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SOME RESULTS ON ϕ -(k,n)-CLOSED SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be a unitary R -module. Let $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called ϕ -semi-n-absorbing if $r^n m \in N \setminus \phi(N)$ where $r \in R, m \in M$ and $n \in \mathbb{Z}^+$, then $r^n \in (N : M)$ or $r^{n-1}m \in N$. Let k and n are positive integers where $k > n$. A proper submodule N of M is called ϕ -(k, n)- closed submodule, if $r^k m \in N \setminus \phi(N)$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$, then $r^n \in (N : M)$ or $r^{n-1}m \in N$. In this work, firstly, we will study some general results when we use the definition ϕ -(k, n)- closed submodule. Moreover, we prove main results of the ϕ -(k, n)- closed submodule for various modules.

1. Introduction

In this work all rings are commutative with identity and all modules are unitary. Let M be an R-module and N be a submodule of M. The ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N : M)$ and ideal $(0 : M)$ will be denoted by $Ann(M)$. A proper ideal I of R is a (m, n)- closed ideal if $a^m \in I$ for $a \in R$ implies $a^n \in I$ (see [4]). Let $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. A proper ideal I of R is called ψ -(m, n)- closed ideal of R if whenever $a \in R$ with $a^m \in I \setminus \psi(I)$, then $a^n \in I$($m > n$) and a proper ideal I of R is said to be ψ-prime if for $a, b \in R$ with $ab \in I \setminus \psi(I)$, then $a \in I$ or $b \in I$. Without loss of generality we may assume that

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$\psi(I) \subseteq I$. In this work, we write $\psi(N : M)$ instead of $\psi((N : M))$.
The generalization of prime ideals play an essential role in the ring theory. This concept has been used by D. Anderson and M. Bataineh (see [5]). Some authors extended various generalized prime ideals and prime submodules (for example see [2], [3], [4], [6], [8] and [9]). N. Zamani defined the concept of ϕ-prime submodule (see [22]). Let M be a unitary R-module, $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called ϕ-prime if $a \in R, x \in M$ with $ax \in N \setminus \phi(N)$, then $a \in (N : M)$ or $x \in N$. Some properties of this concept have been investigated in [22]. Suppose k and n are two positive integers with $k > n$, $S(M)$ be the set of all submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule N of M is called ϕ -(k, n)-closed submodule, if whenever $r \in R, m \in M$ with $r^k m \in N \setminus \phi(N)$, then $r^n \in (N : M)$ or $r^{n-1}m \in N$. Some results of (k, n)-closed submodules have been studied in [21].
We use some concepts of (k, n)-closed submodules for ϕ-(k, n)-closed submodules. Moreover, we recall the concepts of compactly packed submodules and finitely compactly packed modules (see [18], [7], [1]) and we state Corollaries 2.21, 2.22, and Theorems 2.23, 2.24 in connection with these concepts.

2. Main results of ϕ-(k, n)-Closed submodules

In this section, we have proved some results of ϕ-(k, n)-closed submodules.

Proposition 2.1. *Let M be an R-module and N be a proper submodule of M. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$, $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ are two functions where $S(M)$ is the set of all submodules of M and $\mathcal{I}(R)$ is the set of all ideals of R with $\psi(N : M) \subseteq (\phi(N) : m)$, for every $m \in M$ such that $(N : M)$ be a ψ-(k, n)-closed ideal of R. If N is a ϕ-prime submodule of M, then N is a ϕ-(k, n)- closed submodule of M $(k > n)$.*

Proof. Let N be a proper submodule of M and $r^k m \in N \setminus \phi(N)$ where $r \in R$ and $m \in M$. Since N is a ϕ- prime submodule of M, then $r^k \in (N : M)$ or $m \in N$. If $m \in N$, then $r^{n-1}m \in N$. From $r^k \in (N : M)$, it follows that $r^k \in (N : M) \setminus \psi(N : M)$, because $r^k m \notin \phi(N)$ and $\psi(N : M) \subseteq (\phi(N) : m)$ for all $m \in M$. Since $(N : M)$ is a ψ-(k, n)-closed ideal of R, then $r^n \in (N : M)$, as required. □

Proposition 2.2. *Let M be a unitary R -module and $\phi_1, \phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions, where $S(M)$ is the set of all submodules of M with $\phi_1 \leq \phi_2$ (i.e., for every submodule N of M , $\phi_1(N) \subseteq \phi_2(N)$).*

If N is a ϕ_1-(k,n)-closed submodule of M, then N is a ϕ_2-(k,n)-closed submodule of M.

