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ON GP-FLATNESS PROPERTY

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Abstract. It is well-known that, using principal weak flatness property, some important monoids are characterized, such as regular monoids, left almost regular monoids, and so on. In this article, we recall a generalization of principal weak flatness called GP-flatness, and characterize monoids by this property of their right (Rees factor) acts. Also we investigate GP coherent monoids.

1. Introduction

In [6] Husheng and Chongqing introduced GP-flatness property as a generalization of principal weak flatness, and characterized monoids by this property of their right acts in some cases. In this paper first we give an equivalent definition of GP-flatness and give some basic results. Then we investigate GP-flatness property for (mono)cyclic right S-acts. Also we give a characterization of monoids by comparing this property of their ((mono)cyclic, Rees factor) right acts with other properties. Finally we characterize GP coherent monoids.

Throughout this paper S stands for a monoid. We refer the reader to [7] for basic definitions and terminology relating to semigroups and acts over monoids.

Recall that a monoid S is called *right (left) reversible* if for every $s, t \in S$, there exist $u, v \in S$ such that $us = vt$ ($su = tv$). A monoid S is called *left (right) collapsible* if for every $s, t \in S$ there exists $z \in S$

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such that $zs = zt(sz = tz)$. A submonoid P of S is called *weakly left collapsible* if for every $s, t \in P, z \in S$, $sz = tz$ implies the existence of $u \in P$ such that $us = ut$. A right ideal K_S of a monoid S is called *left stabilizing* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

A non-empty set A is called a right (left) S-act, usually denoted A_S $({}_SA)$, if S acts on A unitarily from the right (left), that is, there exists a mapping $A \times S \to A$, $(S \times A \to A)$, $(a, s) \mapsto as$ $((s, a) \mapsto sa)$, satisfying the conditions $(as)t = a(st)$ $(s(ta) = (st)a)$ and $a1 = a$ $(1a = a)$, for all $a \in A$, and all $s, t \in S$. A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A_S satisfies *Condition* (P') if for all $a, a' \in A$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. A right S-act A_S satisfies *Condition* (E) if for all $a \in A_S$, $s, t \in S$, $as = at$ implies that there exist $a' \in A_S$, and $u \in S$ such that $a = a'u$ and $us = ut$. A_S satisfies *Condition* (E') if for all $a \in A_S$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, and $u \in S$ such that $a = a'u$ and $us = ut$. A right S-act A_S satisfies *Condition* (EP) if for all $a \in A_S, s, t \in S$, $as = at$ implies that there exist $a' \in A_S, u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A right S-act A_S satisfies *Condition* $(E'P)$ if for all $a \in A_S, s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A_S$, and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. It is obvious that $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$, and in [1, 2] it was shown that the converses are not true in general.

2. Main results

In this section first, we recall a generalization of principal weak flatness property, called GP-flatness, and give some basic results. Recall from [7] that an act A_S is called *principally weakly flat* if the functor $A_S \otimes {}_S-$ preserves all embeddings of principal left ideals into S. This is equivalent to say that $as = a's$ for $a, a' \in A_S, s \in S$ implies $a \otimes s = a' \otimes s$ in the tensor product $A_S \otimes {}_SSs$.

Definition 2.1. [6] A right S-act A_S is called *GP-flat* if $as = a's$ for $a, a' \in A_S, s \in S$ implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes Ss^n$.

It is clear that every principally weakly flat S-act is GP-flat, but Example 2.19 shows that the converse is not true in general.

Theorem 2.2. *A right S-act A_S is GP-flat if and only if $as = a's$ for $a, a' \in A_S, s \in S$ implies that there exist $n, m \in \mathbb{N}$ $(n \geq m)$ such that $a \otimes s^n = a' \otimes s^m$ and $a \otimes s^m = a' \otimes s^n$ in $A_S \otimes Ss^m$.*

Proof. One way round is clear. Suppose that $as = a's$ for $a, a' \in A_S, s \in S$. By assumption there exist $n, m \in \mathbb{N}$ $(n \geq m)$ such that $a \otimes s^n = a' \otimes s^m$ and $a \otimes s^m = a' \otimes s^n$ in $A_S \otimes Ss^m$. If $n = m$ then obviously A_S is GP-flat, otherwise $n > m$ implies that $as^{n-m} = a's^{n-m}$, thus

$$a \otimes s^n = a \otimes s^{n-m}s^m = as^{n-m} \otimes s^m = a's^{n-m} \otimes s^m = a' \otimes s^{n-m}s^m = a' \otimes s^n$$

in $A_S \otimes Ss^m$. Now

$$a \otimes s^m = a' \otimes s^n = a \otimes s^n = a' \otimes s^m$$

in $A_S \otimes Ss^m$, and so A_S is GP-flat as required. □

Lemma 2.3. *[7] Let A_S be a right S-act and ${}_SM$ be a left S-act. Then $a \otimes m = a' \otimes m'$ for $a, a' \in A_S$ and $m, m' \in {}_SM$, if and only if there exist $s_1, ..., s_k, t_1, ...t_k \in S, b_1, ..., b_{k-1} \in A_S, n_1, ..., n_k \in {}_SM$ such that*

$$\begin{array}{rr} & s_1n_1 = m \\ as_1 = b_1t_1 & s_2n_2 = t_1n_1 \\ b_1s_2 = b_2t_2 & s_3n_3 = t_2n_2 \\ ... & ... \\ b_{k-1}s_k = a't_k & m' = t_kn_k. \end{array}$$

Lemma 2.4. *[6] Let S be a monoid and A_S be a right S-act. The following statements are equivalent:*
(1) A_S is GP-flat;
(2) $a \otimes s = a' \otimes s$, $a, a' \in A_S$ and $s \in S$, implies that there exist $s_1, ..., s_k, t_1, ...t_k \in S, b_1, ..., b_{k-1} \in A_S, n \in \mathbb{N}$ such that

$$\begin{array}{rr} & s_1s^n = s^n \\ as_1 = b_1t_1 & s_2s^n = t_1s^n \\ b_1s_2 = b_2t_2 & s_3s^n = t_2s^n \\ ... & ... \\ b_{k-1}s_k = a't_k & s^n = t_ks^n. \end{array}$$

k is called the length of the above S-tossing and the minimum length of the existing S-tossing is denoted by $d_{s^n}(a, a')$.

Remark 2.5. The following statements are easy consequences of the definition:

1. if $\{A_i \mid i \in I\}$ is a chain of subacts of an act A_S, and every $A_i, i \in I$ is GP-flat, then $\bigcup_{i \in I} A_i$ is GP-flat;
2. $A = \coprod_{i \in I} A_i$ is GP-flat, if and only if every $A_i, i \in I$, is GP-flat;

(3) the right S-act S_S is GP-flat;
(4) the one element right S-act Θ_S is GP-flat.

