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ON SOME TOTAL GRAPHS ON FINITE RINGS

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Abstract. We give a decomposition of the total graph $T(\Gamma(\mathbb{Z}_{2^n p^m}))$ where p is a prime and m, n are positive integers. We also studied some graph theoretical properties with some of its fundamental subgraphs.

1. Introduction

Let R be a commutative ring and $Z(R)$ and $Reg(R)$ be the sets of zero-divisors and regular elements of R, respectively. Let $T(\Gamma(R))$ denote the total graph of R and let $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ be the induced subgraphs of $T(\Gamma(R))$ with vertices in $Z(R)$ and $Reg(R)$, respectively. In this paper, we study the decomposition of total graphs on some finite commutative rings $R = \mathbb{Z}_m$, where the set of zero-divisors of R is not an ideal. For $m \geq 2$, it is well-known that $Z(\mathbb{Z}_m)$ is an ideal of $\mathbb{Z}_m$ if and only if $m = p^n$ for some prime p and integer $n \geq 1$.

For a simple graph G, let $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. For a nonnegative integer r, the graph G is called *r-regular* if all vetices have the same degree r. Recall that the *complement* of a graph G is a graph denoted by $\bar{G}$ on the same vertex set as G, where two distinct vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A graph is called *planar* if it can be drawn in the plane without crossing edges. A *tree* is a connected graph with no cycles. A *claw* the star graph $K_{1,3}$. A *claw-free* is one that does not

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have a claw as an induced subgraph. A *caterpillar* is a tree in which a single path is incident to every edge. A *k-coloring* of G is a function $f : V(G) \longrightarrow \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. A k-coloring is said to be *proper* if adjacent vertices are colored differently. A graph is called *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y. $d(x, x) = 0$; and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is defined as $diam(G) = sup\{d(x, y)|$ x and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G. $gr(G) = \infty$ if G contains no cycles. We say that two (induced) subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). An *independent set* of G is a set of vertices, no two of which are adjacent. *The independence number* of G is defined as the maximum size of an independent set of vertices, denoted by $\alpha(G)$. A *vertex cover* of a graph G is a set $Q \subset V(G)$ that contains at least one endpoint of all edges. We denote the minimum size of vertex covers in G by $\beta(G)$. A *dominating set* for a graph G is a set $D \subseteq V(G)$ such that every vertex of $V(G) - D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the number of vertices of a smallest dominating set for G. A set $I \subseteq V(G)$ is an *independent dominating set* of G if I is both an independent and dominating set. The cardinality of a minimum independent dominating set of G is called the *independent domination number* of G and is denoted by $i(G)$. A graph G is called *domination perfect* if $\gamma(H) = i(H)$ for every induced subgraph H of G. A graph is *well-covered* if $i(G) = \alpha(G)$. A *matching* of G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are saturated by M. A *perfect matching* is a matching that saturates every vertex. The maximum size of matchings in G is denoted by $\alpha'(G)$. An *edge cover* of G is a set L of edges such that every vertex of G is incident to some edge of L. The minimum size of edge covers is denoted by $\beta'(G)$. Let $\omega(G)$ denote the number of components of a graph G. A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* of G, written as $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A *disconnecting set* of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The *edge-connectivity* of G, written as $\kappa'(G)$, is the minimum size of a disconnecting set. More terminologies can be seen in [5].

Throughout, we assume that p is an odd prime number and n, m are natural numbers. Note that $Z(\mathbb{Z}_{2^n p^m})$ is not an ideal of $\mathbb{Z}_{2^n p^m}$. So, by [2] we have

(1) $T(\Gamma(\mathbb{Z}_{2^n p^m}))$ is a connected graph with $diam(T(\Gamma(\mathbb{Z}_{2^n p^m}))) = 2$ and $gr(T(\Gamma(\mathbb{Z}_{2^n p^m})) = gr(Z(\Gamma(\mathbb{Z}_{2^n p^m}))) = 3$.

(2) $Z(\mathbb{Z}_{2^n p^m})$ is a set of $2^{n-1}p^{m-1}(p+1)$ elements which can be classified into two subsets of even and odd zero-divisors with $2^{n-1}p^m$ and $2^{n-1}p^{m-1}$ elements, respectively. In addition, it is obvious that $Reg(\mathbb{Z}_{2^n p^m})$ is the set of odd elements that are not multiples of p and has $2^{n-1}p^{m-1}(p-1)$ elements.

(3) By (2) and Lemma 2.4 of [4], $T(\Gamma(\mathbb{Z}_{2^n p^m}))$ is a $(2^{n-1}p^{m-1}(p+1)-1)$-regular graph.

2. Decomposition of $T(\Gamma(\mathbb{Z}_{2^n p^m}))$

Let p be an odd prime number. In this section, we consider the total graph $T(\Gamma(\mathbb{Z}_{2^n p^m}))$ and decompose it in sevsral steps. At first, we give some basic notions require in the contex of this request.