Proof. The proof is evident. □

Proposition 2.3. *Let N be a ϕ-(k,n)-closed submodule of M. Then N is a ϕ-$(k+1,n+1)$-closed submodule of M.*

Proof. Let $r \in R$ and $m \in M$ with $r^{k+1}m \in N \setminus \phi(N)$. Then $r^k(rm) \in N \setminus \phi(N)$. Since N is a ϕ-(k,n)-closed submodule of M, then $r^n \in (N : M)$ or $r^{n-1}(rm) \in N$. Thus $r^{n+1} \in (N : M)$ or $r^n m \in N$. □

Example 2.4. Suppose that $\phi(N) = \emptyset$, we know that if N is a (k,n)-closed submodule of M, then N is a $(k+1,n+1)$-closed submodule of M. But the converse of Proposition 2.3 is not true in general. For example, let $M = \mathbb{Z} \oplus \mathbb{Z}$ be a $\mathbb{Z}$-module and $N =< (3,0) >$ be a submodule of $\mathbb{Z} \oplus \mathbb{Z}$. We have $(< (3,0) >: \mathbb{Z} \oplus \mathbb{Z}) = 0$ and $(18,0) = 3^2(2,0) \in < (3,0) >$, but $3 \notin (< (3,0) >: \mathbb{Z} \oplus \mathbb{Z}) = 0$ and $3^0(2,0) \notin < (3,0) >$. Therefore $< (3,0) >$ is not a $(2,1)$-closed submodule. Now, we show that $< (3,0) >$ is a $(3,2$-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$. Suppose that $r \in \mathbb{Z}$, $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ with $r^3(m,n) \in < (3,0) >$. If $r = 0$, then $0 = r^2 \in (< (3,0) >: \mathbb{Z} \oplus \mathbb{Z}) = 0$ or $r^{2-1}(m,n) \in < (3,0) >$. So $< (3,0) >$ is a $(3,2)$-closed submodule. Now, let $r \neq 0$,so $0 \neq r^2 \notin (< (3,0) >: \mathbb{Z} \oplus \mathbb{Z}) = 0$. We have $(r^3m, r^3n) = (3k, 0)$ for some $k \in \mathbb{Z}$, hence $n = 0$ and $3 \mid r^3m$. If $3 \mid m$, then $r^{2-1}(m,0) \in < (3,0) >$. If $3 \nmid m$, then $3 \mid r^3$. So $3 \mid r$, therefore $r^{2-1}(m,0) \in < (3,0) >$. Thus $< (3,0) >$ is a $(3,2)$-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$.

Remark 2.5. Let $\varphi : R \to S$ be a ring homomorphism and M be a S-module. It is easy to show that if N is a ϕ-(k,n)-closed submodule of S-module M, then N is a ϕ- (k,n)-closed submodule of R-module M.

Proposition 2.6. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M and N_i be a proper submodule of M for $i \in \Lambda$, such that $\phi(\cup_{i\in\Lambda} N_i) \subseteq \phi(\cap_{i\in\Lambda} N_i)$. If N_i is a ϕ-(k,n)-closed submodule of M for each $i \in \Lambda$, then $\cap_{i\in\Lambda} N_i$ is a ϕ-(k,n)-closed submodule of M.*

Proof. Let $r^k m \in \cap_{i\in\Lambda} N_i \setminus \phi(\cap_{i\in\Lambda} N_i)$ where $r \in R$ and $m \in M$. Then $r^k m \in \cap_{i\in\Lambda} N_i$ and $r^k m \notin \phi(\cap_{i\in\Lambda} N_i)$. By our assumption $\phi(\cup_{i\in\Lambda} N_i) \subseteq \phi(\cap_{i\in\Lambda} N_i)$, so $r^k m \in N_i \setminus \phi(N_i)$ for each $i \in \Lambda$. Since N_i is a ϕ-(k,n)-closed submodule of M, then $r^n \in (N_i, : M)$ or $r^{n-1}m \in N_i$ for every $i \in \Lambda$. Since $(\cap_{i\in\Lambda} N_i : M) = \cap_{i\in\Lambda}(N_i : M)$, then $r^n \in (\cap_{i\in\Lambda} N_i : M)$ or $r^{n-1}m \in \cap_{i\in\Lambda} N_i$. This means that $\cap_{i\in\Lambda} N_i$ is a ϕ-(k,n)-closed submodule of M. □

The next theorem is a generalization of Theorem 2.3 in [21].

Theorem 2.7. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}, \psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M and $\mathcal{I}(R)$ is the set of all ideals of R. Let N be a proper submodule of R-module M.*
(1) If N is a ϕ-(k,n)-closed submodule of M with $(\phi(N) : m) \subseteq \psi(N : m)$ for each $m \in M \setminus N$, then $(N : m)$ is a ψ-(k,n)-closed ideal of R $(k > n)$.
(2) If $(N : m)$ is a ψ-(k,n)-closed ideal of R with $\psi(N : m) \subseteq (\phi(N) : m)$ for each $m \in M \setminus N$, then N is a ϕ-$(k,n+1)$-closed submodule of M $(k > n+1)$.
(3) If N is a ϕ-(k,n)-closed submodule of M with $(\phi(N) : m) \subseteq \psi(N : M)$ for all $m \in M$, then $(N : M)$ is a ψ-(k,n)-closed ideal of R $(k > n)$.