Theorem 2.6. *[10] For any family $\{A_i|\ i \in I\}$ of right S-acts, if $\prod_{i\in I} A_i$ is GP-flat, then for every $i \in I$, A_i is GP-flat.*

Theorem 2.7. *Any retract of a GP-flat right S-act is GP-flat.*

Proof. Let A_S be a retract of B_S, and B_S is a GP-flat right S-act. Suppose that $as = a's$, for $a, a' \in A_S, s \in S$, since A_S is a retract of B_S, there are homomorphisms $\varphi : A_S \to B_S$ and $\varphi' : B_S \longrightarrow A_S$ such that $\varphi'\varphi = 1_A$. Then we have $\varphi(as) = \varphi(a's)$ or $\varphi(a)s = \varphi(a')s$. Since $\varphi(a), \varphi(a') \in B_S$, by assumption there exists $n \in \mathbb{N}$ such that $\varphi(a) \otimes s^n = \varphi(a') \otimes s^n$ in the tensor product $B_S \otimes {}_SSs^n$. Suppose now that this equality is realized by a tossing

$$\begin{array}{cc} & s_1 s^n = s^n \\ \varphi(a)s_1 = b_1t_1 & s_2s^n = t_1s^n \\ b_1s_2 = b_2t_2 & s_3s^n = t_2s^n \\ \dots & \dots \\ b_{k-1}s_k = \varphi(a')t_k & s^n = t_ks^n, \end{array}$$

of length k, where $s_1, ..., s_k, t_1, ..., t_k \in S$, $b_1, ..., b_{k-1} \in B_S$. Then $\varphi'(\varphi(a)s_1) = \varphi'(b_1t_1)$, and so $as_1 = \varphi'(b_1)t_1$. Similarly we have $\varphi'(b_{i-1})$ $s_i = \varphi'(b_i)t_i$, $2 \le i \le k-1$, and $\varphi'(b_{k-1})s_k = a't_k$. Let $\varphi'(b_i) = a_i$, for $1 \le i \le k-1$. Substituting elements $\varphi'(b_i)$ by a_i, we obtain a tossing realizing the equality $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes {}_SSs^n$. □

The following proposition is an easy consequence of the definition.

Proposition 2.8. *Let S be an idempotent monoid. Then every GP-flat right S-act is principally weakly flat.*

Lemma 2.9. *[7] Let ρ be a right and λ a left congruence on a monoid S. Then $[u]_\rho \otimes [s]_\lambda = [v]_\rho \otimes [t]_\lambda$ in $S/\rho \otimes S/\lambda$ for $u, v, s, t \in S$, if and only if $us(\rho \vee \lambda)vt$.*

Lemma 2.10. *[7] Let ${}_SA$ be a left S-act and $a \in {}_SA$. Then $g : S/ker\rho_a \to Sa$ with $g([t]) = ta$ for every $t \in S$ is an S-isomorphism.*

The following result is an immediate corollary of Lemmas 2.9 and 2.10.

Corollary 2.11. *Let ρ be a right congruence on a monoid S and $s \in S$. Then $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$, $u, v \in S$ and $n \in \mathbb{N}$, if and only if $u(\rho \vee ker\rho_{s^n})v$.*

Theorem 2.12. *Let S be a monoid and ρ be a right congruence on S. Then the right S-act S/ρ is GP-flat if and only if for all $u, v, s \in S$ with $(us)\rho(vs)$, there exists $n \in \mathbb{N}$, such that $u(\rho \vee ker\rho_{s^n})v$.*

Proof. Necessity. Suppose that the right S-act S/ρ is GP-flat and let $(us)\rho(vs)$ for $u, v, s \in S$. That is, $[u]_\rho s = [v]_\rho s$, and so by hypothesis there exists $n \in \mathbb{N}$, such that $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$. Hence $u(\rho \vee ker\rho_{s^n})v$, by Lemma 2.11.

Sufficiency. Let $[u]_\rho s = [v]_\rho s$, for $u, v, s \in S$ and $n \in \mathbb{N}$. That is, $(us)\rho(vs)$, and so by hypothesis there exists $n \in \mathbb{N}$, such that $u(\rho \vee ker\rho_{s^n})v$. Now $[u]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $S/\rho \otimes Ss^n$, by Lemma 2.11, and so the right S-act S/ρ is GP-flat. □

Corollary 2.13. *The right ideal $zS, z \in S$, is GP-flat if and only if for all $x, y, s \in S, zxs = zys$ implies that there exists $n \in \mathbb{N}$ such that $x(ker\lambda_z \vee ker\rho_{s^n})y$.*

Proof. Since $zS \cong S/ker\lambda_z$, apply Theorem 2.12 for $\rho = ker\lambda_z, z \in S$. □

We recall from [7] that a monoid S is called regular, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$.

Definition 2.14. [6] Let $s \in S$. s is called *right (left) generally regular*, if there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n(s^n = s^nxs)$. A monoid S is called right generally regular if every $s \in S$ is right generally regular.

It is clear that the class of generally right regular monoids contains all regular monoids.

Theorem 2.15. *Let $s \in S$. If the monocyclic right S-act $S/\rho(s^2, s)$ is GP-flat, then s is a right generally regular.*

Proof. Let $\rho = \rho(s^2, s)$. Since S/ρ is GP-flat and $(ss)\rho(1s)$, by Theorem 2.12 there exists $n \in \mathbb{N}$ such that $s(\rho \vee ker\rho_{s^n})1$. Let $\sigma = \rho \circ ker\rho_{s^n}$ and k be the smallest non-negative integer such that $s\sigma^k 1$. Then there exist $u_1, v_1, ..., u_k, v_k \in S$ such that

$$s = u_1\rho v_1 \quad u_2\rho v_2 \quad \dots \quad u_k\rho v_k$$
$$v_1s^n = u_2s^n \quad v_2s^n = u_3s^n \quad \dots \quad v_ks^n = s^n.$$

By [7, III, 8.5] there exist $m_i, p_i \geq 0$ $(1 \leq i \leq k)$ such that $s^{m_i}u_i = s^{p_i}v_i$ and $s^{j_i}u_i, s^{l_i}v_i \in sS$, for $0 \leq j_i < m_i, 0 \leq l_i < p_i$. The following cases may occur:

Case 1. $p_k > 0$. Let $l_k = 0$, then there exists $x \in S$ such that $v_k = sx$ and so $s^n = sxs^n$.

Case 2. $p_k = 0$. Then $s^{m_k}u_k = v_k$, and so $s^n = s^{m_k}u_k s^n$. If $m_k > 0$ then $s^n = ss^{m_k-1}u_k s^n$, otherwise $m_k = 0$ and so $u_k = v_k$ which contradicts to minimality of k. □

Definition 2.16. Let S be a monoid. A (proper) right ideal K_S of S is called *G-left stabilizing* if

$$(\forall s \in S)(\forall z \in S \setminus K_S)(zs \in K_S \Rightarrow \exists n \in \mathbb{N}, k \in K_S : zs^n = ks^n)$$

It is clear that every left stabilizing right ideal is G-left stabilizing, but Example 2.19 will show the converse is not true in general.

Proposition 2.17. *If the right ideal sS, $s \in S$, is G-left stabilizing, then s is right generally regular.*

Proof. Suppose that the right ideal $sS, s \in S$ is G-left stabilizing. So there exist $k \in sS, n \in \mathbb{N}$ such that $s^n = ks^n$. Since $k \in sS$ there exists $x \in S$ such that $k = sx$, and so s is right generally regular. □

Theorem 2.18. *[6] Let S be a monoid and K_S be a right ideal of S. The right Rees factor S-act S/K_S is GP-flat if and only if K_S is a G-left stabilizing right ideal.*

Example 2.19. Let $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. It is easy to check that K_S is G-left stabilizing and so the right Rees factor S-act S/K_S is GP-flat, but it is not principally weakly flat.