Lemma 2.1. *The only odd zero-divisor of the ring $\mathbb{Z}_{2p}$ is p.*

Proof. It is obvious that p is an odd zero-divisor of $\mathbb{Z}_{2p}$. Let a be an odd zero-divisor of $\mathbb{Z}_{2p}$. Then there exists a $0 \neq b \in \mathbb{Z}_{2p}$ such that $ab = 0$. It is clear that $p \nmid b$. Hence, $p \mid a$ and so, $p = a$. □

Theorem 2.2. *The clique number of $T(\Gamma(\mathbb{Z}_{2p}))$ is p.*

Proof. The set of zero-divisors of $\mathbb{Z}_{2p}$ contains p and all even elements. Thus, in $T(\Gamma(\mathbb{Z}_{2p}))$ every even vertex of $T(\Gamma(\mathbb{Z}_{2p}))$ is adjacent to the other even vertices and similarly every odd vertex of $\mathbb{Z}_{2p}$ is adjacent to the other odd vertices. Since we have exactly p even and p odd elements in $\mathbb{Z}_{2p}$, so there are exactly two complete graphs K_p in the total graph, one with even vertices and the other with odd vertices. Now we show that there isn't any complete graph with more than p vertices. We claim that there isn't any odd vertex adjacent to all even vertices. Let x be an odd vertex adjacent to all even vertices, so $x + y \in Z(\mathbb{Z}_{2p})$ for any even vertex y. Thus, by Lemma 2.1, $x + y = p$ for any even vertex y which is a contradiction. Similarly, there isn't any even vertex adjacent to all odd vertices. Hence $\omega(T(\Gamma(\mathbb{Z}_{2p}))) = p$. □

Definition 2.3. A *decomposition* of a graph G is a set of subgraphs $G_1, G_2, \ldots, G_r$ that partitions the edges of G such that $\bigcup_{1 \leq i \leq r} E(G_i) = E(G)$ and $E(G_i) \cap E(G_j) = \emptyset$ for all $i \neq j$. If there is a decomposition

$G_1, G_2, \cdots, G_r$ for G, we say that G is decomposed by $G_1, G_2, \ldots, G_r$ and denote it by $G = G_1 + G_2 + \cdots + G_r$.

Our next theorem provides a decomposition of the total graph which will be useful in the sequel.

Theorem 2.4. *$T(\Gamma(\mathbb{Z}_{2p}))$ has the following decomposition;*

$$T(\Gamma(\mathbb{Z}_{2p})) = 2K_p + pK_{1,1}.$$

Proof. By the argument of Theorem 2.2, we have exactly two complete graphs K_p in the decomposition of $T(\Gamma(\mathbb{Z}_{2p}))$. By Lemma 2.1, p is the only odd zero divisor of ring $\mathbb{Z}_{2p}$. Moreover, there are exactly p distinct pairs of even-odd vertices in $\mathbb{Z}_{2p}$ such that the sum of each is p. So we have p edges between even vertices and odd vertices. □

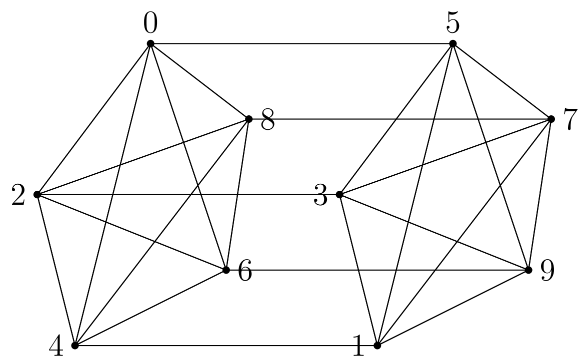


Figure 1. $T(\Gamma(\mathbb{Z}_{10}))$

Theorem 2.5. *The followings hold.*

(i) $Z(\Gamma(\mathbb{Z}_{2p})) = K_p + K_{1,1}$.

(ii) *$Reg(\Gamma(\mathbb{Z}_{2p}))$ is a complete graph with $p - 1$ vertices.*

Proof. (i) It's clear that in $Z(\Gamma(\mathbb{Z}_{2p}))$, there exist p even vertices adjacent to each other. So we have a complete graph K_p. By Lemma 2.1, p is the only odd zero-divisor in $\mathbb{Z}_{2p}$ which is adjacent to 0.

(ii) In $Reg(\Gamma(\mathbb{Z}_{2p}))$, all odd vertices of $T(\Gamma(\mathbb{Z}_{2p}))$ are adjacent to each other except p. So, we have the complete graph K_{p-1}. □

Theorem 2.6. *For all $n \geq 1$ and $p \geq 3$, we have the following decomposition*

$$T(\Gamma(\mathbb{Z}_{2^n p})) = 2K_{2^{n-1}p} + pK_{2^{n-1},2^{n-1}}.$$

Proof. As mentioned in the proof of Theorem 2.2, $T(\Gamma(\mathbb{Z}_{2^n p}))$ induces two complete subgraphs $K_{2^{n-1}p}$, one consists of even vertices and the other of odd vertices. The proof is completed by showing that the remaining edges of $T(\Gamma(\mathbb{Z}_{2^n p}))$ appears in p induced subgraphs $K_{2^{n-1},2^{n-1}}$

with parts as even and odd vertices. We'll refer to these partitions as *even-odd partitions*.

We proceed by induction on n. For $n = 1$, in view of Theorem 2.4, there are p induced subgraphs $K_{1,1}$ in the decomposition of $T(\Gamma(\mathbb{Z}_{2p}))$. Let the assertion be true for a $n > 1$. It is enough to show that $T(\Gamma(\mathbb{Z}_{2^{n+1}p}))$ has p induced subgraphs $K_{2^n,2^n}$.

