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TOPICS ON CONTINUOUS INVERSE ALGEBRAS

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Abstract. In this paper, we first provide some counterexamples and derive some new results concerning the usual and singular spectrum of an element in continuous inverse algebras. Then we continue our investigation about the characterizations of multiplicative linear maps and their related results in these algebras.

1. Introduction

Non-normed topological algebras were initially introduced around the year 1950 for the investigation of certain classes of these algebras that appeared naturally in mathematics and physics. Some results concerning such topological algebras had been published earlier in 1947 by R. Arens [7]. It was in 1952 that Arens and Michael [5, 7] independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras. Here, we would like to mention about the predictions made by the famous Soviet mathematician M.A. Naimark, an expert in the area of Banach algebras, in 1950 regarding the importance of non-normed algebras and the development of their related theory. During his study concerning cosmology, G. Lassner [5] realized that the theory of normed topological algebras was insufficient for his study purposes. An important class of topological algebras namely continuous inverse algebras have propertise similar to Banach algebras. Hence, we can

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extent and prove some propertise of Banach algebras to continuous inverse algebras. In this paper, we first provide some counterexamples and derive some new results concerning the usual and singular spectrum of an element in continuous inverse algebras. Then we investigate about the characterizations of multiplicative linear maps and their related results in these algebras.
This paper is divided into the following sections. In section 2, we have gathered a collection of definitions and known results, and in section 3, we provide some counterexamples and obtain some new results in continuous inverse algebras. In section 4, the characterizations of multiplicative linear maps and their related results are discussed in these algebras.

Throughout this paper, all algebras will be assumed unital and the units will be denoted by e.

2. Definitions and known results

In this section, we present a collection of definitions and known results, which are included in the list of our references.

Definition 2.1. [2] Let A be a Hausdorff topological algebra . An element $a \in A$ is said to be a left (right) topological zero divisor if there exists a sequence $\{x_n\}$ in A such that $x_n \nrightarrow 0$ *and* $ax_n \to 0$ ($x_n a \to 0$) *as* $n \to \infty$. A topological zero divisor is both a left and a right topological zero divisor. If A is commutative, then every left topological zero divisor or right topological zero divisor is a topological zero divisor.

Let A be an algebra. The set of all invertible elements of A is denoted by $Inv(A)$, and the set of all singular elements by $Sing(A)$.

Definition 2.2. Let A be a Hausdorff topological algebra. The usual and singular spectrum of $x \in A$ are denoted by $sp(x)$ *and* $\sigma(x)$ respectively, and defined in the following:

$$sp(x) = \{\lambda \in C \ : \ x - \lambda e \notin Inv(A)\},$$

$$\sigma(x) = \{\lambda \in C \ : \ x - \lambda e \text{ is a topological zero divisor}\}.$$

Definition 2.3. [3] A locally convex algebra is an associative algebra A with a locally convex Hausdorff vector space topology such that the algebra multiplication is separately continuous. A Frechet algebra is a locally convex algebra in which the topology is completely metrizable.

Definition 2.4. [3] A locally convex algebra in which the topology can be described by a family of sub-multiplicative seminorms is called locally multiplicatively convex, or locally m-convex for short.

Definition 2.5. [3] A continuous inverse algebra is a locally convex algebra in which the set of invertible elements is a neighbourhood of e and inversion is continuous at e.

The following lemma follows from Theorem 2.2 of [3].

Lemma 2.6. *Let A be a commutative continuous inverse algebra and Γ_A be the Gelfand spectrum of A. Then*

$$radA = \bigcap_{\varphi \in \Gamma_A} ker\varphi,$$

where $radA$ is the Jacobson radical of A.

The algebra A is called semisimple if $radA = \{0\}$.