Proof. (1) Assume that $r^k \in (N : m)\setminus\psi(N : m)$. We have $r^k \in (N : m)$ and $r^k \notin \psi(N : m)$. Since $(\phi(N) : m) \subseteq \psi(N : m)$ for every $m \in M\setminus N$, then $r^k m \in N \setminus \phi(N)$. Thus $r^n \in (N : M)$ or $r^{n-1}m \in N$. Since $(N : M) \subseteq (N : m)$, then $r^n \in (N : m)$. From $r^{n-1}m \in N$, we get $r^n m \in N$. This means that $(N : m)$ is a ψ-(k,n)-closed ideal of R.
(2) Let $r^k m \in N \setminus \phi(N)$ where $r \in R$ and $m \in M \setminus N$. Then $r^k m \in N$ and $r^k m \notin \phi(N)$. Since $\psi(N : m) \subseteq (\phi(N) : m)$, then $r^k \in (N : m) \setminus \psi(N : m)$. Therefore $r^n \in (N : m)$ and hence $r^n m = r^{(n+1)-1}m \in N$. Thus N is a ϕ-$(k,n+1)$-closed submodule of M.
(3) Assume that $r \in R$ with $r^k \in (N : M) \setminus \psi(N : M)$ but $r^n \notin (N : M)$. Then there is an element $m' \in M$ such that $r^n m' \notin N$ which means that $r^{n-1}m' \notin N$. On the other hand, since $r^k \notin \psi(N : M)$, then $r^k \notin \phi(N : m)$, for all $m \in M$. Hence $r^k \notin (\phi(N) : m')$. Therefore $r^k m' \in N \setminus \phi(N)$ and so $r^n \in (N : M)$ or $r^{n-1}m' \in N$, this is a contradiction. Thus $(N : M)$ is a ψ-(k,n)-closed ideal of R. □

We recall that an R-module M is called a multiplication module if for every submdule N of M, we have $N = IM$, where I is an ideal of R. We say that I is a presentation ideal of N or, for short, a presentation of N and we denote the set of all presentation ideals of N by $\mathcal{P}r(N)$. Clearly $(N : M)$ is a presentation ideal of N.

Corollary 2.8. *Let the situation be as described in Theorem 2.7 and M be a multiplication R-module such that $(\phi(N) : m) \subseteq \psi(N : M)$ for every $m \in M$. If N is a ϕ-(k,n)-closed submodule of M, then $(N : M)$ is a ψ-(k,n)-closed ideal of R.*

Proof. Since $(N : M)$ is a presentation ideal of N and $(\phi(N) : m) \subseteq \psi(N : M)$, by Theorem 2.7 (3), then $(N : M)$ is a ψ-(k,n)-closed ideal of R. □

Now, let F be a free R-module and $\{m_\alpha\}_{\alpha\in\Lambda}$ be a basis for F, then it is clear that submodule IF is of the form $IF = \{\sum_{f.s} e_i m_{\alpha_i} | e_i \in I, m_{\alpha_i} \in \{m_\alpha\}_{\alpha\in\Lambda}\}$, where I is an ideal of R. Also, if $a \in F$ so a has a unique representation in the form $a = \sum_{\alpha\in\Lambda} r_\alpha m_\alpha$ where $r_\alpha \in R$ and $r_\alpha = 0$ for almost all $\alpha \in \Lambda$. Hence we can write $a = \sum_{f.s} r_\alpha m_\alpha$ where $r_\alpha \in R$ and by the way IF is defined, we have $(IF : F) = I$. In light of above explanation, we state the following theorem.

Theorem 2.9. *Let F be a free R-module, $\phi : S(F) \to S(F) \cup \{\emptyset\}, \psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be two functions where $S(F)$ is the set of all submodules of F and $\mathcal{I}(R)$ is the set of all ideals of R. If I is a ψ-prime ideal of R with $\psi(I)F \subseteq \phi(IF)$ and $\sqrt{I} = I$, then IF is a ϕ-(k, n)-closed submodule of F*

Proof. Let $r^k m \in IF \setminus \phi(IF)$ where $r \in R$ and $m \in F$. Suppose that $\{m_\alpha\}_{\alpha\in\Lambda}$ be a basis for F. We have $r^k m \in IF$ and $r^k m \notin \phi(IF)$. Since $m \in F$, then $m = \sum_{f.s} r_\alpha m_\alpha$ where $r_\alpha \in R$ and hence $r^k m = \sum_{f.s} (r^k r_\alpha) m_\alpha$. But $r^k m \in IF$ implies that $r^k m = \sum_{f.s} s_\alpha m_\alpha$ where $s_\alpha \in I$. Then $\sum_{f.s} (r^k r_\alpha) m_\alpha = \sum_{f.s} s_\alpha m_\alpha$ and since $\{m_\alpha\}_{\alpha\in\Lambda}$ is a basis for F, we must have $r^k r_\alpha = s_\alpha$ and hence $r^k r_\alpha \in I$. On the other hand $r^k m \notin \phi(IF)$, since $\psi(I)F \subseteq \phi(IF)$, then $r^k m \notin \psi(I)F$. It follows that $r^k r_\alpha \notin \psi(I)$. Thus $r^k r_\alpha \in I \setminus \psi(I)$. Because I is an ideal ψ-prime of R, so $r^k \in I$ or $r_\alpha \in I$ for all $\alpha \in \Lambda$. Since $r^k \in I$ and $\sqrt{I} = I$, then $r \in I$ implies $r^n \in I = (IF : F)$. If $r_\alpha \in I$ for all $\alpha \in \Lambda$, we have $\sum_{f.s} r_\alpha m_\alpha \in IF$, so $m \in IF$ implies $r^{n-1} m \in IF$. Thus IF is a ϕ-(k, n)-closed submodule of F. □