Since $|V(T(\Gamma(\mathbb{Z}_{2^{n+1}p}))| = 2|V(T(\Gamma(\mathbb{Z}_{2^np}))|$, any induced subgraph $K_{2^n,2^n}$ of $T(\Gamma(\mathbb{Z}_{2^{n+1}p}))$ can be constructed by appending 2^{n-1} even vertices to the even part elements and 2^{n-1} odd vertices to odd part elements of an induced subgraph $K_{2^{n-1},2^{n-1}}$ of $T(\Gamma(\mathbb{Z}_{2^np}))$. Let $K = \{0, \dots, 2^{n-1}-1\}, K^{'} = \{2^{n-1}, \dots, 2^n-1\}$ and $\{2kp; \quad k \in K\} \bigcup \{(2k+1)p; \quad k \in K\}$,$\{2kp + 2; \quad k \in K\} \bigcup \{(2k + 1)p - 2; \quad k \in K\}$, ... ,$\{2kp+2^{p-1}; \quad k \in K\} \bigcup \{(2k+1)p-2^{p-1}; \quad k \in K\}$ be even-odd partitions of p subgraphs $K_{2^{n-1},2^{n-1}}$. Consider the subgraph $K_{2^{n-1},2^{n-1}}$ of $T(\Gamma(\mathbb{Z}_{2^np}))$ with $A = \{2kp; \quad k \in K\}$ as even part and $B = \{(2k + 1)p; \quad k \in K\}$ as odd part; and append to A and B, 2^{n-1} even vertices of $A' = \{2kp; \quad k \in K^{'}\}$ and 2^{n-1} odd vertices of $B' = \{(2k + 1)p; \quad k \in K^{'}\}$, respectively. We have to show that for any $x \in A$, $y \in B$, $x' \in A'$ and $y' \in B'$, x' is adjacent to both y' and y and also, x is adjacent to y'.

Note that for any $x' \in A'$, there is $x \in A$ such that $x' = x + 2^np$. Similarly, for any $y' \in B'$, $y' = y + 2^np$ for some $y \in B$. By induction hypothesis $x + y \in Z(\mathbb{Z}_{2^np})$; it follows that, $x' + y' = (x + 2^np) + (y + 2^np) = (x + y) + 2^{n+1}p \equiv^{2^{n+1}p} x + y \in Z(\mathbb{Z}_{2^np}) \subseteq Z(\mathbb{Z}_{2^{n+1}p})$. Hence x' is adjacent to y'.

If $x \in A$, then there is an odd number $k \in \{1, \dots, 2^n - 1\}$ such that $x + y = kp$. So, one sees immediately that $x' + y = (x + 2^np) + y = kp + 2^np = (k + 2^n)p$ in which $k + 2^n$ is an odd number belonging to $\{1 + 2^n, \dots, 2^{n+1} - 1\}$. Thus $x' + y \in Z_{odd}(\mathbb{Z}_{2^{n+1}p})$, where the index means the odd zero-divisors. Similarly x is adjacent to y'. □

Theorem 2.7. *The followings hold.*

(i) $Z(\Gamma(\mathbb{Z}_{2^np})) = K_{2^{n-1}p} + K_{2^{n-1},2^{n-1}} + K_{2^{n-1}}$ *for* $n > 1$.
(ii) $Reg(\Gamma(\mathbb{Z}_{2^np}))$ *is a complete graph with* $2^{n-1}(p-1)$ *vertices.*

Proof. (i) There are $2^{n-1}p$ even zero-divisors and 2^{n-1} odd zero-divisors in $\mathbb{Z}_{2^np}$. Clearly, the $2^{n-1}p$ even vertices in $Z(\Gamma(\mathbb{Z}_{2^np}))$ are adjacent to each other which induce the complete graph $K_{2^{n-1}p}$. Furthermore, the set of odd zero-divisors of $\mathbb{Z}_{2^np}$ is $\{p, 3p, \dots, (2^n - 1)p\}$ which forms $K_{2^{n-1}}$, (see Figure 2.b). Considering the sets $\{2kp; \quad 0 \le k \le 2^{n-1}-1\}$ and $\{(2k+1)p; \quad 0 \le k \le 2^{n-1}-1\}$ as two parts, the complete bipartite subgraph $K_{2^{n-1},2^{n-1}}$ will be formed, (see Figure 2.a).

(ii) The vertices of $Reg(\Gamma(\mathbb{Z}_{2^n p}))$ are the odd vertices of $T(\Gamma(\mathbb{Z}_{2^n p}))$ which are non-zero-divisors, so they are not multiple of p, and $|V(Reg(\Gamma(\mathbb{Z}_{2^n p})))| = 2^{n-1}(p-1)$. One can clearly see that they form the complete graph $K_{2^{n-1}(p-1)}$. □

Theorem 2.8. *For all $n \geq 1$ and $p \geq 3$, one has*

$$T(\Gamma(\mathbb{Z}_{2^n p^m})) = 2K_{2^{n-1}p^m} + pK_{2^{n-1}p^{m-1},2^{n-1}p^{m-1}}.$$

