X and Y, $\overline{\nu} \in \mathsf{FMF}(Y)$. Then $f^{-1}(\overline{\nu}) \in \mathsf{FMF}(X)$.*

Proof. Assume that $\overline{\nu} \in \mathsf{FMF}(Y)$ and $x, y, z \in X$. Then we have

$$\begin{aligned}
f^{-1}(\overline{\nu})(x \to y) &= \overline{\nu}(f(x \to y)) \\
&= \overline{\nu}(f(x) \to f(y)) \\
&\geq \min\{\overline{\nu}(f(x) \to f(z)), \overline{\nu}(f(z) \to f(y))\} \\
&= \min\{\overline{\nu}(f(x \to z)), \overline{\nu}(f(z \to y))\} \\
&= \min\{f^{-1}(\overline{\nu})(x \to z), f^{-1}(\overline{\nu})(z \to y)\}.
\end{aligned}$$

Similarly, $f^{-1}(\overline{\nu})(x \rightsquigarrow y) \geq \min\{f^{-1}(\overline{\nu})(x \rightsquigarrow z)), f^{-1}(\overline{\nu})(z \rightsquigarrow y)\}$. Also, since f is a homomorphism, we have $f(1) = 1$, and so

$$f^{-1}(\overline{\nu})(1) = \overline{\nu}(f(1)) \geq \overline{\nu}(f(x)) = f^{-1}(\overline{\nu})(x).$$

Therefore, $f^{-1}(\overline{\nu}) \in \mathsf{FMF}(X)$. $\square$

Let $\overline{\mu}$ be a fuzzy set of X and $t \in [0,1]$. A fuzzy set defined by $\overline{\mu}^{(t)}(x) = \min\{\overline{\mu}(x), t\}$ is called *t-fuzzy set* of X.

Proposition 3.35. *If $\overline{\mu} \in \mathsf{FMF}(X)$, then $\overline{\mu}^{(t)} \in \mathsf{FMF}(X)$, for all $t \in [0,1]$.*

Proof. Assume that $\overline{\mu} \in \mathsf{FMF}(X)$. Then we have

$$\begin{aligned}
\overline{\mu}^{(t)}(x \to y) &= \min\{\overline{\mu}(x \to y), t\} \\
&\geq \min\{\min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}, t\} \\
&= \min\{\min\{\overline{\mu}(x \to z), t\}, \min\{\overline{\mu}(z \to y), t\}\} \\
&= \min\{\overline{\mu}^{(t)}(x \to z), \overline{\mu}^{(t)}(z \to y)\}.
\end{aligned}$$

Thus, $\overline{\mu}$ satisfies (FMF_1). Similarly, $\overline{\mu}$ also, satisfies (FMF_2). Consequently, $\overline{\mu} \in \mathsf{FMF}(\mathfrak{X})$. $\square$

Let $\overline{\mu}$ and $\overline{\nu}$ be two fuzzy sets of X and Y respectively. The cartesian product $\overline{\mu} \times \overline{\nu}$ is a fuzzy set of $X \times Y$ defined by $(\overline{\mu} \times \overline{\nu})(x, y) = \min\{\overline{\mu}(x), \overline{\nu}(y)\}$, for all $x \in X$ and $y \in Y$.

Theorem 3.36. *Let $\overline{\mu}$ and $\overline{\nu}$ be two fuzzy medial filters of X and Y respectively. Then $\overline{\mu} \times \overline{\nu}$ is a fuzzy medial filter of $X \times Y$.*

Theorem 3.37. *Let $\overline{\mu} \times \overline{\nu}$ be a fuzzy medial filter of $X \times Y$. Then*