For a submodule L of M, let $\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\}$ be defined by $\phi_L(\frac{N}{L}) = \frac{\phi(N)+L}{L}$ with $L \subseteq N$ (and $\phi_L(\frac{N}{L}) = \emptyset$ if $\phi(N) = \emptyset$) where $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function and $S(\frac{M}{L})$ is the set of all submodules of $\frac{M}{L}$. Now, we state the generalization of Corollary 2.34 in [21].

Theorem 2.10. *Let M be an R-module and $L \subseteq N$ be a proper submodule of M. Suppose that $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\}$ be two functions. Then the following statements hold.*
(1) If N is a ϕ-(k, n)-closed submodule of M, then $\frac{N}{L}$ is a ϕ_L-(k, n)-closed submodule of $\frac{M}{L}$.
(2) If $L \subseteq \phi(N)$ and $\frac{N}{L}$ is a ϕ_L-(k, n)-closed submodule of $\frac{M}{L}$, then N is a ϕ-(k, n)-closed submodule of M.

Proof. (1) Let $r \in R$ and $m + L \in \frac{M}{L}$ with $r^k(m + L) \in \frac{N}{L} \setminus \phi_L(\frac{N}{L})$. It follows that $r^k m \in N \setminus \phi(N)$. Since N is a ϕ-(k, n)-closed submodule of M, then $r^n \in (N : M)$ or $r^{n-1}m \in N$. Thus $r^n \in (\frac{N}{L} : \frac{M}{L})$ or $r^{n-1}(m + L) \in \frac{N}{L}$, as required.

(2) Let $r \in R$ and $m \in M$ with $r^k m \in N \setminus \phi(N)$. Since $L \subseteq \phi(N)$, then $r^k m + L \notin \frac{\phi(N)+L}{L}$. So $r^k(m + L) \in \frac{N}{L} \setminus \phi_L(\frac{N}{L})$. Since $\frac{N}{L}$ is a ϕ_L-(k, n)-closed submodule of $\frac{M}{L}$, then $r^n \in (\frac{N}{L} : \frac{M}{L})$ or $r^{n-1}(m + L) \in \frac{N}{L}$. It follows that $r^n \in (N : M)$ or $r^{n-1}m \in N$. □

We recall that a proper submodule N of M is called weakly-(k, n)-closed submodule if $0 \neq r^k m \in N$ where $r \in R$ and $m \in M$, then $r^n \in (N : M)$ or $r^{n-1}m \in M$ $(k > n)$. Ebrahimpour and Mirzaee use the following proposition for ϕ-semiprime submodules and weakly semiprime submodules (see [10, Proposition 2.15]).

Proposition 2.11. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper submodule of M. Then, N is a ϕ-(k, n)-closed submodule of M if and only if $\frac{N}{\phi(N)}$ is a weakly-(k, n)-closed submodule of $\frac{M}{\phi(N)}$.*

Proof. The proof of this proposition is straightforward. □

The next proposition is a generalization of Lemma 2.4 in [21].

Proposition 2.12. *Let M be a finitely generated R-module such that $M = Rm_1 + ... + Rm_t$, N be a proper submodule of M and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R. Then*
(1) If $(N : m_i)$ is a ψ-(k, n)-closed ideal of R with $\psi(N : m_i) \subseteq \psi(N : M)$ for each $i = 1, ..., t$, then $(N : M)$ is a ψ-(k, n)-closed ideal of R.
(2) If $(N : M)$ is a ψ- (k, n)-closed ideal of R, then $(N : m_i)$ is a ψ-(k, n)-closed ideal of R for each $i = 1, ..., t$

Proof. (1) Let $r \in R$ with $r^k \in (N : M) \setminus \psi(N : M)$ and $r^n \notin (N : M)$. So $r^n \notin (N : m_j)$ for some $j \in \{1, ..., t\}$, because $(N : \sum_{i=1}^{t} Rm_i) = \cap_{i=1}^{t}(N : Rm_i) = \cap_{i=1}^{t}(N : m_i)$. Since $r^k \notin \psi(N : M)$, then $r^k \notin \psi(N : m_i)$ for all $i \in \{1, ..., t\}$. It follows that $r^k \in (N : m_j) \setminus \psi(N : m_j)$ for some $j \in \{1, ..., t\}$. Since $(N : m_j)$ is a ψ-(k, n)-closed ideal of R, then $r^n \in (N : m_j)$ which contradicts with our assumption. Thus $(N : M)$ is a ψ-(k, n)-closed ideal of R.
(2) Assume that $(N : M)$ is a ψ- (k, n)-closed ideal of R. Let $r \in R$ with $r^k \in (N : m_i) \setminus \psi(N : m_i)$ for all $i \in \{1, ..., t\}$. We have $r^k \in \cap_{i=1}^{t}(N : m_i) = (N : \sum_{i=1}^{t} Rm_i) = (N : M)$ and because of $\psi(\cap_{i=1}^{t}(N : m_i)) \subseteq \cap_{i=1}^{t}\psi(N : m_i) \subseteq \psi(N : m_i)$, for all $i \in \{1, ..., t\}$, $r^k \notin \psi(N : m_i)$ implies that $r^k \notin \psi(\cap_{i=1}^{t}(N : m_i)) = \psi(N : M)$. It follows that $r^k \in (N : M) \setminus \psi(N : M)$. Thus $r^n \in (N : M)$ and so $r^n \in \cap_{i=1}^{t}(N : m_i)$, therefore $r^n \in (N : m_i)$ for all $i \in \{1, ..., t\}$. □