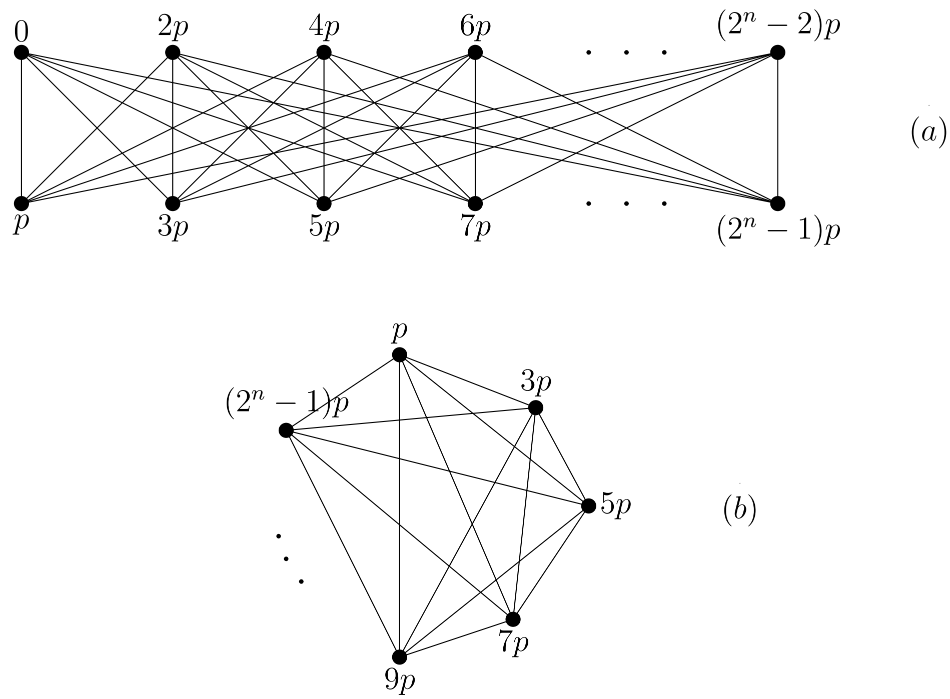


Figure 2.

Proof. We do by induction on m. For $m = 1$, in view of Theorem 2.6, $T(\Gamma(\mathbb{Z}_{2^n p})) = 2K_{2^{n-1}p} + pK_{2^{n-1},2^{n-1}}$. Let the assertion be true for a $m > 1$. Analogue of the proof of Theorem 2.6, $T(\Gamma(\mathbb{Z}_{2^n p^{m+1}}))$ induces two complete subgraphs as $K_{2^{n-1}p^{m+1}}$, one consists of even vertices and the other of odd vertices. In the similar way to the proof of Theorem 2.6, consider the subgraph $K_{2^{n-1}p^{m-1},2^{n-1}p^{m-1}}$ of $T(\Gamma(\mathbb{Z}_{2^n p^m}))$ with $A = \{2kp; \quad k = 0, \ldots, 2^{n-1}p^{m-1} - 1\}$ as even part and $B = \{(2k+1)p; \quad k = 0, \ldots, 2^{n-1}p^{m-1} - 1\}$ as odd part, and append to A and B even vertices $A' = \{2kp; \quad k = 2^{n-1}p^{m-1}, \ldots, 2^{n-1}p^m - 1\}$ as odd vertices $B' = \{(2k+1)p; \quad k = 2^{n-1}p^{m-1}, \ldots, 2^{n-1}p^m - 1\}$, respectively. Let $x' \in A'$ and $y' \in B'$, then we have $x' = x + a2^n p^m$ and $y' = y + a2^n p^m$ for some $x \in A, y \in B$ and $1 \leq a \leq p-1$. By induction hypothesis $x + y \in Z(\mathbb{Z}_{2^n p^m})$; so, it follows that, $x' + y' =$

$(x+y)+2a2^np^m \in Z(\mathbb{Z}_{2^np^m}) \subseteq Z(\mathbb{Z}_{2^np^{m+1}})$. Hence, x' is adjacent to y'. It is easy to check that for any $x' \in A'$ and $y \in B$, x' is adjacent to y, too. □

Corollary 2.9. *The following statements hold.*

(i) $Z(\Gamma(\mathbb{Z}_{2^np^m})) = K_{2^{n-1}p^m} + K_{2^{n-1}p^{m-1},2^{n-1}p^{m-1}} + K_{2^{n-1}p^{m-1}}$ *for* $n > 1$.

(ii) $Reg(\Gamma(\mathbb{Z}_{2^np^m}))$ *is a complete graph with* $2^{n-1}p^{m-1}(p-1)$ *vertices.*

3. Some properties of $G = T(\Gamma(\mathbb{Z}_{2^np^m}))$

In this section, we show that $G = T(\Gamma(\mathbb{Z}_{2^np^m}))$ is a claw-free graph and determine some graph theoretical properties of G and $\bar{G}$. We also study the structure of $\overline{ZR}(\Gamma(R))$, the spanning subgraph of $G = T(\Gamma(R))$ with edge set $E(G)-E(H)$ where $H = Z(\Gamma(R)) \cup Reg(\Gamma(R))$.