Let J be a proper right ideal of a monoid S. If x, y and z denote elements not belonging to S, define

$$A(J) = (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$$

and define a right S-action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us) & us \notin J \\ (z, us) & us \in J \end{cases}$$

$$(y, u)s = \begin{cases} (y, us) & us \notin J \\ (z, us) & us \in J \end{cases}$$

$$(z, u)s = (z, us).$$

$A(J)$ is a right S-act, which is usually denoted by $S_S \coprod^J S_S$, and we have

Theorem 2.20. *[6] $A(J)$ is GP-flat if and only if the right ideal J is G-left stabilizing.*

From Theorems 2.18 and 2.20 we have

Corollary 2.21. *Let J be a proper right ideal of a monoid S. S/J_S is GP-flat if and only if $A(J)$ is GP-flat.*

3. Characterization of monoids by GP-flatness property of ((mono)cyclic) right S-acts

In this section we give a characterization of monoids by GP-flatness property. Qiao and Wei in [6], Theorems 3.4 and 3.7 have provided the condition for a monoid so that all right S-acts are GP-flat. In the next theorem we add some more equivalent statements.

Theorem 3.1. *For any monoid S the following statements are equivalent:*

(1) *all right S-acts are GP-flat;*
(2) *all finitely generated right S-acts are GP-flat;*
(3) *all cyclic right S-acts are GP-flat;*
(4) *all monocyclic right S-acts are GP-flat;*
(5) *all monocyclic right S-acts of the form $S/\rho(s^2, s)$, $s \in S$, are GP-flat;*
(6) *all right Rees factor S-acts are GP-flat;*
(7) *all right Rees factor S-acts of the form S/sS, $s \in S$, are GP-flat;*
(8) *S is a right generally regular monoid.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) and (3) ⇒ (6) ⇒ (7) are obvious.

(5) ⇒ (8) Let $s \in S$. Then by assumption the monocyclic right S-act $S/\rho(s^2, s)$ is GP-flat, and so s is right generally regular by Theorem 2.15.

(7) ⇒ (8) Let $s \in S$. Then by assumption the right Rees factor S-act S/sS is GP-flat, and so the right ideal sS is G-left stabilizing by Theorem 2.18. Then s is right generally regular by Proposition 2.17.

(8) ⇒ (1) It follows from [6, 3.4]. □

Theorem 3.2. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts are free;*
(2) *all finitely generated GP-flat right S-acts are free;*
(3) *all cyclic GP-flat right S-acts are free;*
(4) *all monocyclic GP-flat right S-acts are free;*
(5) *all GP-flat right S-acts are projective generators;*
(6) *all finitely generated GP-flat right S-acts are projective generators;*
(7) *all cyclic GP-flat right S-acts are projective generators;*
(8) *all monocyclic GP-flat right S-acts are projective generators;*

(9) *all GP-flat right S-acts are projective;*
(10) *all finitely generated GP-flat right S-acts are projective;*
(11) *all GP-flat right S-acts are strongly flat;*
(12) *all finitely generated GP-flat right S-acts are strongly flat;*
(13) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (8), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8), (1) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (12) and (1) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious.

(8) $\Rightarrow$ (13) If all monocyclic GP-flat right S-acts are projective generators, then all monocyclic right S-acts satisfying Condition (P) are projective generators and so by [7, IV, 12.8], $S = \{1\}$.

(12) $\Rightarrow$ (13) Assume all finitely generated GP-flat right S-acts are strongly flat. Then S is aperiodic by [7, IV, 10.2]. Let $1 \neq s \in S$, then there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$ and so $e = s^n$ is an idempotent different from 1. It is easy to see that eS is a G-left stabilizing right ideal and so the right S-act $S_S \coprod^{eS} S_S$ is GP-flat by Theorem 2.20. Thus by assumption it is strongly flat (satisfies Condition (P)), which is a contradiction [see 7, III, 13.14]. So $S = \{1\}$.

(13) $\Rightarrow$ (1) It is obvious. ☐

Lemma 3.3. *If all monocyclic GP-flat right S-acts are strongly flat, then all monocyclic right S-acts are strongly flat.*

Proof. Suppose that all GP-flat monocyclic right S-acts are strongly flat, then all monocyclic right S-acts satisfying Condition (P) are strongly flat and so S is aperiodic by [7, IV, 10.2]. Thus for every $s \in S$ there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$, which gives that S is generally right regular. Now by Theorem 3.1, all right S-acts are GP-flat. ☐

Theorem 3.4. *For any monoid S the following statements are equivalent:*
(1) all cyclic GP-flat right S-acts are projective;
(2) all monocyclic GP-flat right S-acts are projective;
(3) all cyclic GP-flat right S-acts are strongly flat;
(4) all monocyclic GP-flat right S-acts are strongly flat;
(5) $S = \{1\}$ *or* $S = \{1, 0\}$.

Proof. Implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By assumption all monocyclic GP-flat right S-acts are strongly flat, and so by Lemma 3.3, all monocyclic right S-acts are strongly flat. Thus by [7, IV, 10.10], $S = \{1\}$ or $S = \{1, 0\}$.

(2) $\Leftrightarrow$ (4) It follows from [7, III, 17.13].

(5) $\Rightarrow$ (1) It follows from [7, IV, 11.14]. $\square$

From [4, 1.12] and [4, 1.8] we have the following.

Theorem 3.5. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts are regular;*
(2) *all finitely generated GP-flat right S-acts are regular;*
(3) *all cyclic GP-flat right S-acts are regular;*
(4) *$S = \{1\}$ or $S = \{0, 1\}$.*

Theorem 3.6. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts satisfy Condition (E);*
(2) *all finitely generated GP-flat right S-acts satisfy Condition (E);*
(3) *all cyclic GP-flat right S-acts satisfy Condition (E);*
(4) *all monocyclic GP-flat right S-acts satisfy Condition (E);*
(5) *$S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) It follows from Theorem 3.4.

(5) $\Rightarrow$ (1) It is straightforward. $\square$

We recall from [9] that a right S-act A_S is *principally weakly kernel flat* if and only if it satisfies Condition (PWP) and the following condition holds:

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z, x \in S \ : ker\rho_x \subseteq ker\rho_z)((asx = a's'x,$$
$$sz = tz), (atx = a't'x, s'z = t'z)) \Rightarrow a \otimes (sx, tx) = a' \otimes (s'x, t'x)$$

in $A_S \otimes {}_SP$, where ${}_SP = \{(ux, vx) | u, v \in S, uz = vz\}$. A right S-act A_S is *translation kernel flat* if and only if it satisfies Condition (PWP) and the following condition holds:

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z \in S)$$
$$((as = a's', sz = tz), (at = a't', s'z = t'z)) \Rightarrow a \otimes (s, t) = a' \otimes (s', t')$$

in $A_S \otimes {}_S(ker\rho_s)$.

A monoid S is called a left PSF monoid if every principal left ideal of S is strongly flat (as a left S-act), equivalently if and only if for $su = tu$, $s, t, u \in S$ there exists $r \in S$ such that $ru = u$ and $sr = tr$.

Lemma 3.7. *[5] Let S be a monoid, and J be a proper right ideal of S. Then $A(J) = S_S \coprod^J S_S$ fails to satisfy Condition (PWP).*

Theorem 3.8. *For any monoid S the following statements are equivalent:*

(1) *S is right cancellative;*
(2) *S is left PSF and all GP-flat right S-acts are principally weakly kernel flat;*
(3) *S is left PSF and all GP-flat right S-acts are translation kernel flat;*
(4) *S is left PSF and all GP-flat right S-acts satisfy Condition (P');*
(5) *S is left PSF and all GP-flat right S-acts satisfy Condition (PWP).*

Proof. Since principally weakly kernel flat $\Rightarrow$ translation kernel flat $\Rightarrow$ Condition (PWP), and Condition (P') $\Rightarrow$ Condition (PWP), implications $(2) \Rightarrow (3) \Rightarrow (5), (4) \Rightarrow (5)$ are obvious.

