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ON DERIVATIONS OF PSEUDO-BL ALGEBRA

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Abstract. Pseudo-BL algebras are a natural generalization of BL-algebras and of pseudo-MV algebras. In this paper the notions of five different types of derivations on a pseudo-BL algebra as generalizations of derivations of a BL-algebra are introduced. Moreover, as an extension of derivations of a pseudo-BL algebra, the notions of (φ, ψ)-derivations are defined on these types. Finally, several related properties are discussed.

1. Introduction

The concept of a pseudo-BL algebra first introduced by A. Di Nola et al. [6, 7] as a noncommutative extension of Hájek's BL-algebra [10] and as a generalization of an MV-algebra [5]. Hájek was the first to propose a complete theory of BL-algebra as algebraic structures to illustrate the completeness theorem of basic logic in 1998 [10]. MV-algebras, which introduced by Chang [5], are contained in the class of BL-algebras. In [6, 7, 23] the main properties of the pseudo-BL algebras were discussed in detail. The most recognized classes of BL-algebras are MV-algebras, Gödel algebras and product algebras. More over Georgescu and Iorgules [9] were the first to study pseudo-MV algebras as a noncommutative generalization of MV algebras. A pseudo-BL algebra is a pseudo-MV algebra if and only if $(x^-)^\sim = (x^\sim)^- = x$, for all x.

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process of developing Galois theory and the theory of invariants and is a very interesting and important field of many researchers. In 1957 the notion of derivations was first given in rings by E. C. Posner [15]. Subsequently, the concept of derivation has been studied on lattices [16, 8, 4], BCI-algebras [11, 24, 13], MV-algebra [2, 3, 19, 20] and lattice implication by Lee and Yong [12, 22]. In 2013 Torkzadeh et al. applied the notion of derivations to BL-algebras [17]. Inspired by this, several researchers have extended this notion in [21, 1, 14].

In this paper, five kinds of derivations of a pseudo-BL algebra are introduced. These derivatons are defined as $(\otimes, \vee)$-derivation, $(\ominus, \otimes)$-derivation, $(©, \otimes)$-derivation and two implicative derivation as $(\rightarrow, \vee)$-derivation and $(\rightsquigarrow, \vee)$-derivation on pseudo-BL algebras. We have generalized the notion of derivation on a pseudo-BL algebra A to (φ, ψ)-derivations on A by using two functions φ and ψ of A into itself. These derivations are extended by introducing the notions of (φ, ψ)-derivations of type 1, 2, 3, $\overset{\rightsquigarrow}{(\varphi, \psi)}$-derivation, $\overset{\rightarrow}{(\varphi, \psi)}$-derivation and also study some related properties.

2. Preliminaries

In this section, we recall the concept of a pseudo-BL algebra and then present some definitions and properties which we will need in the next sections.

Definition 2.1. A *pseudo-BL algebra* is a structure $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ where A is a non-empty set, $\vee, \wedge, \otimes, \rightarrow, \rightsquigarrow$ are binary operation and 0,1 are constant satisfying:

(PBL-1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
(PBL-2) $(A, \otimes, 1)$ is a monoid;
(PBL-3) $x \otimes y \ \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$;
(PBL-4) $x \wedge y = (x \rightarrow y) \otimes x = x \otimes (x \rightsquigarrow y)$;
(PBL-5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, for all $x, y \in A$.

In the sequal, we shall agree that the operations $\vee, \wedge, \otimes$ have priority towards the operations $\rightarrow, \ \rightsquigarrow$. A pseudo-BL algebra $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ is nontrivial if and only if $0 \neq 1$. Let A be a pseudo-BL algebra. We set $x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$. For all $x \in A$ we define the auxiliary operations $\oslash, \ominus$ and © as follows $x \oslash y = x^\sim \rightarrow y, x \ominus y = x \otimes y^-, x © y = y^\sim \otimes x$.
Now we give some examples of pseudo-BL algebras.

Example 2.2. [6, Example 2.21] Consider an arbitrary l-group $(G, \vee, \wedge, +, -, 0, 1)$ and let $u \in G, u \leq 0$. We put:

$$x \otimes y = (x + y) \vee u, x \to y = (y - x) \vee 0, x \rightsquigarrow y = (-x + y) \wedge 0.$$

Then it can be proved, $A = ([u, 0], \vee, \wedge, \otimes, \to, \rightsquigarrow, 0 = u, 1 = 0)$ is a pseudo-BL algebra.

We recall that a lattice-ordered group (l-group) [6] is a structure $(G, \vee, \wedge, +, -, 0)$ verifying the following:

(1) $(G, +, -, 0)$ is a group,
(2) $(G, \vee, \wedge)$ is a lattice,
(3) If $\leq$ denotes the partial order on G induced by $\vee, \wedge$, then for all $a, b, x \in G$, if $a \leq b$ then $a + x \leq b + x$ and $x + a \leq x + b$.