(i) *either $\overline{\mu}(1) \geq \overline{\mu}(x)$ or $\overline{\nu}(1) \geq \nu(y)$, for all $x \in X$ and $y \in Y$,*
(ii) *if $\overline{\mu}(1) \geq \overline{\mu}(x)$, for all $x \in X$, then either $\overline{\nu}(1) \geq \overline{\mu}(x)$, for all $x \in X$ or $\overline{\mu}(1) \geq \overline{\nu}(y)$, for all $y \in Y$,*
(iii) *if $\overline{\nu}(1) \geq \overline{\nu}(y)$, for all $y \in Y$, then either $\overline{\mu}(1) \geq \overline{\nu}(y)$, for all $y \in Y$ or $\overline{\nu}(1) \geq \overline{\mu}(x)$, for all $x \in X$,*
(iv) *either $\overline{\mu}$ is a fuzzy medial filter of X or $\overline{\nu}$ is a fuzzy medial filter of Y.*

Proof. (i) Assume that $\overline{\mu} \times \overline{\nu}$ is a fuzzy medial filter of $X \times Y$. Hence (FF_1) holds, and so $(\overline{\mu} \times \overline{\nu})(1,1) \geq (\overline{\mu} \times \overline{\nu})(x, y)$, for all $x \in X$ and $y \in Y$.
By the contrary, let there exist $a \in X$ and $b \in Y$, such that $\overline{\mu}(a) > \overline{\mu}(1)$ and $\overline{\nu}(b) > \overline{\nu}(1)$.
Thus, $(\overline{\mu} \times \overline{\nu})(a, b) = \min\{\overline{\mu}(a), \overline{\nu}(b)\} > \min\{\overline{\mu}(1), \overline{\nu}(1)\} = (\overline{\mu} \times \overline{\nu})(1, 1)$, which is a contradiction.

(ii) By the contrary, let there exist $a \in X$ and $b \in Y$ such that $\overline{\nu}(1) < \overline{\mu}(a)$ and $\overline{\mu}(1) < \overline{\nu}(b)$.
Then $(\overline{\mu} \times \overline{\nu})(1,1) = \min\{\overline{\mu}(1), \overline{\nu}(1)\} = \overline{\nu}(1)$, since $\overline{\nu}(1) < \overline{\mu}(a) \leq \overline{\mu}(1)$.
Thus, $(\overline{\mu} \times \overline{\nu})(a,b) = \min\{\overline{\mu}(a), \overline{\nu}(b)\} > \overline{\nu}(1) = (\overline{\mu} \times \overline{\nu})(1,1)$, which is a contradiction.

Similarly, (iii) holds.

(iv) Applying (i) and (ii), if $\overline{\mu}(1) \geq \overline{\mu}(x)$ and $\overline{\nu}(1) \geq \overline{\mu}(x)$, for all $x \in X$, then we have

$$\begin{aligned}
(\overline{\mu} \times \overline{\nu})[(x,1) \to (y,1)] &= (\overline{\mu} \times \overline{\nu})(x \to y, 1 \to 1) \\
&= \min\{\overline{\mu}(x \to y), \overline{\nu}(1 \to 1)\} \\
&= \min\{\overline{\mu}(x \to y), \overline{\nu}(1)\} \\
&= \overline{\mu}(x \to y).
\end{aligned}$$

Similarly, $(\overline{\mu} \times \overline{\nu})[(x,1) \to (z,1)] = \overline{\mu}(x \to z)$ and
$(\overline{\mu} \times \overline{\nu})[(z,1) \to (y,1)] = \overline{\mu}(z \to y)$.
Now, since $\overline{\mu} \times \overline{\nu}$ is a fuzzy medial filter, for all $z_1 \in X, z_2 \in Y$, we get
$(\overline{\mu} \times \overline{\nu})[(x,1) \to (y,1)] \geq$
$\min\{(\overline{\mu} \times \overline{\nu})[(x,1) \to (z_1,z_2)], (\overline{\mu} \times \overline{\nu})[(z_1,z_2) \to (y,1)]\}$.
Take $z_1 := z$ and $z_2 := 1$. Thus,

$$\begin{aligned}
\overline{\mu}(x \to y) &= (\overline{\mu} \times \overline{\nu})[(x,1) \to (y,1)] \\
&\geq \min\{(\overline{\mu} \times \overline{\nu})[(x,1) \to (z,1)], (\overline{\mu} \times \overline{\nu})[(z,1) \to (y,1)]\} \\
&= \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.
\end{aligned}$$

Similarly, $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}(z \rightsquigarrow y)\}$.
Therefore, $\overline{\mu}$ is a fuzzy medial filter of X.