Proposition 2.7. [3, 1.2] *Let A be a continuous inverse algebra. Then $Inv(A)$ is an open subset of A, and inversion is a continuous map from $Inv(A)$ into itself.*

Proposition 2.8. [3, 1.5] *Let A be a continuous inverse algebra. Then every element of A has non-empty compact spectrum.*

3. Counterexamples and new results

In this section, we provide some counterexamples and derive some new results concerning the usual and singular spectrum of an element in continuous inverse algebras.
It is well known that every Banach algebra is a continuous inverse algebra, but the following example shows that the converse may be false in general.

Example 3.1. Consider the algebra $A = C^\infty[0,1]$ of all C^∞- functions on $[0,1]$ with topology τ defined by the algebra seminorms

$$p_n(f) = \sup_{0 \leq t \leq 1} [\sum_{k=0}^{n} \frac{|f^k(t)|}{k!}].$$

Then (A, τ) is a locally m- convex Frechet algebra whose the set of invertible elements is open. So A is a continuous inverse algebra, but not a Banach algebra.

Zelazko [12] constructed a continuous inverse Frechet algebra which is not locally m- convex. However, Turpin [11] proved that every commutative continuous inverse algebra is locally m- convex. The following example shows that the converse is false in general.

Example 3.2. The algebra $C(\mathbb{R})$ of all continuous complex-valued functions on the real line $\mathbb{R}$ with the sequence $(p_n)_n$ of seminorms defined by $p_n(f) = \sup_{|x|\leq n} \left|f(x)\right|$ is a is locally m- convex Frechet algebra, but not a continuous inverse algebra (see [9, 3.6]).

In Banach algebra A, every boundary point of $Inv(A)$ is a topological zero divisor [10, 42.9]. Here we prove a similar result for continuous inverse algebras.

Theorem 3.3. *Let A be a continuous inverse algebra. If $x \in \partial Inv(A)$, then x is a topological zero divisor.*

Proof. If $x \in \partial Inv(A)$, then there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$. By [2, 3.7.24], the set $E = \{x_n^{-1} : n \in N\}$ is unbounded. So there exists a neighbourhood V of zero and a divergent sequence of scalars r_n such that no r_nV contains E. Choose $x_n^{-1} \in E$ such that $x_n^{-1} \notin r_nV$. Then no $r_n^{-1}x_n^{-1}$ is in V, so that $\{r_n^{-1}x_n^{-1}\}$ does not converge to zero, i.e., $r_n^{-1}x_n^{-1} \not\to 0$ *as* $n \to \infty$. By the continuity of multiplication in topological algebras, we have $\lim\limits_{n\to\infty} xr_n^{-1}x_n^{-1} = \lim\limits_{n\to\infty} x_nr_n^{-1}x_n^{-1} = \lim\limits_{n\to\infty} r_n^{-1}e = 0$, and so x is a left topological zero divisor. Similarly, x is a right topological zero divisor. Thus x is a topological zero divisor. $\square$

In the following we give an example of continuous inverse algebra A and an element a in A such that a is a topological zero divisor and $0 \notin \partial sp(a)$.

Example 3.4. Let Δ be the closed unit disc in the complex plane, let A be the continuous inverse algebra of all complex-valued continuous functions on Δ, and let $a \in A$ be defined by $a(z) := z$. Then, clearly, the spectrum of a equals Δ. As a first consequence, 0 is not in the boundary of the spectrum of a. But, since a is not invertible, and A is a C^*-algebra, it follows that a is a topological zero divisor.

It is well known that the Gelfand-Mazur theorem holds true for Banach algebras :

Theorem 3.5. [10, 42.10] *Let A be a Banach algebra. If A has no non-zero topological zero divisor, then $A = \mathbb{C}e$.*

The following example shows that the Gelfand-Mazur theorem may be false for continuous inverse algebras in general.

Example 3.6. Let A be the algebra of all formal power series $x = \sum_{k=1}^{\infty} \xi_k(x)t^k$ with the topology of pointwise convergence of the coefficients $\xi_k(x)$ and with the Cauchy multiplication of power series. It is a comuutative locally m- convex Frechet algebra with seminorms

$$\|x\|_k = \sum_{i=0}^{k-1} |\xi_i(x)|.$$

Since $Inv(A)$ is open, A is a continuous inverse algebra. It has no topological zero divisor and yet it is not equal to the field of complex numbers.