Now, let M_i be an R_i-module for $i = 1,2$, where R_i is a commutative ring. We know that $M_1 \times M_2$ be an $R_1 \times R_2$-module. Assume that $N_1 \times N_2$ be a proper submodule of $M_1 \times M_2$, where N_i is a proper submodule of M_i for $i = 1,2$. Let $\phi : S(M_1 \times M_2) \to S(M_1 \times M_2) \cup \{\emptyset\}$, $\phi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be functions with $\phi(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$ for $i = 1,2$. Now, we state two following theorems.

Theorem 2.13. *Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and N_i be a proper submodule of M_i for $i = 1,2$. If $N_1 \times N_2$ is a ϕ-(k,n)-closed submodule of $M_1 \times M_2$, then N_i is a ϕ_i-(k,n)-closed submodule of M_i for $i = 1,2$ $(k > n)$*

Proof. Let $i = 1$, $N_1 \neq M_1$, $r_1 \in R_1$ with $r_1^k m_1 \in N_1 \setminus \phi_1(N_1)$. So $(r_1^k m_1, 0) \in N_1 \times N_2$. Since $r_1^k m_1 \notin \phi_1(N_1)$, then $(r_1^k m_1, 0) \notin \phi_1(N_1) \times \phi_2(N_2)$. Thus $(r_1^k m_1, 0) \in N_1 \times N_2 \setminus \phi_1(N_1) \times \phi_2(N_2)$.Since $(r_1, 1)^k (m_1, 0) = (r_1^k m_1, 0)$ and $N_1 \times N_2$ is a ϕ-(k,n)-closed submodule of $M_1 \times M_2$, then $(r_1, 1)^n \in (N_1 \times N_2 : M_1 \times M_2)$ or $(r_1, 1)^{n-1}(m_1, 0) \in N_1 \times N_2$. It follows that $r_1^n \in (N_1 : M_1)$ or $r_1^{n-1} m_1 \in N_1$, as required. □

Theorem 2.14. *Let $M_1 \times M_2$ be an $R_1 \times R_2$-module and $\phi : S(M_1 \times M_2) \to S(M_1 \times M_2) \cup \{\emptyset\}$ be a function with $\phi(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$ where $\phi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ is a function such that $(\phi_i(M_i) : M_i) = R_i$ for $i = 1,2$. If N_i is a ϕ_i-(k,n)-closed submodule of M_i for $i = 1,2$, then $N_1 \times M_2$ and $M_1 \times N_2$ are ϕ-(k,n)-closed submodules of $M_1 \times M_2$ $(k > n)$.*

Proof. Let $(r_1, r_2)^k (m_1, m_2) \in N_1 \times M_2 \setminus \phi(N_1 \times M_2)$ where $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$. We have $r_1^k m_1 \in N_1$, $r_2^k m_2 \in M_2$ and $(r_1^k m_1, r_2^k m_2) \notin \phi_1(N_1) \times \phi_2(M_2)$. Since $R_2 = (\phi_2(M_2) : M_2)$, then $r_2^k m_2 \in \phi_2(M_2)$ and hence $r_1^k m_1 \notin \phi_1(N_1)$. Therefore $r_1^k m_1 \in N_1 \setminus \phi_1(N_1)$. So $r_1^n \in (N_1, M_1)$ or $r_1^{n-1} m_1 \in N_1$. Thus $(r_1^n, r_2^n) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(r_1^{n-1} m_1, r_2^{n-1} m_2) \in N_1 \times M_2$, as required. □

The following theorem is the generalization of Theorem 2.33 in [21].

Theorem 2.15. *Let $f : M \to M'$ be an epimorphism R-module, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*
(1) If N is a ϕ-(k,n)-closed submodule of M with $ker f \subseteq N$ and $f(\phi(N)) \subseteq \phi'(f(N))$, then $f(N)$ is a ϕ'-(k,n)-closed submodule of M' $(k > n)$.
(2) If L is a ϕ'-(k,n)-closed submodule of M' and $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then $f^{-1}(L)$ is a ϕ-(k,n)-closed submodule of M $(k > n)$.