If $\overline{\mu}(1) \geq \overline{\mu}(x)$, for all $x \in X$ and $\overline{\mu}(1) \geq \overline{\nu}(y)$, for all $y \in Y$, then, for all $y, t \in Y$ we have

$$\begin{aligned}
(\overline{\mu} \times \overline{\nu})[(1,y) \to (1,t)] &= (\overline{\mu} \times \overline{\nu})(1 \to 1, y \to t) \\
&= \min\{\overline{\mu}(1 \to 1), \overline{\nu}(y \to t)\} \\
&= \min\{\overline{\mu}(1), \overline{\nu}(y \to t)\} \\
&= \overline{\nu}(y \to t).
\end{aligned}$$

Similarly, $(\overline{\mu} \times \overline{\nu})[(1,y) \to (1,z)] = \overline{\nu}(y \to z)$ and
$(\overline{\mu} \times \overline{\nu})[(1,z) \to (1,t)] = \overline{\nu}(z \to t)$.
Now, since $\overline{\mu} \times \overline{\nu}$ is a fuzzy medial filter, for all $z_1 \in X, z_2 \in Y$, we have
$(\overline{\mu} \times \overline{\nu})[(1,y) \to (1,t)] \geq$
$\min\{(\overline{\mu} \times \overline{\nu})[(1,y) \to (z_1,z_2)], (\overline{\mu} \times \overline{\nu})[(z_1,z_2) \to (1,t)]\}$.

Take $z_1 := 1$ and $z_2 := z$. Thus,

$$\begin{aligned}\overline{\nu}(y \to t) &= (\overline{\mu} \times \overline{\nu})[(1, y) \to (1, t)] \\ &\geq \min\{(\overline{\mu} \times \overline{\nu})[(1, y) \to (1, z)], (\overline{\mu} \times \overline{\nu})[(1, z) \to (1, t)]\} \\ &= \min\{\overline{\nu}(y \to z), \overline{\nu}(z \to t)\}.\end{aligned}$$

Similarly, $\overline{\nu}(y \rightsquigarrow t) \geq \min\{\overline{\nu}(y \rightsquigarrow z), \overline{\nu}(z \rightsquigarrow t)\}$.
Therefore, $\overline{\nu}$ is a fuzzy medial filter of Y.

Finally, if $\overline{\nu}(1) \geq \nu(y)$, for all $y \in Y$, then applying (iii) and by a similar argument, we get either $\overline{\nu}$ is a fuzzy medial filter of Y or $\overline{\mu}$ is a fuzzy medial filter of X. □

4. Fuzzy implicative filters of pseudo BE-algebras

In this section, we introduce the notion of the *fuzzy implicative filter* of a pseudo BE-algebra and show that every fuzzy implicative filter is a fuzzy medial filter, but the converse may not be valid in general.

Here we recall the definition of the *implicative pseudo-filter* of pseudo BCK-algebra X was defined by Zhang and Jun (see [19]).
We *redefine* it for a pseudo BE-algebra X as follows:

Definition 4.1. A nonempty subset F of X is called an *implicative filter* of X if it satisfies (F_1) and the following conditions:
(IF_1) $x \rightsquigarrow (z \to y) \in F$ and $x \to z \in F$ imply $x \to y \in F$,
(IF_2) $x \to (z \rightsquigarrow y) \in F$ and $x \rightsquigarrow z \in F$ imply $x \rightsquigarrow y \in F$, for all $x, y, z \in X$.

If $x := 1$, then every implicative filter is a filter of X.

Let $\mathsf{IF}(X)$ be the set of all implicative filters of X.