If A is a closed subalgebra of a continuous inverse algebra B and also A and B do not have the same unit element, then we have:

Theorem 3.7. *Let B be a continuous inverse algebra with idempotent $0 \neq p \neq e$. If $A := pBp$, then the following statements hold for all x in A.*
(i) $\sigma_A(x) \subseteq \sigma_B(x)$,
(ii) $sp_B(x) = sp_A(x) \cup \{0\}$.

Proof. (i) By [3, 1.7], A is a continuous inverse algebra. Let $\lambda \in \sigma_A(x)$. Then $x - \lambda p$ is a topological zero divisor in A. If $x - \lambda p$ is a left topological zero divisor in A, then there exists a sequence $\{z_n\}$ in A with $z_n \not\to 0$ and $(x-\lambda p)z_n \to 0$ *as* $n \to \infty$. Since $(x-\lambda e)z_n{=}(x-\lambda p)z_n \to 0$ *as* $n \to \infty$ in B, i.e., $x - \lambda e$ is a left topological zero divisor in B. It follows similarly that $x - \lambda e$ is a right topological zero divisor in B and so $\lambda \in \sigma_B(x)$.
(ii) It is enough to show that $\mathbb{C}^*\backslash sp_A(x) = \mathbb{C}\backslash sp_B(x)$, where $\mathbb{C}^* = \mathbb{C}\backslash \{0\}$. Let $\lambda \in \mathbb{C}^*\backslash sp_A(x)$. Then there exists an element a in A such that $(\lambda p - x)a = a(\lambda p - x) = p$. On the other hand, we have

$$\begin{aligned}(\lambda e - x)(a + \lambda^{-1}(e-p)) &= ((\lambda p - x) + \lambda(e-p))(a + \lambda^{-1}(e-p)) \\ &= p + (e-p) = e.\end{aligned}$$

Similarly, $(a + \lambda^{-1}(e-p))(\lambda e - x) = e$. Hence, $\lambda e - x$ has the inverse $a + \lambda^{-1}(e-p)$ in B, and so $\lambda \notin sp_B(x)$. Conversely, let $\lambda \in \mathbb{C}\backslash sp_B(x)$. Then there exists b in B such that $(\lambda e - x)b = b(\lambda e - x) = e$. On the other hand, we have

$$(\lambda p - x)pb = p(\lambda e - x)b = p, \quad pb(\lambda p - x) = b(\lambda e - x)p = p.$$

Thus, $\lambda p - x$ has inverse in A and so $\lambda \in \mathbb{C}\backslash sp_A(x)$. Moreover, if $\lambda = 0$, then $xb = bx = -e$, therefore x is invertible in B. Since $x \in A$, we obtain $x(e-p) = 0$. The invertibility of x implies that $e - p = 0$ *or* $p = e$, this is a contradiction. Thus $\lambda \neq 0$, and so $\lambda \in \mathbb{C}^*\backslash sp_A(x)$. ∎

Lemma 3.8. *Let A be a Hausdorff topological algebra and $0 \neq x, y \in A$ and $0 \neq \lambda \in \mathbb{C}$. If $\{z_n\}$ is a sequence in A with $z_n \nrightarrow 0$ and $(xy - \lambda e)z_n \to 0$ as $n \to \infty$, then $yz_n \nrightarrow 0$.*

Proof. we have

$$\lambda z_n = xyz_n - (xy - \lambda e) z_n$$

or

$$z_n = \lambda^{-1} x (yz_n) - \lambda^{-1} (xy - \lambda e) z_n.$$

Now, if $yz_n \to 0$, then

$$\lambda^{-1} x (yz_n) \to 0 \ \ and\ \lambda^{-1} (xy - \lambda e) z_n \to 0.$$

Therefore $z_n \to 0$, which is a contradiction. So $yz_n \nrightarrow 0$. □

Now, we would like to prove the following theorem for singular spectrum in topological algebras. The corresponding result for the usual spectrum is well known[1, 3.1.2].