Corollary 3.1. *Let* $G = T(\Gamma(\mathbb{Z}_{2^np^m}))$, *then*

$$\kappa(G) = \kappa'(G) = 2^{n-1}p^{m-1}(p+1) - 1.$$

Proof. By decomposition of G proved in Theorem 2.8, each vertex contributes in exactly one complete subgraph $K_{2^{n-1}p^m}$ and one complete bipartite subgraph $K_{2^{n-1}p^{m-1},2^{n-1}p^{m-1}}$. So, to obtain the number of smallest vertex cut, first we pick up all vertices of $K_{2^{n-1}p^m}$ except one, say v. By the way, v has $2^{n-1}p^{m-1}$ neighbours in the other $K_{2^{n-1}p^m}$, which should be picked up. So, $\kappa(G) = (2^{n-1}p^m - 1) + 2^{n-1}p^{m-1} = 2^{n-1}p^{m-1}(p+1)-1$. Also, to determine the number of smallest disconnecting set of edges, we should pick up all edges incident on a vertex. Thus, for an arbitrary vertex v on G, $\kappa(G) = \kappa'(G) = deg(v)$ which is $2^{n-1}p^{m-1}(p+1) - 1$. □

Proposition 3.2. $T(\Gamma(\mathbb{Z}_{2^np^m}))$ *is a claw-free graph.*

Proof. By decomposition of G in Theorem 2.8, some stars $K_{1,3}$ are visible in $T(\Gamma(\mathbb{Z}_{2^np^m}))$. One can show that they are not induced subgraphs of $T(\Gamma(\mathbb{Z}_{2^np^m}))$, by the light discussion in the following cases.

Case 1: All vertices of $K_{1,3}$ are present in one $K_{2^{n-1}p^m}$.

Case 2: The center vertex of $K_{1,3}$ is in one of $K_{2^{n-1}p^m}$ and the leaves are in the other one.

Case 3: There is a leaf of $K_{1,3}$ in one $K_{2^{n-1}p^m}$ and the other vertices are in the other one.

By the adjacencies in complete graphs we are done. □

Lemma 3.3. *(See [5], Lemma 3.1.21) In a graph* G, $S \subseteq V(G)$ *is an independent set if and only if* $\bar{S}$ *is a vertex cover, and hence* $\alpha(G) + \beta(G) = n(G)$.

Theorem 3.4. *(See [5], Theorem 3.1.22) If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.*

Proposition 3.5. *(See [5], Corollary 3.1.24) If G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$.*

Corollary 3.6. *Let $G = T(\Gamma(\mathbb{Z}_{2^n p^m}))$, then*

(i) $\alpha(G) = \gamma(G) = 2$,
(ii) $\beta(G) = 2^n p^m - 2$,
(iii) $\alpha'(G) = \beta'(G) = 2^{n-1}p^m$,
(iv) $i(G) = 2$, *and G is domination perfect.*
(v) *G is well-covered.*

Proof. Using the structure of G decomposed in Theorem 2.8, it is a simple matter to verify that the largest independent sets are of size two, which contain a pair of even and odd nonadjacent vertices present in the different two $K_{2^{n-1}p^m}$ s, like $\{(2k+1)p, (2k+1)p-1\}$. Also, they can be the smallest dominating sets. So, $\alpha(G) = \gamma(G) = 2$. By Proposition 3.2, G is claw-free. So by the main theorem in [1] for claw-free graphs, $i(G) = \gamma(G)$ and G is domination perfect. Also, since $i(G) = \alpha(G)$, G is a well-covered graph. Furthermore, the maximum size of matchings is the number of independent edges between two $K_{2^{n-1}p^m}$ s, which is equal to $|V(K_{2^{n-1}p^m})|$, i.e. $\alpha'(G) = 2^{n-1}p^m$. The equalities mentioned in Lemma 3.3 and Theorem 3.4 yield the remaining items. □

Remark 3.7. The *chromatic number* of complete bipartite graphs and complete graphs are well-known by [5] and [3], as follows

$$\chi(K_{m,n}) = 2, \quad \chi(K_n) = n.$$

Corollary 3.8. *Let $G = T(\Gamma(\mathbb{Z}_{2^n p^m}))$, then $\chi(G) = 2^{n-1}p^m$.*

Proof. According to the decomposition of Theorem 2.8 and Corollary 3.6, there are $2^{n-1}p^m$ independent sets of vertices of size two in G, each contain a pair of even and odd nonadjacent vertices from different two $K_{2^{n-1}p^m}$ s. Thus, we can color the vertices of each independent set by the same color. Since each independent set contains a vertex of a complete graph, by Remark 3.7, we need $2^{n-1}p^m$ different colors for coloring of the vertices of G. Thus, $\chi(G) = 2^{n-1}p^m$. □

In the next two results, we mention some simple properties about the complement of G. Anderson and Badawi in [2], described the total graph of R completely. They showed that if $2 \in Z(R)$, then the structure of $Reg(\Gamma(R))$ is different from that without 2 in $Z(R)$. And since in this paper, $2 \in Z(\mathbb{Z}_{2^n p^m}))$, we need the next lemma.

Definition 3.9. Let $G = T(\Gamma(R))$. Then $\bar{G}$, the complement of G, is a graph with vertex set $V(G)$ and two vertices x and y are adjacent if and only if $x + y \notin Z(R)$.