Proposition 4.2. *Let $F \in \mathsf{IF}(X)$. Then*
(i) $x \rightsquigarrow (x \to y) \in F$ *implies* $x \to y \in F$,
(ii) $x \to (x \rightsquigarrow y) \in F$ *implies* $x \rightsquigarrow y \in F$.

Proof. By ($psBE_1$), (F_1) and take $z := x$ the proofs are obvious. □

Definition 4.3. A fuzzy set $\overline{\mu}$ is called *fuzzy implicative filter,* if it satisfies (FF_1) and the following conditions:
(FIF_1) $\overline{\mu}(x \to y) \geq \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\}$,
(FIF_2) $\overline{\mu}(x \rightsquigarrow y) \geq \min\{\overline{\mu}(x \rightsquigarrow z), \overline{\mu}[x \to (z \rightsquigarrow y)]\}$,
for all $x, y, z \in X$.

Let $\mathsf{FIF}(X)$ be the set of all fuzzy implicative filters of X.

As an immediate consequence, we obtain the following theorem.

Theorem 4.4. *Let $\overline{\mu} \in \mathsf{FIF}(X)$. Then*
(i) $\overline{\mu} \in \mathsf{FF}(X)$,

(ii) $\overline{\mu} \in \mathsf{FMF}(X)$.

Proof. (i) Let $y, z \in X$ and take $x := 1$. Applying (psBE_3) and (FIF_1), we get

$$\begin{aligned}\overline{\mu}(y) &= \overline{\mu}(1 \to y)\\ &\geq \min\{\overline{\mu}(1 \to z), \overline{\mu}[1 \rightsquigarrow (z \to y)]\}\\ &= \min\{\overline{\mu}(z), \overline{\mu}(z \to y)\}.\end{aligned}$$

(ii) Assume that $\overline{\mu} \in \mathsf{FIF}(X)$ and $x, y, z \in X$. Using (p_1), we get $z \to y \leq x \rightsquigarrow (z \to y)$, and so by Propositions 3.17 and 3.5, we have

$$\overline{\mu}(z \to y) \leq \overline{\mu}[x \rightsquigarrow (z \to y)].$$

Thus, $\min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\} \leq \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\}$. Since $\overline{\mu}$ is a fuzzy implicative filter, we have

$$\begin{aligned}\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}[x \rightsquigarrow (z \to y)]\}\\ &\geq \min\{\overline{\mu}(x \to z), \overline{\mu}(z \to y)\}.\end{aligned}$$

Thus, $\overline{\mu}$ satisfies (FMF_1). Similarly, $\overline{\mu}$ also satisfies (FMF_2). Consequently, $\overline{\mu} \in \mathsf{FMF}(X)$. □

The following example shows that the converse of Theorem 4.4(ii), is not valid, in general.

Example 4.5. Consider the fuzzy medial filter $\overline{\mu}$ given in Example 3.7. It is not a fuzzy implicative filter, since

$$\begin{aligned}\overline{\mu}(c \rightsquigarrow b) = \overline{\mu}(b) = 0.2 &\ngeq \min\{\overline{\mu}(c \rightsquigarrow c), \overline{\mu}[c \to (c \rightsquigarrow b)]\}\\ &= \min\{\overline{\mu}(1), \overline{\mu}(a)\}\\ &= \overline{\mu}(a)\\ &= 0.6.\end{aligned}$$

Theorem 4.6. *Let $\overline{\mu} \in \mathsf{FIF}(X)$. Then*

$$\overline{\mu}(x \to y) = \overline{\mu}(x \rightsquigarrow y) = \overline{\mu}[x \rightsquigarrow (x \to y)] = \overline{\mu}[x \to (x \rightsquigarrow y)].$$

Proof. Let $\overline{\mu} \in \mathsf{FIF}(X)$. Applying ($\mathrm{FF}_1$), ($\mathrm{FIF}_1$) and take $z := x$, we deduced that

$$\begin{aligned}\overline{\mu}(x \to y) &\geq \min\{\overline{\mu}(x \to x), \overline{\mu}[x \rightsquigarrow (x \to y)]\}\\ &= \min\{\overline{\mu}(1), \overline{\mu}[x \rightsquigarrow (x \to y)]\}\\ &= \overline{\mu}[x \rightsquigarrow (x \to y)].\end{aligned}$$

On the other hand, by (p_2) since $x \to y \leq x \rightsquigarrow (x \to y)$, we get

$$\overline{\mu}(x \to y) \leq \overline{\mu}[x \rightsquigarrow (x \to y)].$$

Thus, $\overline{\mu}(x \to y) = \overline{\mu}[x \rightsquigarrow (x \to y)]$.