Proof. (1) Let $r \in R$ and $m' \in M'$ with $r^k m' \in f(N) \setminus \phi'(f(N))$. There exists $m \in M$ such that $f(m) = m'$. Hence $r^k f(m) \in f(N)$ and $r^k f(m) \notin \phi'(f(N))$. It follows that $r^k m \in N$ and $r^k m \notin \phi(N)$, because $r^k f(m) \notin f(\phi(N))$. Thus $r^k m \in N \setminus \phi(N)$, so $r^n \in (N : M)$ or $r^{n-1}m \in N$. Therefore $r^n \in (f(N) : M')$ or $r^{n-1}f(m) \in f(N)$.
(2) Let $r^k m \in f^{-1}(L) \setminus \phi(f^{-1}(L))$ where $m \in M$ and $r \in R$. So $r^k m \in f^{-1}(L)$ and $r^k m \notin \phi(f^{-1}(L))$, thus $r^k f(m) \in L \setminus \phi'(L)$. Therefore $r^n \in (L : M')$ or $r^{n-1}f(m) \in L$, since L is a ϕ'-(k, n)-closed submodule of M'. Thus $r^n \in (f^{-1}(L) : M)$ or $r^{n-1}m \in f^{-1}(L)$, as required. □

Let S be a multiplicatively closed subset of R. We know that every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $\phi_S : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $\phi_S(S^{-1}N) = S^{-1}\phi(N)$ and $\phi_S(S^{-1}N) = \emptyset$ if $\phi(N) = \emptyset$ where N is a submodule of M.
The following theorem has been proved for (k, n)-closed submodules and semi n-absorbing submodules (see [21, Theorem 2.30]).

Theorem 2.16. *Let M be an R-module and S be a multiplicatively closed subset of R such that $S^{-1}N \neq S^{-1}M$ and $S^{-1}(\phi(N)) \subseteq \phi_S(S^{-1}N)$. If N is a ϕ-(k, n)-closed submodule of M with $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a ϕ_S-(k, n)-closed submodule of $S^{-1}M$.*

Proof. Let $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$ with $(\frac{r}{s})^k \frac{m}{t} \in S^{-1}N \setminus \phi_S(S^{-1}N)$. We have $\frac{r^k m}{s^k t} \in S^{-1}N$ and $\frac{r^k m}{s^k t} \notin \phi_S(S^{-1}N)$. Hence, there exists $u \in S$ such that $ur^k m \in N$ and $ur^k m \notin \phi(N)$. Therefore $\frac{r^n}{s^n} \in S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$ or $\frac{r^{n-1}}{s^{n-1}} \frac{m}{t} \in S^{-1}N$. □

Now, we consider $S^{-1}M$ as an R-module. Let $\pi : M \to S^{-1}M$ be given by $m \mapsto \frac{m}{1}$. Then π is R-homomorphism. We show that if T is a ϕ_S-(k, n)-closed submodule of $S^{-1}M$, then $\pi^{-1}(T)$ is a ϕ-(k, n)-closed submodule of M.

Proposition 2.17. *Let M be an R-module and S be a multiplicatively closed subset in R. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and define $\phi_S : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ by $\phi_S(T) = S^{-1}\phi(\pi^{-1}(T))$ (and $\phi_S(T) = \emptyset$ when $\phi(\pi^{-1}(T)) = \emptyset$) for every submodule T of $S^{-1}M$. If T is a ϕ_S-(k, n)- closed submodule of $S^{-1}M$ such that $\frac{m}{1} \notin T$ for some $m \in M$, then $\pi^{-1}(T)$ is a ϕ-(k, n)-closed submodule of M.*

Proof. Since $\frac{m}{1} \notin T$ for some $m \in M$, then $\pi^{-1}(T) \neq M$. Let $r \in R$, $m \in M$ with $r^k m \in \pi^{-1}(T) \setminus \phi(\pi^{-1}(T))$. Then $r^k m \in \pi^{-1}(T)$ and $r^k m \notin \phi(\pi^{-1}(T))$. Thus $\frac{r^k m}{1} \in T$ and $\frac{r^k m}{1} \notin S^{-1}\phi(\pi^{-1}(T))$. So $\frac{r^k m}{1} \in T \setminus \phi_S(T)$. Since T is a ϕ_S-(k, n)-closed submodule of $S^{-1}M$, then $\frac{r^n}{1} \in (T : S^{-1}M)$ or $\frac{r^{n-1}m}{1} \in T$. Thus $r^n \in (\pi^{-1}(T) : M)$ or $\pi(r^{n-1}m) \in T$, hence $\pi^{-1}(T)$ is a ϕ-(k,n)-closed submodule of M. □

Definition 2.18. Let M be an R-module and N be a submodule of M. Then N is called relatively divisible submodule denoted by RD-submodule, if $rN = N \cap rM$ for each $r \in R$. M as an R-module is said to be prime if $rm = 0$ where $r \in R$ and $m \in M$, then $r \in Ann(M)$ or $m = 0$. Now, we give the following proposition.