Lemma 3.10. *Let R be a finite ring such that $|R| = n$. Let $G = T(\Gamma(R))$. If $2 \in Z(R)$, then $\bar{G}$ is a $(n - |Z(R)|)$-regular graph.*

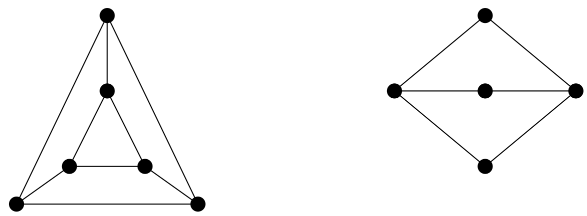
Proof. By Lemma 2.4 of [4], G is a $(|Z(R)| - 1)$-regular graph. Let v be an arbitrary vertex of G. Evidently, by definition of complement, $deg_{\bar{G}}(v) = deg_{K_n}(v) - deg_G(v) = (n-1) - (|Z(R)| - 1) = n - |Z(R)|$. □

Corollary 3.11. *Let $G = T(\Gamma(\mathbb{Z}_{2^n p^m}))$. Then the following statements hold.*

(i) *$\bar{G}$ is a $2^{n-1}p^{m-1}(p-1)$-regular graph.*
(ii) *$\bar{G}$ is a connected bipartite graph.*
(iii) *$\bar{G}$ is triangle-free.*
(iv) *$diam(\bar{G}) = 3$, except for $G = T(\Gamma(\mathbb{Z}_6))$.*
(v) *$gr(\bar{G}) = 4$, except for $G = T(\Gamma(\mathbb{Z}_6))$.*
(vi) *$\alpha(\bar{G}) = \beta(\bar{G}) = \alpha'(\bar{G}) = \beta'(\bar{G}) = 2^{n-1}p^m$.*

Proof. By Lemma 3.10, the regularity of $\bar{G}$ is $2^n p^m - 2^{n-1}p^{m-1}(p+1) = 2^{n-1}p^{m-1}(p-1)$. Since every two even (odd) vertices of G are adjacent, they are nonadjacent in $\bar{G}$. Hence, $\bar{G}$ is a triangle-free graph with no odd cycle and thus, $\bar{G}$ is a bipartite graph with parts of even and odd vertices. In view of Example 3.2, one can obtain the complement of $G = T(\Gamma(\mathbb{Z}_6))$ and see that $\bar{G} \cong C_6$ whose diameter is 2 and girth is 6. This is the only exception graph for parts $[(iv)]$ and $[(v)]$. For $G \neq T(\Gamma(\mathbb{Z}_6))$, any pair of even vertices has a common neighbour and so does any pair of odd vertices. If x and y are two nonadjacent even and odd vertices,respectively, then they don't have any common neighbour, so $diam(\bar{G}) > 2$. On the other hand, there exist odd and even vertices z and t, respectively such that $x \sim z \sim t \sim y$. Therefore $\bar{G}$ is connected and $diam(\bar{G}) = 3$. Furthermore, for $G \neq T(\Gamma(\mathbb{Z}_6))$, there is a C_4 in $\bar{G}$, like $0 \sim 1 \sim (2k+1)p + 1 \sim (2k+1)p + 2 \sim 0$ where $0 \leq k \leq (p-1)/2$. Thus $gr(\bar{G}) = 4$. For the last part, it is easy to see that the set of even vertices and the set of odd vertices are two only largest independent sets in $\bar{G}$, so $\alpha(\bar{G}) = 2^{n-1}p^m$. Looking at Lemma 3.3, we conclude that $\beta(\bar{G}) = 2^{n-1}p^m$. It is easy to check that $\alpha'(\bar{G}) = 2^{n-1}p^m$ and in view of Lemma 3.4, $\beta'(\bar{G}) = 2^{n-1}p^m$. □

Remark 3.12. Let $G = T(\Gamma(\mathbb{Z}_{2^n p^m}))$. In view of Theorem 2.8, one can see some $K_{3,3}$ in $K_{2^{n-1}p^{m-1}, 2^{n-1}p^{m-1}}$ or some K_5 in $K_{2^{n-1}p^m}$ where $G \neq T(\Gamma(\mathbb{Z}_6))$. So, by Proposition 6.2.2 in [5], G is not planar. In Figure 3, we show a planar embedding for $T(\Gamma(\mathbb{Z}_6))$ and draw its dual which is isomrphic to $K_{2,3}$.



$T(\Gamma(\mathbb{Z}_6))$ $T^*(\Gamma(\mathbb{Z}_6))$

Figure 3. $T(\Gamma(\mathbb{Z}_6))$ and its dual.

Definition 3.13. Let R be a finite ring. Let $H = Z(\Gamma(R)) \cup Reg(\Gamma(R))$. The spanning subgraph of $G = T(\Gamma(R))$ with edge set $E(G) - E(H)$ is denoted by $\overline{ZR}(\Gamma(R))$.

Example 3.14. In Figure 4, we see an edge decomposition of $T(\Gamma(\mathbb{Z}_6))$ by $Z(\Gamma(\mathbb{Z}_6))$ as dashed lines, $Reg(\Gamma(\mathbb{Z}_6))$ as a bold line, and $\overline{ZR}(\Gamma(\mathbb{Z}_6))$ as ordinary lines.