$(1) \Rightarrow (4)$ Since S is right cancellative, S is left PSF. Also in this case by [3, 2.2] all torsion free right S-acts satisfy Condition (P'), and so all GP-flat right S-acts satisfy Condition (P').

$(5) \Rightarrow (1)$ If S is right cancellative, then we are done. Otherwise, if $J = \{s \in S \mid s \text{ is not right cancellable}\}$, then J is a proper right ideal of S. Let $j \in J$, then there exist $s, t \in S$ such that $s \neq t$ and $sj = tj$. Since S is left PSF, there exists $r \in S$ such that $rj = j$ and $sr = tr$. If $r \notin I$, then r is right cancellable and so $s = t$, a contradiction. Thus $r \in J$, and so the right ideal J is left stabilizing, thus the right S-act $S_S \coprod^J S_S$ is GP-flat by Theorem 2.20. Hence by assumption $S_S \coprod^J S_S$ satisfies Condition (PWP), which contradicts Lemma 3.7.

$(1) \Rightarrow (2)$ Suppose that S is right cancellative and A_S is GP-flat. By $(1) \Leftrightarrow (5)$, S is left PSF and A_S satisfies Condition (PWP) . Now let $asx = a's'x, sz = tz$ and $atx = a't'x, s'z = t'z$, for $a, a' \in A_S, s, s', t, t', z, x \in S$. Since S is right cancellative and A_S is torsion free, we have $as = a's'$, and so $a \otimes s = a' \otimes s'$ in $A_S \otimes {}_SS$ by [7, II, 5.13]. It is obvious that $a \otimes s = a' \otimes s'$ in $A_S \otimes {}_SS$ if and only if $a \otimes (s, s) = a' \otimes (s', s')$ in $A_S \otimes {}_S\Delta$. But S is right cancellative and so ${}_SP = \{(ux, vx) | u, v \in S, uz = vz\} = {}_S\Delta$, as required. □

For a monoid S we have the following implications:
right cancellative $\Rightarrow$ left almost regular $\Rightarrow$ left PP $\Rightarrow$ left PSF, so in Theorem 3.8 we can replace left PSF by left PP or left almost regular. Also notice that Theorem 3.8 is true for finitely generated right S-acts, and right S-acts generated by at most (exactly) two elements.

From [3, Theorem 2.9], [5, Theorem 2.9] and Theorem 3.8 we have the following.

Corollary 3.9. *For any monoid S the following statements are equivalent:*

(1) *S is right cancellative;*
(2) *there exists a regular left S-act and all GP-flat right S-acts are principally weakly kernel flat;*
(3) *there exists a regular left S-act and all GP-flat right S-acts are translation kernel flat;*
(4) *there exists a regular left S-act and all GP-flat right S-acts satisfy Condition (P');*
(5) *there exists a regular left S-act and all GP-flat right S-acts satisfy Condition (PWP).*

We recall from [7] that an S-act A_S is divisible if $Ac = A$ for any left cancellable element $c \in S$.

Theorem 3.10. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts are divisible;*
(2) *all finitely generated GP-flat right S-acts are divisible;*
(3) *all cyclic GP-flat right S-acts are divisible;*
(4) *all monocyclic GP-flat right S-acts are divisible;*
(5) *all left cancellable elements of S are left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By Theorem 2.5, $S_S \cong S/\rho(1,1)$ is GP-flat, as a monocyclic right S-act. Thus by assumption S_S is divisible, and so $Sc = S$, for any left cancellable element $c \in S$. That is, there exists $x \in S$ such that $xc = 1$.

(5) $\Rightarrow$ (1) It is clear from [7, III, 2.2]. □

We recall from [7] that a right S-act A_S is (strongly) faithful, if for $s, t \in S$ the equality $as = at$, for all (some) $a \in A_S$, implies that $s = t$. It is obvious that every strongly faithful act is faithful.

Theorem 3.11. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts are (strongly) faithful;*
(2) *all finitely generated GP-flat right S-acts are (strongly) faithful;*
(3) *all cyclic GP-flat right S-acts are (strongly) faithful;*
(4) *all GP-flat right Rees factor S-acts are (strongly) faithful;*

(5) $S = \{1\}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5) By Theorem 2.5, the one-element right S-act $\Theta_S = S/S_S$ is GP-flat, and so by assumption it is (strongly) faithful. By [7, I, 5.25], for every $1 \neq s \in S$, there exists $u \in S$ such that $[us]_{\rho_S} \neq [u]_{\rho_S}$, a contradiction. Thus $S = \{1\}$, as required.

(5) ⇒ (1) It is obvious. □

Theorem 3.12. *For any monoid S the following statements are equivalent:*

(1) *all GP-flat right S-acts are completely reducible;*
(2) *all finitely generated GP-flat right S-acts are completely reducible;*
(3) *all cyclic GP-flat right S-acts are completely reducible;*
(4) *all monocyclic GP-flat right S-acts are completely reducible;*
(5) *S is a group.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5) By Theorem 2.5, $S_S \cong S/\rho(1, 1)$ is GP-flat as a monocyclic right S-act, and so by assumption S_S is completely reducible. Thus S is a group by [7, I, 5.33].

(5) ⇒ (1) It follows from [7, I, 5.34]. □

Definition 3.13. [6] An element s of S is called generally left almost regular if there exist elements $r, r_1, ..., r_m, s_1, ..., s_m \in S$, right cancellable elements $c_1, ...c_m \in S$, and $n \in \mathbb{N}$ such that

$$\begin{gathered} s_1c_1 = sr_1 \\ s_2c_2 = s_1r_2 \\ \cdots \\ s_mc_m = s_{m-1}r_m \\ s^n = s_mrs^n. \end{gathered}$$

A monoid S is called generally left almost regular if all its elements are generally left almost regular.

It is obvious that every left almost regular monoid is generally left almost regular.

Theorem 3.14. *[6] For any monoid S the following statements are equivalent:*

(1) *all torsion free right S-acts are GP-flat;*

(2) *all cyclic torsion free right S-acts are GP-flat;*
(3) *all torsion free right Rees factor S-acts are GP-flat;*
(4) *S is a generally left almost regular monoid.*

We recall from [11] that a right S-act A_S is $\Re$-torsion free if $ac = a'c$ and $a\Re a'$, for $a, a' \in A_S, c \in S$, c right cancellable, imply that $a = a'$. We also recall that $\rho_{\Re TF}(u, v)$ is the smallest right congruence containing (u, v), such that $S/\rho_{\Re TF}(u, v)$ is $\Re$-torsin free.

Theorem 3.15. *For any monoid S, the following statements are equivalent:*

(1) *all cyclic $\Re$-torsion free right S-acts are GP-flat;*
(2) *for any $u, v, s \in S$ there exists $n \in \mathbb{N}$ such that $(u, v) \in (\rho_{\Re TF}(us, vs) \vee ker\rho_{s^n})$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $u, v, s \in S$. Then the cyclic right S-act $S/\rho_{\Re TF}(us, vs)$ is $\Re$-torsin free, and so it is GP-flat. Since $(us, vs) \in \rho_{\Re TF}(us, vs)$, by Theorem 2.12 there exists $n \in \mathbb{N}$ such that $(u, v) \in (\rho_{\Re TF}(us, vs) \vee ker\rho_{s^n})$.