Also, by (FF_1), (FIF_2) and take $z := x$, we have

$$\begin{aligned}\overline{\mu}(x \rightsquigarrow y) &\geq \min\{\overline{\mu}(x \rightsquigarrow x), \overline{\mu}[x \to (x \rightsquigarrow y)]\} \\ &= \min\{\overline{\mu}(1), \overline{\mu}[x \to (x \rightsquigarrow y)]\} \\ &= \overline{\mu}[x \to (x \rightsquigarrow y)].\end{aligned}$$

Also, since $x \rightsquigarrow y \leq x \to (x \rightsquigarrow y)$, we get

$$\overline{\mu}(x \rightsquigarrow y) \leq \overline{\mu}[x \to (x \rightsquigarrow y)].$$

Thus, $\overline{\mu}(x \rightsquigarrow y) = \overline{\mu}[x \to (x \rightsquigarrow y)]$.

Using (psBE_4), since $x \to (x \rightsquigarrow y) = x \rightsquigarrow (x \to y)$, we get

$$\overline{\mu}[x \to (x \rightsquigarrow y)] = \overline{\mu}[x \rightsquigarrow (x \to y)].$$

Thus, $\overline{\mu}(x \to y) = \overline{\mu}[x \rightsquigarrow (x \to y)] = \overline{\mu}[x \to (x \rightsquigarrow y)] = \overline{\mu}(x \rightsquigarrow y)$. □

5. Conclusions

BE-algebras were studied by researchers, and some classification is given. It is well known that the fuzzy structure with special properties plays an essential role in the algebraic structures. In this paper, the notion of the *fuzzy medial filter* in a pseudo BE-algebra is discussed. Several conditions to every fuzzy filter could be a fuzzy medial filter are given. Also, the notion of the *fuzzy implicative filter* is defined and showed that every fuzzy implicative filter is a fuzzy medial filter. By Theorem 3.23(ii), and Proposition 3.17 we obtain $\mathsf{FIF}(X) \subset \mathsf{FMF}(X) \subset \mathsf{FF}(X)$. Moreover, if X is a commutative pseudo BE-algebra, then $\mathsf{FCF}(X) = \mathsf{FMF}(X) = \mathsf{FF}(X)$ follows from Corollary 3.26.

Problem 5.1. Is it true that every fuzzy commutative filter is a fuzzy implicative filter?

Problem 5.2. Is it true that every fuzzy implicative filter is a fuzzy commutative filter?

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FUZZY MEDIAL FILTERS OF PSEUDO BE-ALGEBRAS

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فیلترهای میدآل فازی در شبه BE-جبرها

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در این مقاله، مفهوم فیلترهای میدآل فازی در یک شبه BE-جبر تعریف و برخی خواص آن‌ها بررسی شده است. نشان داده‌خواهد شد که مجموعه همه فیلترهای میدآل فازی یک شبه BE-جبر، مشبکه کامل می‌باشد. علاوه‌برآن، ثابت خواهد شد که در شبه BE-جبرهای جابجایی فیلترهای فازی و فیلترهای میدآل فازی بر هم منطبق هستند. در پایان، مفهوم فیلتر استلزامی فازی تعریف و ثابت خواهد شد که هر فیلتر استلزامی فازی یک فیلتر میدآل فازی است ولی عکس آن در حالت کلی برقرار نیست.

کلمات کلیدی: شبه BE-جبر (جابجایی)، فیلتر میدآل (فازی)، فیلتر استلزامی (فازی).