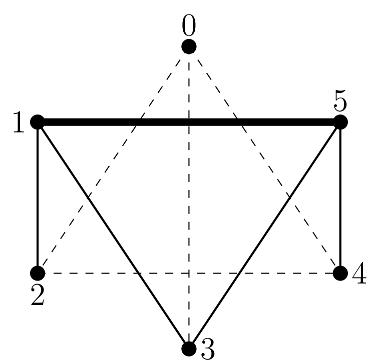


Figure 4. $Z(\Gamma(\mathbb{Z}_6))$, $Reg(\Gamma(\mathbb{Z}_6))$, and $\overline{ZR}(\Gamma(\mathbb{Z}_6))$ in $T(\Gamma(\mathbb{Z}_6))$

Theorem 3.15. $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$ *is a disconnected graph consisting of two components; a tree and an isolated vertex.*

Proof. First, in the graph $Z(\Gamma(\mathbb{Z}_{2p}))$, zero vertex is adjacent to the other zero-divisor vertices, so it is an isolated vertex in $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$. We claim that the other $2p-1$ vertices induce a connected graph with $2p-2$ edges, so it is a tree, by Theorem 2.1.4 in [5]. Let u, v be two arbitrary non-adjacent vertices of $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$. We show that there is a u, v-path in $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$. The following cases can be considered:

(i) If both u and v are even vertices, then there exist distinct odd vertices $x \neq p$ and $y \neq p$ such that $x \nsim y$ and $x \sim u$, $y \sim v$ in $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$, because the zero element is the only even vertex adjacent to p, and all other odd vertices are adjacent in $Reg(\Gamma(\mathbb{Z}_{2p}))$. So we have $x \sim p \sim y$, and the u, v-path $u \sim x \sim p \sim y \sim v$.

(ii) If both u and v are odd vertices other than p, then we have the u, v-path $u \sim p \sim v$.

(iii) If one of u or v is p, say $u = p$, then v is an even vertex, if not, $u \sim v$. Therefore, there exists odd vertex x such that $x \sim v$. So, we have $u \sim x \sim v$.

(iv) If u, v are even and odd vertices, respectively, and none of them is p, then there exists an odd vertex x such that $u \sim x \sim p$. So, we have the u, v-path $u \sim x \sim p \sim v$.

Furthermore, as we mentioned in Theorems 2.4, 2.5,

$$\begin{aligned} T(\Gamma(\mathbb{Z}_{2p})) &= 2K_p + pK_{1,1}, \\ Z(\Gamma(\mathbb{Z}_{2p})) &= K_p + K_{1,1}, \\ Reg(\Gamma(\mathbb{Z}_{2p})) &= K_{p-1}. \end{aligned}$$

It follows that,

$$\begin{aligned} |E(\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))| &= |E(T(\Gamma(\mathbb{Z}_{2p})))| - |E(Z(\Gamma(\mathbb{Z}_{2p})) \cup Reg(\Gamma(\mathbb{Z}_{2p})))| \\ &= 2\frac{p(p-1)}{2} + p - \left(\frac{p(p-1)}{2} + 1 + \frac{(p-1)(p-2)}{2}\right) \\ &= 2p - 2. \end{aligned}$$

So, $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}) - \{0\}$ is a graph with $2p-1$ vertices which is connected with $2p-2$ edges, and it is a tree. □

It is proved in Theorem 2.2.19 of [5] that a tree is a caterpillar if and only if it does not contain the tree Y below.

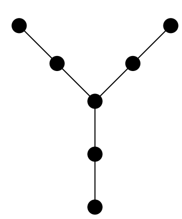


Figure 5. The Tree Y

Theorem 3.16. *The tree in* $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$ *is not a caterpillar for* $p \neq 3$.

Proof. It is clear that the tree which consists of $ZR(\Gamma(\mathbb{Z}_6))$ without its isolated vertex 0, shown by bold lines $2 \sim 1 \sim 3 \sim 5 \sim 4$ in Example 3.14, is a P_5 which obviously doesn't contain the tree Y. So it is a caterpillar. Consider $\overline{ZR}(\Gamma(\mathbb{Z}_{2p}))$ without its isolated vertex 0. As the argument in Theorem 3.15, it is a tree. Let $p \neq 3$. Since p is adjacent to all of the other odd vertices, and there are more than three odd vertices in $ZR(\Gamma(\mathbb{Z}_{2p}))$ where $p > 3$, one can see the star subgraph $K_{1,3}$ whose center is p. Moreover, there is an even vertex adjacent to each one of the neighbours of p in this $K_{1,3}$, the degree of these neighbours is not one. So, Y shown in Figure 5 exists. □

Definition 3.17. Let G be a graph. The subgraph of G obtained by removing all isolated vertices is called *sociable* subgraph of G, and denoted by $S(G)$.

Theorem 3.18. *$\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p}))$ is a disconnected graph consisting of a bipartite graph and 2^{n-1} isolated vertices.*

Proof. In view of Remark 1, $Z(\mathbb{Z}_{2^n p}) = \{2k; \quad k = 0, \dots, 2^{n-1}p - 1\} \cup \{(2k+1)p; \quad k = 0, \dots, 2^{n-1} - 1\}$. So $Reg(\mathbb{Z}_{2^n p}) = \{2k+1; \quad k = 0, \dots, 2^{n-1}p - 1\} - \{(2k+1)p; \quad k = 0, \dots, 2^{n-1}p - 1\}$. By the argument of Theorem 2.7 and looking at Figure 2, one can see that $A = \{2kp; \quad k = 0, \dots, 2^{n-1} - 1\}$ is the set of isolated vertices in $\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p}))$. Let x be a vertex of $\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p}))$ and $x = 2kp$, then it is obvious that

$$deg_{T(\Gamma(\mathbb{Z}_{2^n p}))}(x) = deg_{Z(\Gamma(\mathbb{Z}_{2^n p}))}(x) = 2^{n-1}(p+1) - 1.$$

So, $deg_{\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p}))}(x) = 0$. We put aside these isolated vertices, so it remains to show that $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))$ is a bipartite graph. By considering $V_1 = Z(\mathbb{Z}_{2^n p}) - A$ and $V_2 = Reg(\mathbb{Z}_{2^n p})$ as a partition for $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))$, we are done. □

One may easily generalize the result for $\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m}))$ as the following corollary.