(2) $\Rightarrow$ (1) Let τ be a right congruence on S, such that S/τ is $\Re$-torsion free and let $(us, vs) \in \tau$. By assumption there exists $n \in \mathbb{N}$ such that $(u, v) \in (\rho_{\Re TF}(us, vs) \vee ker\rho_{s^n})$, since $\rho_{\Re TF}(us, vs) \subseteq \tau$, we have $(u, v) \in \tau \vee ker\rho_{s^n}$. Thus S/τ is GP-flat by Theorem 2.12. □

Theorem 3.16. *For any monoid S, the following statements are equivalent:*

(1) *all $\Re$-torsion free right S-acts are GP-flat;*
(2) *S is right generally regular.*

Proof. It follows from [11, Lemma 4.1] □

Example 3.17. Let $S = (\mathbb{N}, \cdot)$ be the monoid of natural numbers with multiplication and let $A_S = S_S \coprod^{S\setminus\{1\}} S_S$. Since there exist no $x \in S \setminus \{1\}, n \in \mathbb{N}$ such that $2^n = x2^n$, A_S is not GP-flat by Theorem 2.20. But A_S satisfies Condition (E) by [7, III, 14.3(3)]. Thus it is natural to ask for monoids over which Condition (E) implies GP-flatness.

Theorem 3.18. *For any monoid S, the following statements are equivalent:*

(1) *all right S-acts satisfying Condition $(E'P)$ are GP-flat;*
(2) *all right S-acts satisfying Condition (EP) are GP-flat;*
(3) *all right S-acts satisfying Condition (E') are GP-flat;*
(4) *all right S-acts satisfying Condition (E) are GP-flat;*

(5) *S is right generally regular.*

Proof. Since $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$, implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) Let $s \in S$. If $sS = S$ then s is obviously regular. Suppose that $sS \neq S$. Then by [7, III, 14.3(3)] the right S-act $A_S = S_S \coprod^{sS} S_S$ satisfies Condition (E), and so by assumption it is GP-flat. Now by Theorem 2.20 the right ideal sS is G-left stabilizing and so s is right generally regular by Proposition 2.17.

(5) $\Rightarrow$ (1) It follows from Theorem 3.1. □

Note that above theorem is also true for finitely generated (at most (exactly) by two elements) right S-acts.

Theorem 3.19. *[10] Let S be a left PSF monoid. Then the right S-act A_S is GP-flat if and only if for $a, a' \in A_S$ and $s \in S$ with $as = a's$, there exist $n \in \mathbb{N}$ and $r \in S$ such that $rs^n = s^n$ and $ar = a'r$.*

Theorem 3.20. *Let S be a left PSF monoid. Then the following statements are equivalent:*

(1) *all divisible right S-acts are GP-flat;*
(2) *all principally weakly injective right S-acts are GP-flat;*
(3) *all fg-weakly injective right S-acts are GP-flat;*
(4) *all weakly injective right S-acts are GP-flat;*
(5) *all injective right S-acts are GP-flat;*
(6) *all cofree right S-acts are GP-flat;*
(7) *S is right generally regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (7) Let $s \in S$. If $sS = S$ then s is obviously regular, otherwise $I = sS$ is a proper right ideal of S. By assumption and [7, II, 4.3] the right S-act S/I can be embedded into a GP-flat right S- act, and so the equality $[1]_I s = [s]_I s$ implies that there exist $n \in \mathbb{N}$ and $r \in S$ such that $rs^n = s^n$ and $[1]_I r = [s]_I r$ by Theorem 3.19. The last equality implies that $r = sr$ or $r \in I$, and so s is right generally reqular.

(7) $\Rightarrow$ (1) It follows from Theorem 3.1. □

Lemma 3.21. *[7] Let A_S be a right S-act. Then A_S is a generator if and only if there exists an epimorphism $\pi : A_S \to S_S$.*

Theorem 3.22. *For any monoid S the following statements are equivalent:*

(1) *all generators right S-acts are GP-flat;*

(2) *all finitely generated generators right S-acts are GP-flat;*
(3) *all generators right S-acts that generated at most by three elements are GP-flat;*
(4) $S \times A_S$ *is GP-flat, for each right S-act A_S;*
(5) $S \times A_S$ *is GP-flat, for each finitely generated right S-act A_S;*
(6) $S \times A_S$ *is GP-flat, for each cyclic right S-act A_S;*
(7) $S \times A_S$ *is GP-flat, for each monocyclic right S-act A_S;*
(8) $S \times A_S$ *is GP-flat, for each monocyclic right S-act A_S of the form $S/\rho(s^2, s)$;*
(9) $S \times A_S$ *is GP-flat, for each right Rees factor S-act A_S;*
(10) $S \times A_S$ *is GP-flat, for each right Rees factor S-act A_S of the form $S/sS, s \in S$;*
(11) $S \times A_S$ *is GP-flat, for each generator right S-act A_S;*
(12) $S \times A_S$ *is GP-flat, for each finitely generated generator right S-act A_S;*
(13) $S \times A_S$ *is GP-flat, for each generator right S-act A_S, that generated at most by three elements;*
(14) *a right S-act A_S is GP-flat, if $Hom(A_S, S_S) \neq \phi$;*
(15) *a finitely generated right S-act A_S is GP-flat, if $Hom(A_S, S_S) \neq \phi$;*
(16) *a right S-act A_S that generated at most by two elements is GP-flat, if $Hom(A_S, S_S) \neq \phi$;*
(17) *S is right generally regular.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3), (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8), (6) \Rightarrow (9) \Rightarrow (10), (4) \Rightarrow (11) \Rightarrow (12) \Rightarrow (13)$ and $(14) \Rightarrow (15) \Rightarrow (16)$ are obvious.

$(3) \Rightarrow (1)$ Suppose that A_S is a generator and that $as = a's$ for $a, a' \in A_S, s \in S$. By Lemma 3.21, there exists an epimorphism $\pi : A_S \to S_S$, and so there exists $a'' \in A_S$ such that $\pi(a'') = 1$. Assume that $B_S = aS \cup a'S \cup a''S$, then by Lemma 3.21, B_S is a generator and so by assumption it is GP-flat. Thus there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $B_S \otimes Ss^n \subseteq A_S \otimes Ss^n$.

$(1) \Rightarrow (4)$ Suppose that all generators are GP-flat, and A_S is a right S-act. By Lemma 3.21, the coproduct $S_S \coprod (S \times A_S)$ is a generator, and so by assumption it is GP-flat. Thus $S \times A_S$ is GP-flat by Theorem 2.5.

$(8) \Rightarrow (17)$ let $s \in S$. By assumption $S \times S/\rho(s^2, s)$ is GP-flat and so by Theorem 2.6, $S/\rho(s^2, s)$ is GP-flat. Thus by Theorem 2.15, s is right generally regular.

$(10) \Rightarrow (17)$ It is similar to the proof of $(8) \Rightarrow (17)$.

(13) ⇒ (1) Suppose that A_S is a generator, a similar argument as in the proof of (3) ⇒ (1) shows that $S \times B_S$ is GP-flat. Thus B_S is GP-flat by Theorem 2.6, which implies GP-flatness of A_S.

(4) ⇒ (14) Let A_S be a right S-act such that $Hom(A_S, S_S) \neq \phi$. In view of Lemma 3.21, A_S is a retract of $S \times A_S$, which by our assumption is GP-flat. Thus A_S is GP-flat by Theorem 2.7.

(16) ⇒ (1) Suppose that A_S is a generator and that $as = a's$ for $a, a' \in A_S, s \in S$. By Lemma 3.21, there exists an epimorphism $\pi : A_S \to S_S$. Let $\pi^* = \pi|_{B_S = aS \cup a'S}$, then by assumption B_S is GP-flat, and so there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $B_S \otimes Ss^n \subseteq A_S \otimes Ss^n$.