Theorem 3.9. *Let A be a Hausdorff topological algebra and $0 \neq x, y \in A$ and $0 \neq \lambda \in \mathbb{C}$. Then $xy - \lambda e$ is a topological zero divisor if and only if $yx - \lambda e$ is a topological zero divisor, i.e.,*

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

Proof. If $xy - \lambda e$ is a topological zero divisor, it is a left topological zero divisor. Hence there is a sequence $\{z_n\}$ in A such that $z_n \nrightarrow 0$ and $(xy - \lambda e)z_n \to 0$ *as* $n \to \infty$. By Lemma 3.8, $yz_n \nrightarrow 0$, therefore

$$(yx - \lambda e)(yz_n) = y(xy - \lambda e)z_n \to 0 \ \ as \ \ n \to \infty.$$

Consquently $yx - \lambda e$ is a left topological zero divisor. It follows similarly that $yx - \lambda e$ is a right topological zero divisor and so $yx - \lambda e$ is a topological zero divisor. □

4. Multiplicative linear maps and their related results

In this section, we investigate about the characterizations of multiplicative linear maps and their related results in continuous inverse algebras.

A characterization of multiplicative linear functionals in Banach algebras was given by Gleason-Kahane-Zelazko [4]. A version of the Gleason-Kahane-Zelazko theorem was also proved for certain topological algebras [8, 4.2]. Now we investigate it for continuous inverse algebras.

Theorem 4.1. *Let A be a continuous inverse algebra and $\varphi : A \longrightarrow \mathbb{C}$ be a linear functional on A. If $\varphi(x) \in sp(x)$, for all $x \in A$, then φ is multiplicative.*

Proof. According to the Theorem 4 of [6] it is evidence. □

Corollary 4.2. *Let A be a continuous inverse algebra and $\varphi : A \longrightarrow \mathbb{C}$ be a linear functional on A such that*

$$\varphi(e) = 1 \;\; and \;\;\; ker\, \varphi \subseteq \; Sing(A).$$

Then φ is multiplicative.

Proof. Since $\varphi(\varphi(x)e - x) = 0$, our assumption implies that $\varphi(x)e - x \notin Inv(A)$. Thus $\varphi(x) \in sp(x)$ for all $x \in A$. By Theorem 4.1, φ is multiplicative. □

Theorem 4.3. *Let A and B be two continuous inverse algebras with the unit elements e_A and e_B respectively, and B be commutative and semisimple. If $T : A \to B$ is a linear map such that $T(e_A) = e_B$ and $T(a) \in Inv(B)$ for $a \in Inv(A)$, then T is multiplicative.*

Proof. Let φ be be a multiplicative linear functional on B and $x, y \in A$. Then we have

$$\varphi o T(a) = \varphi(T(a)) \neq 0 \; and \; \varphi o T(e_A) = 1,$$

this implies that $ker\, \varphi o T \subseteq Sing(A)$. By Corollary 4.2, we conclude that $\varphi o T$ is multiplicative. Now, we have

$$\varphi(T(xy) - T(x)T(y)) = \varphi o T(xy) - \varphi o T(x)\varphi o T(y) = 0.$$

Since B is semisimple and φ is arbitrary, it follows that

$$T(xy) = T(x)T(y).$$

□

Theorem 4.4. *Let A and B be two continuous inverse algebras and B be semisimple. Suppose that T is a linear map from A into B such that*

$$sp(T(x)) \subseteq sp(x), \;\; for\ all\ x \in A.$$

Then T is multiplicative.