Corollary 3.19. *$\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m}))$ is a disconnected graph consisting of a bipartite graph and $2^{n-1}p^{m-1}$ isolated vertices.*

Proof. One can see that $A = \{2kp; \quad k = 0, \dots, 2^{n-1}p^{m-1} - 1\}$ is the set of isolated vertices in $\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m}))$, and $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m})))$ is partitioned by $V_1 = Z(\mathbb{Z}_{2^n p^m}) - A$ and $V_2 = Reg(\mathbb{Z}_{2^n p^m})$. □

Corollary 3.20. *The following statements hold for $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))$.*

(i) *The even zero-divisor vertices are of degree 2^{n-1}.*
(ii) *The odd zero-divisor vertices are of degree $2^{n-1}(p-1)$.*
(iii) *The regular vertices are of degree 2^n.*

Proof. (i) The even zero-divisors of $\mathbb{Z}_{2^n p}$ are of the form $2k$, where $0 \le k \le 2^{n-1}p - 1$. Let $x = 2k$ be an even zero-divisor vertex. Then

$$\begin{aligned} deg_{S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))}(x) &= deg_{T(\Gamma(\mathbb{Z}_{2^n p}))}(x) - deg_{Z(\Gamma(\mathbb{Z}_{2^n p}))}(x) \\ &= 2^{n-1}(p+1) - 1 - (2^{n-1}p - 1) \\ &= 2^{n-1}. \end{aligned}$$

(ii) The odd zero-divisors of $\mathbb{Z}_{2^n p}$ are of the form $(2k+1)p$, where $0 \le k \le 2^{n-1} - 1$. Let x be an odd zero-divisor vertex and $x = (2k+1)p$, then by Theorem 2.7,

$$\begin{aligned} deg_{Z(\Gamma(\mathbb{Z}_{2^n p}))}(x) &= deg_{K_{2^{n-1},2^{n-1}}}(x) + deg_{K_{2^{n-1}}}(x) \\ &= 2^{n-1} + (2^{n-1} - 1) \\ &= 2^n - 1. \end{aligned}$$

It follows that

$$\begin{aligned} deg_{S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))}(x) &= deg_{T(\Gamma(\mathbb{Z}_{2^n p}))}(x) - deg_{Z(\Gamma(\mathbb{Z}_{2^n p}))}(x) \\ &= 2^{n-1}(p+1) - 1 - (2^n - 1) \\ &= 2^{n-1}(p-1). \end{aligned}$$

(iii) Let x be a regular vertex, then

$$\begin{aligned} deg_{S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p})))}(x) &= deg_{T(\Gamma(\mathbb{Z}_{2^n p}))}(x) - deg_{Reg(\Gamma(\mathbb{Z}_{2^n p}))}(x) \\ &= 2^{n-1}(p+1) - 1 - (2^{n-1}(p-1) - 1) \\ &= 2^n. \end{aligned}$$

□

At the end of article, one can easily generalize the recent results to $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m})))$.

Corollary 3.21. *The following statements hold for* $S(\overline{ZR}(\Gamma(\mathbb{Z}_{2^n p^m})))$.

(i) *The even zero-divisor vertices are of degree* $2^{n-1}p^{m-1}$.
(ii) *The odd zero-divisor vertices are of degree* $2^{n-1}p^{m-1}(p-1)$.
(iii) *The regular vertices are of degree* $2^n p^{m-1}$.

Proof. In the light of Theorem 2.5, Theorem 2.7, Corollary 2.9, and Theorem 3.20, one can easily check the items. □

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References

[1] R. B. Allan and R. Laskar, On domination and independent domination numbers of a graph, *Discrete Math.*, **23** (1978), 73–76.
[2] D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra*, **320** (2008), 2706–2719.
[3] M. Behzad, G. Chartrand and J. K. Cooper, The colour numbers of complete graphs, *J. London Math. Soc.*, **42** (1967), 226–228.
[4] M. H. Shekarriz, M. H. Shirdareh Haghighi and H. Sharif, On the Total graph of a finite commutative ring, *Comm. Algebra*, **40**(8) (2012), 2798–2807.
[5] D. B. West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, Upper Saddle river, NJ, 1996.

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ON SOME TOTAL GRAPHS ON FINITE RINGS

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درباره گراف‌های جامع روی حلقه‌های متناهی

مژگان تقی‌دوست لسکوکلایه[۱] ، مونا غلام‌نیا طالشانی[۲] و احمد عباسی[۳]

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فرض کنیم p یک عدد اول و m و n اعداد صحیح مثبت باشند. در این مقاله گراف جامع $T(\Gamma(\mathbb{Z}_{۲^n p^m}))$ را تجزیه کرده، خواص آن و زیرگراف‌های اساسی آن را مورد مطالعه قرار می‌دهیم.

کلمات کلیدی: گراف جامع، تجزیه در گراف، گراف‌های روی حلقه‌های تعویض‌پذیر.