Proposition 2.19. *Let M be a prime R-module and N be a proper submodule of M. If N is a RD-submodule of M with $Ann(M) \subseteq (\phi(N) : M)$, then N is a ϕ-(k,n)-closed submodule of M.*

Proof. Let $r \in R$ and $m \in M$ with $r^k m \in N \setminus \phi(N)$. Since N is a RD-submodule, then $r^k M \cap N = r^k N$. So $r^k m \in r^k M \cap N = r^k N$, hence $r^k m = r^k s$, for some $s \in N$. Thus $r^k(m - s) = 0$. Since M is prime, then $r^k \in Ann(M)$ or $m - s = 0$. But if $r^k \in Ann(M)$, then $r^k \in (\phi(N) : M)$. So $r^k m \in \phi(N)$ which contradicts with our assumption. Thus $m - s = 0$, hence $m \in N$ and so $r^{n-1}m \in N$, as required. □

Definition 2.20. A proper submodule N of M is called *finitely compactly packed* if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of prime submodules of M with $N \subseteq \cup_{\alpha\in\Lambda} N_\alpha$, there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $N \subseteq \cup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called *compactly packed* . A module M is said to be *finitely compactly packed* (*compactly packed*), if every proper submodule N of M is finitely compactly packed (compactly packed) submodule (see [1]).
We will call a proper submodule N of M as ϕ-(k,n)-*closed finitely compactly packed* if for each family $\{P_\alpha\}_{\alpha\in\Lambda}$ of ϕ-(k,n)-closed submodules of M with $N \subseteq \cup_{\alpha\in\Lambda} P_\alpha$, there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $N \subseteq \cup_{i=1}^{n} P_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called ϕ-(k,n)-*closed compactly packed*. A module M is said to be ϕ-(k,n)-*closed finitely compactly packed* (*compactly packed*) if every proper submodule is a ϕ-(k,n)-closed finitely compactly packed (compactly packed).
For more details concerning finitely compactly packed (compactly packed) submodule of a module refer to [1], [7] and [18].

Corollary 2.21. *Let M be an R-module and ϕ_1, $\phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all submodules of M with $\phi_1 \leqslant \phi_2$. If M is a ϕ_2-(k,n)-closed finitely compactly packed (compactly packed) module, then M is a ϕ_1-(k,n)-closed finitely compactly packed (compactly packed) module.*

Proof. Clear by Proposition 2.2. □

Corollary 2.22. *Every ϕ-$(k+1,n+1)$-closed finitely compactly packed (compactly packed) module is a ϕ-(k,n)-closed finitely compactly packed (compactly packed) module.*

Proof. Apply Proposition 2.3. □

Theorem 2.23. *Let $f : M \to M'$ be an epimorphism R-module, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \to S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold:*
(1) *If M is a ϕ-(k,n)-closed finitely compactly packed module such that for every ϕ'-(k,n)-closed submodule L of M' we have $f^{-1}(\phi'(L)) \subseteq \phi(f^{-1}(L))$, then M' is a ϕ'-(k,n)-closed finitely compactly packed module.*
(2) *If M' is a ϕ'-(k,n)-closed finitely compactly packed module such that for every ϕ-(k,n)-closed submodule P of M we have $ker f \subseteq P$ and $f(\phi(P)) \subseteq \phi'(f(P))$, then M is a ϕ-(k,n)-closed finitely compactly packed module.*

Proof. (1) Let N' be a proper submodule of M'. Suppose that $N' \subseteq \cup_{\alpha\in\Lambda} P'_\alpha$, where P'_α is a ϕ'-(k,n)-closed submodule of M' for each $\alpha \in \Lambda$. We have $f^{-1}(N') \subseteq f^{-1}(\cup_{\alpha\in\Lambda} P'_\alpha)$, so $f^{-1}(N') \subseteq \cup_{\alpha\in\Lambda} f^{-1}(P'_\alpha)$. Since P'_α is a ϕ'- (k,n)-closed submodule of M' and $f^{-1}(\phi'(P'_\alpha)) \subseteq \phi(f^{-1}(P'_\alpha))$ for each $\alpha \in \Lambda$, by Theorem 2.15, we get $f^{-1}(P'_\alpha)$ is a ϕ-(k,n)-closed submodule of M for each $\alpha \in \Lambda$. But M is a ϕ-(k,n)-closed finitely compactly packed module, thus there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $f^{-1}(N') \subseteq \cup_{i=1}^n f^{-1}(P'_{\alpha_i})$, hence $f^{-1}(N') \subseteq f^{-1}(\cup_{i=1}^n P'_{\alpha_i})$. Since f is an epimorphism R-module, then $N' \subseteq \cup_{i=1}^n P'_{\alpha_i}$. Thus N' is a ϕ'-(k,n)-closed finitely compactly packed submodule of M' and hence M' is a ϕ'-(k,n)-closed finitely compactly packed module.
(2) Suppose that N is a proper submodule of M with $N \subseteq \cup_{\alpha\in\Lambda} P_\alpha$ where P_α is a ϕ-(k,n)-closed submodule of M for every $\alpha \in \Lambda$. We have $f(N) \subseteq f(\cup_{\alpha\in\Lambda} P_\alpha)$. Since P_α is a ϕ-(k,n)- closed submodule of M, $f(\phi(P_\alpha)) \subseteq \phi'(f(P_\alpha))$ and $ker f \subseteq P_\alpha$ for each $\alpha \in \Lambda$, by Theorem 2.15, we get $f(P_\alpha)$ is a ϕ'-(k,n)-closed submodule of M'. Since M' is a ϕ'-(k,n)-closed finitely compactly packed module, then there exist $\alpha_1, ..., \alpha_n \in \Lambda$ such that $f(N) \subseteq \cup_{i=1}^n f(P_{\alpha_i})$. Now, assume that $n \in N$, therefore $f(n) \in f(\cup_{i=1}^n P_{\alpha_i})$, so $f(n) = f(m)$ for some $m \in \cup_{i=1}^n P_{\alpha_i}$. Thus $n - m \in ker f \subseteq P_{\alpha_j}$ and $m \in P_{\alpha_j}$ for some $\alpha_j \in \{\alpha_1, ..., \alpha_n\}$. Thus $n \in P_{\alpha_j}$ and hence $n \in \cup_{i=1}^n P_{\alpha_i}$. It follows that $N \subseteq \cup_{i=1}^n P_{\alpha_i}$. So N is a ϕ-(k,n)-closed finitely compactly packed submodule of M and hence M is a ϕ-(k,n)-closed finitely compactly packed module. □