Proof. Let φ be be a multiplicative linear functional on B and $x \in A$. Then $\varphi o T$ is a linear functional on A. On the other hand,

$$\varphi o T(x) = \varphi(T(x)) \in sp(T(x)).$$

Hence our assumption implies that

$$\varphi o T(x) \in sp(x).$$

Thus, $\varphi o T$ is multiplicative by Theorem 4.1. The remainder of the proof follows from Theorem 4.3. $\square$

Theorem 4.5. *let A and B be two continuous inverse algebras with the unit elements e_A and e_B respectively. If $T : A \to B$ is a multiplicative linear map between A and B such that $T(e_A) = e_B$, then*

$$sp(T(x)) \subseteq sp(x), \text{ for all } x \in A.$$

Proof. We have

$$e_B = T(e_A) = T(xx^{-1}) = T(x)T(x^{-1}),$$

this implies that $T(x)$ is invetible in B and $T(x^{-1}) = T(x)^{-1}$.

Now let $\lambda \notin sp(x)$, then $x - \lambda e_A$ is invertible in A and so $T(x - \lambda e_A) = Tx - \lambda e_B$ is invertible in B. Thus $\lambda \notin sp(Tx)$. $\square$

Remark 4.6. Note that the assumption $T(e_A) = e_B$ is essential. For example, if we take $T : A \to A \oplus B$, then T is multiplicative, but $T(e_A) \neq e_B$ and $sp(x) \subseteq sp(Tx)$.

Theorem 4.7. *Let A and B be two continuous inverse algebras, with the unit elements e_A and e_B respectively. If $\theta : A \to B$ is injective continuous multiplicative linear map such that $\theta(e_A) = e_B$, then $\sigma(x) \subseteq \sigma(\theta(x))$, for all $x \in A$.*

Proof. Let $\lambda \in \sigma(x)$. Then $x - \lambda e_A$ is a topological zero divisor. If $x - \lambda e_A$ is a left topological zero divisor in A, then there exists a sequence $\{z_n\}$ in A with $z_n \not\to 0$ and $(x - \lambda e_A)z_n \to 0$ *as* $n \to \infty$. Since θ is a continuous multiplicative linear map, $(\theta(x) - \lambda e_B)\theta(z_n) \to 0$. Now, we show that $\theta(z_n) \not\to 0$. Suppose that $\theta(z_n) \to 0$, then

$$\theta(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} \theta(z_n) = \theta(0) = 0.$$

Since θ is injective, $z_n \to 0$, which is a contradiction. Hence, $\theta(z_n) \not\to 0$. This implies that $(\theta(x) - \lambda e_B)$ is a left topological zero divisor. Similarly, it is a right topological zero divisor, and so it is a topological zero divisor. Consequently $\lambda \in \sigma(\theta(x))$. Therefore, $\sigma(x) \subseteq \sigma(\theta(x))$, for all $x \in A$. $\square$

Corollary 4.8. *If A is a closed unital subalgebra of continuous inverse algebra B, then*

$$\sigma_A(x) \subseteq \sigma_B(x)\,, \text{ for all } x \in A\,.$$

Proof. By [3, 1.7], A is a continuous inverse algebra. If we adopt the preceding theorem for inclusion map $i : A \to B$, then the result will be obtained. $\square$

Theorem 4.9. *Let A and B be two continuous inverse Frechet algebras with the unit elements e_A, e_B respectively. Also let B be commutative and semisimple. If $\theta : A \to B$ is injective multiplicative linear map such that $\theta(e_A) = e_B$, then $\sigma(x) \subseteq \sigma(\theta(x))$, for all $x \in A$.*

Proof. By [3, 2.4], θ is continuous. The remainder proof of the theorem follows by the same reasoning as in Theorem 4.7. □

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مباحثی در جبرهای معکوس پیوسته

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در این مقاله، ابتدا چند مثال نقض را ارائه داده و نتایج جدیدی در ارتباط با طیف معمولی و منفرد یک عنصر در جبرهای معکوس پیوسته به‌دست می‌آوریم. سپس بررسی‌مان را در مورد خواص نگاشت‌های خطی ضربی و نتایج مربوط به آنها در این جبرها ادامه می‌دهیم.

کلمات کلیدی: جبر معکوس پیوسته، مقسوم‌علیه توپولوژیک صفر، طیف، نگاشت خطی ضربی.