(17) ⇒ (1) It follows from Theorem 3.1. □

Theorem 3.23. *For any monoid S, the following statements are equivalent:*

(1) *all faithful right S-acts are GP-flat;*
(2) *all finitely generated faithful right S-acts are GP-flat;*
(3) *all faithful right S-acts generated by two elements are GP-flat;*
(4) *S is right generally regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4) If $sS = S$ then obviously s is regular. So suppose that $sS \neq S$, for $s \in S$, and let $A_S = S_S \coprod^{sS} S_S$. As we know A_S is a faithful right S-act generated by two elements, and so by assumption it is GP-flat. Thus sS is G-left stabilizing by Theorem 2.20, and so s is right generally regular by Proposition 2.17.

(4) ⇒ (1) It follows from Theorem 3.1. □

Theorem 3.24. *For any monoid S, the following statements are equivalent:*

(1) *all strongly faithful right S-acts are GP-flat;*
(2) *all finitely generated strongly faithful right S-acts are GP-flat;*
(3) *all strongly faithful right S-acts generated by two elements are GP-flat;*
(4) *either S is not left cancellative or S is right generally regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4) If S is not left cancellative, then the statement is obvious. So, let S be left cancellative, and there exists $s \in S$ such that $sS \neq S$. Suppose that $A_S = S_S \coprod^{sS} S_S$, and let $(w, u)s = (w, u)t$, for $w \in \{x, y, z\}, u, s, t \in S$. So $us = ut$ and since S is left cancellative $s = t$,

which implies A_S is strongly faithful and so by assumption it is GP-flat. Thus by Theorem 2.20, sS is G-left stabilizing, and so by Proposition 2.17, s is right generally regular.

(4) $\Rightarrow$ (1) Suppose that A_S is a strongly faithful right S-act, and $ls = lt$, for $l, s, t \in S$. Then for an arbitrary $a \in A_S$, we have $als = alt$. Since A_S ia a strongly faithful right S-act, and $al \in A_S, s = t$, and so S is left cancellative. So if S is not left cancellative, then there exist no strongly faithful right S-act. Now suppose that S is right generally regular, then by Theorem 3.1 all right S-act are GP-flat as required. ☐

Theorem 3.25. *For any monoid S the following statements are equivalent:*

(1) *all indecomposable right S-acts are GP-flat;*
(2) *all finitely generated indecomposable right S-acts are GP-flat;*
(3) *all indecomposable right S-acts generated at most by two elements are GP-flat;*
(4) *all cyclic indecomposable right S-acts are GP-flat;*
(5) *S is right generally regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By [7, I, 5.8] all cyclic right S-acts are indecomposable, and so by assumption all cyclic right S-act are GP-flat. Thus S is right generally regular by Theorem 3.1.

(5) $\Rightarrow$ (1) It follows from Theorem 3.1. ☐

4. Characterization of monoids by GP-flatness property of right Rees factor acts

In this section we give a characterization of monoids by GP-flatness property of right Rees factor acts. Note that by Theorem 3.14, all torsion free right Rees factor S-acts are GP-flat if and only if S is a generally left almost regular monoid. Now for the Rees factors of the form $S/sS, s \in S$ see the following:

Theorem 4.1. *If all torsion free right Rees factor S-acts of the form S/sS are GP-flat, then either s is a right generally regular element or it satisfies Condition (tcu).*

(tcu) : there exist $u, t, c \in S$, which c is a right cancellable element, such that $t \notin sS$, and $tc = su$.

Proof. Suppose that for $s \in S$ Condition (tcu) is not satisfied. If $sS = S$, then obviously s is a regular element. Otherwise $K_S = sS$ is a proper right ideal of S. Let c be a right cancellable element, and $t \in S$ such that $tc \in K_S$. That is there exists $u \in S$ such that $tc = su$, since Condition (tcu) is not satisfied, $t \in K_S$, and so S/K_S is torsion free by [7, III, 8.10]. Thus by assumption S/K_S is GP-flat, and so $K_S = sS$ is G-left stabilizing by Theorem 2.18. Then s is right generally regular by Proposition 2.17. □

Note that by Example 2.19, GP-flatness of right Rees factor S-acts does not imply principally weakly flat property in general. See the following theorem:

Theorem 4.2. *Let S be a monoid. Then all GP-flat right Rees factor S-acts are principally weakly flat if and only if every G-left stabilizing proper right ideal of S is left stabilizing.*

Proof. Suppose that all GP-flat right Rees factor S-acts are principally weakly flat and let K_S be a G-left stabilizing proper right ideal of S. Then by Theorem 2.18, S/K_S is GP-flat, and so by assumption S/K_S is principally weakly flat. Hence by [7, III, 10.11], K_S is left stabilizing.

Conversely, suppose that for the right ideal K_S of S, S/K_S is GP-flat. Then there are two cases as follow:

Case 1. $K_S = S$. Then $S/K_S \cong \Theta_S$ is principally weakly flat by [7, III, 10.2(2)].

Case 2. $K_S \neq S$. Then by Theorem 2.18, K_S is G-left stabilizing. Thus by assumption K_S is left stabilizing, and so S/K_S is principally weakly flat by [7, III, 10.11]. □

The proof of the following theorem is similar to that of Theorem 4.2.

Theorem 4.3. *Let S be a monoid. Then all GP-flat right Rees factor S-acts are (weakly) flat if and only if S is right reversible and every G-left stabilizing proper right ideal K_S of S is left stabilizing ideal.*

Theorem 4.4. *Let S be a monoid. Then all GP-flat right Rees factor S-acts satisfy Condition (P) if and only if S is right reversible and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.*

Proof. Suppose first that all GP-flat right Rees factor S-acts satisfy Condition (P) and let K_S be a G-left stabilizing proper right ideal of S. Then by Theorem 2.18, S/K_S is GP-flat, and so by assumption S/K_S satisfies Condition (P). Hence by [7, III, 13.9], $|K_S| = 1$. The one element right S-act Θ_S is GP-flat by Theorem 2.5, and so it satisfies Condition (P), then S is right reversible by [7, III, 13.7].

Conversely, suppose that for the right ideal K_S of S, S/K_S is GP-flat. Then there are two cases:

Case 1. $K_S = S$. Since S is right reversible, $S/K_S \cong \Theta_S$ satisfies Condition (P) by [7, III, 13.7].

Case 2. $K_S \neq S$. Then by Theorem 2.18, K_S is G-left stabilizing, and so by assumption $|K_S| = 1$. Thus by [7, III, 13.9], S/K_S satisfies Condition (P) as required. □

Note that by Example 3.17 Condition (E) of right S-acts does not imply GP-flatness in general. But for (cyclic) Rees factor right S-acts Condition (E) coincides to strong flatness and so it implies GP-flatness. The proof of following theorems are essentially the same as to that of Theorem 4.4.

Theorem 4.5. *Let S be a monoid. Then all GP-flat right Rees factor S-acts satisfy Condition (E)(are strongly flat) if and only if, S is left collapsible and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.*

We recall from [8] that a right S-act A_S is *weakly pullback flat* if and only if it satisfies Conditions (P) and (E').

Theorem 4.6. *Let S be a monoid. Then all GP-flat right Rees factor S-acts are weakly pullback flat if and only if S is right reversible and weakly left collapsible, and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.*

Theorem 4.7. *Let S be a monoid. Then all GP-flat right Rees factor S-acts are projective if and only if S contains a left zero, and there exist no G-left stabilizing proper right ideal K_S of S with $|K_S| \geq 2$.*

Theorem 4.8. *Let S be a monoid. Then all GP-flat right Rees factor S-acts are free if and only if $S = \{1\}$.*

Note that by [7, I, 5.22] and [7, III, 18.7], the above theorem is also valid for Rees factor projective generators.