Theorem 2.24. *Let M be an R-module, S be a multiplicatively closed set in R and $\phi : S(M) \to S(M) \cup \{\emptyset\}$, $\phi_S : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$*

be two functions such that $\phi_S(T) = S^{-1}(\phi(\pi^{-1}(T))$ *for every submodule* T *of* $S^{-1}M$ *where* $\pi : M \to S^{-1}M$ *by* $\pi(m) = \frac{m}{1}$ *for each* $m \in M$ *and* $\frac{x}{1} \notin T$ *for some* $x \in M$*. If* M *is a* ϕ*-*(k,n)*-closed compactly packed module, then* $S^{-1}M$ *is a* ϕ_S*-*(k,n)*-closed compactly packed module.*

Proof. Let T be a proper submodule of $S^{-1}M$. Suppose that $T \subseteq \cup_{\alpha\in\Lambda} P_\alpha$ where P_α is a ϕ_S-(k,n)-closed submodule of $S^{-1}M$ for each $\alpha \in \Lambda$. We have $\pi^{-1}(T) \subseteq \pi^{-1}(\cup_{\alpha\in\Lambda} P_\alpha) = \cup_{\alpha\in\Lambda}\pi^{-1}(P_\alpha)$. Since $\pi^{-1}(T)$ is a proper submodule of M and $\pi^{-1}(P_\alpha)$ is a ϕ-(k,n)-closed submodule of M for each $\alpha \in \Lambda$, by Proposition 2.16., we get $\pi^{-1}(T) \subseteq \pi^{-1}(P_\beta)$ for some $\beta \in \Lambda$, because M is a ϕ-(k,n)-closed compactly packed module. On the other hand, we write $S^{-1}(\pi^{-1}(T)) = T$ because $S^{-1}(\pi^{-1}(T)) = \{\frac{m}{s} \mid m \in \pi^{-1}(T), s \in S\} = \{\frac{m}{1}\frac{1}{s} \mid \frac{m}{1} \in T, s \in S\} = T$ (so that we can consider submodule T as $S^{-1}R$-module $S^{-1}M$). Therefore $S^{-1}(\pi^{-1}(T)) \subseteq S^{-1}(\pi^{-1}(P_\beta))$ implies that $T \subseteq P_\beta$ for some $\beta \in \Lambda$. So $S^{-1}M$ is a ϕ_S-(k,n)- closed compactly packed module. □

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Some Results on ϕ -(k,n)-Closed Submodules

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برخی نتایج در مورد زیر مدول های بسته -ϕ-(k, n)-

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فرض کنیم R یک حلقه جابجایی و یکدار، M یک R - مدول یکانی، $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ یک تابع و $S(M)$ مجموعه تمام زیر مدول های M باشد، زیرمدول سره N از M را یک زیرمدول-ϕ - نیم جاذب می نامیم اگر از $r^n m \in N \setminus \phi(N)$ که در آن $r \in R$ ، $m \in M$ و n یک عدد صحیح مثبت می باشد، نتیجه بگیریم $r^n \in (N : M)$ یا $r^{n-1} m \in N$. اکنون فرض کنیم k و n دو عدد صحیح مثبت باشرط $k > n$ باشد، زیرمدول سره N از M را یک زیرمدول - ϕ - (k, n) -بسته می نامیم هرگاه از $r^k m \in N \setminus \phi(N)$ که در آن $r \in R$ ، $m \in M$ می باشد، نتیجه شود $r^n \in (N : M)$ یا $r^{n-1} m \in N$. در این مقاله ابتدا برخی نتایج عمومی و اولیه در مورد زیرمدول های -ϕ - (k, n)- بسته، اثبات می شود. علاوه بر آن، چند نتیجهٔ مهم در مورد این زیرمدول ها از انواع مدول ها بیان و اثبات می شوند.

کلمات کلیدی: زیرمدول - ϕ - (k, n) - بسته، زیرمدول - ϕ - نیم جاذب، ایده آل - (m, n) - بسته، ایده آل - ψ- (m, n) - بسته، زیرمدول - ϕ - اول.