5. GP Coherent monoids

In this section we introduce GP coherent monoids and will give a characterization of these monoids.

The right S-act $(S \times S)_S$ equipped with the right S-action $(s, t)u = (su, tu)$, $s, t, u \in S$ is called the *diagonal act* of S and is denoted by $D(S)$. Also, if S is a monoid and I a non-empty set, then the set of all maps from I to S, equipped with the right S-action $(\alpha s)x = (\alpha(x))s$, for the mapping $\alpha : I \to S$, $s \in S$ and $x \in I$, is a right S-act, which is denoted by $(S^I)_S$.

Theorem 5.1. *Let S be a left PP monoid. Then the right S-act $(S^I)_S$ is principally weakly flat.*

Proof. Suppose that $(\alpha_i)_I s = (\alpha'_i)_I s$ for $\alpha_i, \alpha'_i, s \in S$. Since S is left PP, there exists an idempotent $e \in S$ such that $es = s$ and $\alpha_i s = \alpha'_i s$ implies $\alpha_i e = \alpha'_i e$ for every $i \in I$. That is, $(\alpha_i)_I e = (\alpha'_i)_I e$, and so the right S-act $(S^I)_S$ is principally weakly flat by [7, III, 10.16]. □

From Theorem 5.1 we can easily deduce that:

Corollary 5.2. *Let S be a left PP monoid. Then the right S-act $(S^I)_S$ is GP-flat.*

Corollary 5.3. *Let S be a left PP monoid. Then the diagonal act $D(S)$ is GP-flat.*

It follows from Lemma 2.4 that:

Corollary 5.4. *Let I and J be non-empty sets and $\mid J \mid \leq \mid I \mid$. If S^I is GP-flat then so is S^J.*

Lemma 5.5. *Let $A_S = \prod_{i\in I} A_i$ for a family $\{A_i \mid i \in I\}$ of right S-acts. Take $a_i \in A_i, s, u_i, v_i \in S$ with $u_i s = v_i s$ for each $i \in I$. If $(S^I)_S$ is GP-flat, then there exists $n \in \mathbb{N}$ such that $(a_i u_i)_I \otimes s^n = (a_i v_i)_I \otimes s^n$ in $A_S \otimes Ss^n$.*

Proof. Suppose that $(u_i)_I s = (v_i)_I s$ for $s, u_i, v_i \in S$. Since $(S^I)_S$ is GP-flat, by Lemma 2.4 there exist $s_1, ..., s_k, t_1, ..., t_k \in S$, $(x_{i_1})_I, ..., (x_{i_{k-1}})_I \in (S^I)_S$ and $n \in \mathbb{N}$ such that

$$\begin{array}{rl} & s_1 s^n = s^n \\ (u_i)_I s_1 = (x_{i_1})_I t_1 & s_2 s^n = t_1 s^n \\ (x_{i_1})_I s_2 = (x_{i_2})_I t_2 & s_3 s^n = t_2 s^n \\ \cdots & \cdots \\ (x_{i_{k-1}})_I s_k = (v_i)_I t_k & s^n = t_k s^n. \end{array}$$

Thus

$$\begin{array}{rl} & s_1 s^n = s^n \\ (a_i u_i)_I s_1 = (a_i x_{i_1})_I t_1 & s_2 s^n = t_1 s^n \\ (a_i x_{i_1})_I s_2 = (a_i x_{i_2})_I t_2 & s_3 s^n = t_2 s^n \\ \cdots & \cdots \\ (a_i x_{i_{k-1}})_I s_k = (a_i v_i)_I t_k & s^n = t_k s^n, \end{array}$$

and so $(a_i u_i)_I \otimes s^n = (a_i v_i)_I \otimes s^n$ in $A_S \otimes Ss^n$ as required. □

Theorem 5.6. *The following statements are equivalent on a monoid S:*

(1) $(S^I)_S$ *is* GP*-flat for each non-empty set* I*;*

(2) *for any* $z \in S$*, any non-empty set* I *and* $(s_i)_I, (t_i)_I \in (S^I)_S$*, if* $(s_i)_I z = (t_i)_I z$ *then there exists* $n \in \mathbb{N}$ *such that* $(s_i)_I \otimes z^n = (t_i)_I \otimes z^n$ *in* $(S^I)_S \otimes Sz^n$*;*

(3) *for any* $z \in S$*, there exist* $n, m \in \mathbb{N}$*,* $(u_1, v_1), ..., (u_m, v_m) \in S \times S$ *such that*

(i) $u_1 z^n = z^n = v_m z^n$, $u_{i+1} z^n = v_i z^n$ $(1 \leq i \leq m-1)$,

(ii) if $sz = tz, (s, t \in S)$*, then there exist* $s_1, ..., s_m, x_{i_1}, ...x_{i_{m-1}} \in S$ *such that*

$$\begin{aligned} su_1 &= x_1 v_1 \\ x_1 u_2 &= x_2 v_2 \\ &... \\ x_{m-1} u_m &= t v_m, \end{aligned}$$

(4) $(S^{S\times S})_S$ *is* GP*-flat.*

Proof. Implications (1) ⇒ (2) and (1) ⇒ (4) are clear.

(2) ⇒ (3) Suppose that I is an index for the non-empty set $K = \{(s,t) \in S \times S | sz = tz\}$, so $K = \{(s_i, t_i) | i \in I\}$. Since $(s_i)_I z = (t_i)_I z \in (S^I)_S$, by assumption there exists $n \in \mathbb{N}$ such that $(s_i)_I \otimes z^n = (t_i)_I \otimes z^n$ in $(S^I)_S \otimes Sz^n$, and so by Lemma 2.4, there exist $m \in \mathbb{N}, u_1, v_1, ..., u_m, v_m \in S$ and $(x_{i_1})_I, ..., (x_{i_{k-1}})_I \in (S^I)_S$ such that

$$\begin{array}{rr} & u_1 z^n = z^n \\ (s_i)_I u_1 = (x_{i_1})_I v_1 & u_2 z^n = v_1 z^n \\ (x_{i_1})_I u_2 = (x_{i_2})_I v_2 & u_3 z^n = v_2 z^n \\ ... & ... \\ (x_{i_{m-1}})_I u_m = (t_i)_I v_m & z^n = v_m z^n. \end{array}$$

And clearly (3) holds.

(3) ⇒ (2) Suppose that $z \in S$, I is a non-empty set, and $(s_i)_I, (t_i)_I \in (S^I)_S$ such that $(s_i)_I z = (t_i)_I z$. Then $(s_i z)_I = (t_i z)_I \in ((Sz)^I)_S$, by assumption there exist $n, m \in \mathbb{N}$, $(u_1, v_1), ..., (u_m, v_m) \in S \times S$ such that $u_1 z^n = z^n = v_m z^n$, $u_{i+1} z^n = v_i z^n$, $(1 \leq i \leq m-1)$, and for each $i \in I$ there exist $(x_{i_1})_I, ..., (x_{i_{k-1}})_I \in (S^I)_S$ such that

$$\begin{aligned} (s_i)_I u_1 &= (x_{i_1}) v_1 \\ (x_{i_1}) u_2 &= (x_{i_2}) v_2 \\ &... \\ (x_{i_{m-1}}) u_m &= (t_i)_I v_m. \end{aligned}$$

Thus

$$(s_i)_I \otimes z^n = (s_i)_I \otimes u_1 z^n = (s_i)_I u_1 \otimes z^n = (x_{i_1})_I v_1 \otimes z^n =$$

$$(x_{i_1})_I \otimes v_1 z^n = (x_{i_1})_I \otimes u_2 z^n = \cdots =$$
$$(x_{i_{m-1}})u_m \otimes z^n = (t_i)_I v_m \otimes z^n = (t_i)_I \otimes v_m z^n = (t_i)_I \otimes z^n$$
in $(S^I)_I \otimes Ss^n$.

(4) $\Rightarrow$ (1). Let $z \in S$, I be a non-empty set and $(s_i)_I, (t_i)_I \in (S^I)_I$ such that $(s_i)_I z = (t_i)_I z$. Suppose that $J = \{(s_i, t_i) \mid i \in I\}$, and index J by a set K as $J = \{(s_k, t_k) \mid k \in K\}$. By assumption $(S^{S\times S})_S$ is GP-flat, and so S^K is GP-flat by Corollary 5.4. Since $(s_k)_K z = (t_k)_K z$, it follows from Lemma 2.4 that there exist $n, m \in \mathbb{N}, u_1, v_1, ..., u_m, v_m \in S$ and $(x_{k_1})_K, ..., (x_{k_{m-1}})_K \in (S^K)_S$ such that

$$\begin{array}{rr} & u_1 z^n = z^n \\ (s_k)_K u_1 = (x_{k_1})_K v_1 & u_2 z^n = v_1 z^n \\ (x_{k_1})_K u_2 = (x_{k_2})_K v_2 & u_3 z^n = v_2 z^n \\ \cdots & \cdots \\ (x_{k_{m-1}})_K u_m = (t_k)_K v_m & z^n = v_m z^n. \end{array}$$

Now for each $i \in I$ there exists $k \in K$ such that $(s_i, t_i) = (s_k, t_k)$. Let $y_{i_j} = x_{k_j}$, $1 \leq j \leq m$, then we have the following S-tossing

$$\begin{array}{rr} & u_1 z^n = z^n \\ (s_i)_I u_1 = (y_{i_1})_I v_1 & u_2 z^n = v_1 z^n \\ (y_{i_1})_I u_2 = (y_{i_2})_I v_2 & u_3 z^n = v_2 z^n \\ \cdots & \cdots \\ (y_{i_{m-1}})_I u_m = (t_i)_I v_m & z^n = v_m z^n, \end{array}$$

and so $(S^I)_I$ is GP-flat as required. □

Definition 5.7. A monoid S is called right finite GP-flat if for every $s \in S$, there exist $m, n \in \mathbb{N}$ such that for every GP-flat right S-act A_S, $d_{s^n}(a, a') \leq m$, where $as = a's, a, a' \in A_S$.

Definition 5.8. A monoid S is called GP left (right) coherent if all direct products of non-empty families of GP-flat right (left) S-acts are GP-flat, and it is called GP coherent if it is both GP left coherent and GP right coherent.

Theorem 5.9. *A monoid S is left GP coherent if and only if:*

(1) *(1) $(S^I)_S$ is GP-flat for any non-empty set I, and*
(2) *(2) S is right finite GP-flat.*

Proof. Suppose that S is left GP coherent. It is obvious that (i) holds. If S is not right finite GP-flat, then there exists $s \in S$ such that for any $m, n \in \mathbb{N}$ there exists a GP-flat right S-act A_m and $a_m, a'_m \in A_m$, where $a_m s = a'_m s$ and $d_{s^n}(a_m, a'_m) > m_s$. By assumption $\prod_{m=1}^{\infty} A_m$ is GP-flat, thus $(a_m)_{\mathbb{N}} s = (a'_m)_{\mathbb{N}} s$, implies that there exists $n' \in \mathbb{N}$ such that $(a_m)_{\mathbb{N}} \otimes s^{n'} = (a'_m)_{\mathbb{N}} \otimes s^{n'}$ in $(\prod_{m=1}^{\infty} A_n)_S \otimes {}_S Ss^{n'}$. Suppose the length of the corresponding tossing is equal k. That is, there exists an S-tossing of length k connecting $(a_m, s^{n'})$ to $(a'_m, s^{n'})$ in $A_m \times Ss^{n'}$. But this contradicts $d_{s^{n'}}(a_k, a'_k) > k$.

Conversely, suppose that $\{A_i | i \in I\}$ is a family of GP-flat right S-acts and let $A_S = \prod_{i \in I} A_i$. Let $(a_i)_I s = (a'_i)_I s$, then $a_i s = a'_i s$ for every $i \in I$. Since S is right finite GP-flat, there exists $m, n \in \mathbb{N}$ such that (a_i, s^n) and (a'_i, s^n) are connected by an S-tossing of length m in $A_i \times Ss^n$, for every $i \in I$. Thus for each $i \in I$, there exists an S-tossing of the form

$$\begin{array}{rr} & s_{i_1} s^n = s^n \\ a_i s_{i_1} = b_{i_1} t_{i_1} & s_{i_2} s^n = t_{i_1} s^n \\ b_{i_1} s_{i_2} = b_{i_2} t_{i_2} & s_{i_3} s^n = t_{i_2} s^n \\ \dots & \dots \\ b_{i_{m-1}} s_{i_m} = a'_i t_{i_m} & s^n = t_{i_m} s^n \end{array}$$

where $s_{i_1}, t_{i_1}, \dots, s_{i_m}, t_{i_m} \in S$ and $b_{i_1}, \dots, b_{i_{m-1}} \in A_i$. Let $t_{i_0} = s_{i_{m+1}} = 1$, $b_{i_0} = a_i$ and $b_{i_m} = a'_i$ for $i \in I$. Then $s_{i_j} s = t_{i_{j-1}} s$, for $i \in I$ and $1 \leq j \leq m+1$, and so $(b_{i_{j-1}} t_{i_{j-1}})_I \otimes s^n = (b_{i_{j-1}} s_{i_j})_I \otimes s^n$ in $A_S \otimes Ss^n$ by Lemma 5.5. Thus

$$(a_i)_I \otimes s^n = (b_{i_0} t_{i_0})_I \otimes s^n = (b_{i_0} s_{i_1})_I \otimes s^n = (b_{i_1} t_{i_1})_I \otimes s^n$$
$$= \dots = (b_{i_m} s_{i_{m+1}})_I \otimes s^n = (a'_i)_I \otimes s^n$$

in $A_S \otimes Ss^n$. □

By Theorem 3.19, every left PP monoid is right finite GP-flat. So in this case we have the following theorem as a result of Theorem 5.9, and Corollary 5.2.

Theorem 5.10. *Every left PP monoid is GP left coherent.*

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On GP-FLATNESS PROPERTY

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خاصیت GPـ همواری

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با استفاده از خاصیت همواری به طور اساسی ضعیف، برخی از تکواره های مهم مانند تکواره های منظم، تکواره های تقریبا منظم چپ و ... مشخص سازی شده اند. در این مقاله با یادآوری یک تعمیم از خاصیت همواری به طور اساسی ضعیف، به نام GPـ همواری، به مشخص سازی تکواره ها براساس این خاصیت از سیستم های راست (خارج قسمتی ریس) آنها می پردازیم. همچنین تکواره های GPـ منسجم نیز مورد بررسی قرار می گیرند.

کلمات کلیدی: Gـ پایدار ساز چپ ، GPـ همواری، GPـ منسجم، عموما منظم راست.