Example 2.3. [18, Example 2.13] Let $a, b, c, d \in \mathbb{R}$. We put by definition

$$(a, b) \leq (c, d) \Leftrightarrow a < c \text{ or } (a = c \text{ and } b \leq d).$$

For any $u, v \in \mathbb{R} \times \mathbb{R}$, we define the operations $\vee$ and $\wedge$ as follows: $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$. Let $A = \{\left(\frac{1}{2}, b\right) \in \mathbb{R}^2 : b \geq 0\} \cup \{(a, b) \in \mathbb{R}^2 : \frac{1}{2} < a < 1, b \in \mathbb{R}\} \cup \{(1, b) \in \mathbb{R}^2 : b \leq 0\}$. For any $(a, b), (c, d) \in A$, we put:

$$(a, b) \otimes (c, d) = \left(\frac{1}{2}, 0\right) \vee (ac, bc + d),$$

$$(a, b) \to (c, d) = \left(\frac{1}{2}, 0\right) \vee [\left(\frac{c}{a}, \frac{d - b}{a}\right) \wedge (1, 0)],$$

$$(a, b) \rightsquigarrow (c, d) = \left(\frac{1}{2}, 0\right) \vee [\left(\frac{c}{a}, \frac{ad - bc}{a}\right) \wedge (1, 0)],$$

$$(a, b)^- = (a, b) \to 0_A = (a, b) \to \left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}, 0\right) \vee [\left(\frac{1}{2a}, \frac{-b}{a}\right) \wedge (1, 0)],$$

$$(a, b)^\sim = (a, b) \rightsquigarrow 0_A = (a, b) \rightsquigarrow \left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}, 0\right) \vee [\left(\frac{1}{2a}, \frac{-b}{2a}\right) \wedge (1, 0)].$$

Then it can be shown, $(A, \vee, \wedge, \otimes, \to, \rightsquigarrow, \left(\frac{1}{2}, 0\right), (1, 0))$ is a pseudo-BL algebra.

Now we are able to make the connections of pseudo-BL algebra with BL-algebras. At first glance pseudo-BL algebras appears to differ from

BL-algebras in two major ways: commutativity of $\otimes$ and the difference between $\rightarrow$ and $\rightsquigarrow$. We shall say that a pseudo-BL algebra A is commutative iff $x \otimes y = y \otimes x$, for any $x, y \in A$.

It can be easily shown that a pseudo-BL algebra A is commutative iff $x \rightsquigarrow y = x \rightarrow y$, for any $x, y \in A$. This is equivalent with the statement that $\rightarrow = \rightsquigarrow$. Any commutative pseudo-BL algebra A is a BL-algebra. Then we shall say that a pseudo-BL algebra is *proper* if it is not commutative, i.e. if that is not a BL-algebra.

In Proposition 2.4, we present some elementary properties of this concept.

Proposition 2.4. [23, 6, Proposition 2.2, Proposition 3.1, Proposition 3.9] *In a pseudo-BL algebra A, for all $x, y, z \in A$ the following properties hold:*

(1) $(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z)$ *and* $(y \otimes x) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$*;*
(2) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$;
(3) $x \leq y$ *implies* $x \otimes z \leq y \otimes z$, $z \otimes x \leq z \otimes y$ *and* $x \leq z \rightsquigarrow y, x \leq z \rightarrow y$*;*
(4) $x \otimes y \leq x, y$ *and* $x \otimes y \leq x \wedge y$*;*
(5) $x \leq y$ *implies* $z \rightsquigarrow x \leq z \rightsquigarrow y$, $z \rightarrow x \leq z \rightarrow y$, *and also* $y \rightsquigarrow z \leq x \rightsquigarrow z$, $y \rightarrow z \leq x \rightarrow z$*;*
(6) $x \rightarrow y = x \rightarrow x \wedge y, x \rightsquigarrow y = x \rightsquigarrow x \wedge y$*;*
(7) $x \vee y = ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x) = ((x \rightsquigarrow y) \rightarrow y) \wedge ((y \rightsquigarrow x) \rightarrow x)$*;*
(8) $x \otimes y = 0$ *iff* $x \leq y^-$, $x \leq y^\sim$ *iff* $y \otimes x = 0$*;*
(9) $x \otimes x^\sim = x^- \otimes x = 0$*;*
(10) $1 \rightarrow x = 1 \rightsquigarrow x = x$ *and* $x \rightarrow 1 = x \rightsquigarrow 1 = 1$*;*
(11) $x^- = 1$ *iff* $x^\sim = 1$ *iff* $x = 0$*;*
(12) $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$*;*
(13) $x \rightarrow y \leq y^- \rightsquigarrow x^-, x \rightsquigarrow y \leq y^\sim \rightarrow x^\sim$*;*
(14) $(x \otimes y)^- = x \rightarrow y^-$, $(x \otimes y)^\sim = y \rightsquigarrow x^\sim$*;*
(15) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$, $(y \vee z) \otimes x = (y \otimes x) \vee (z \otimes x)$*;*
(16) $x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$, $(x \wedge y) \otimes z = (x \otimes z) \wedge (y \otimes z)$.

Definition 2.5. Let $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra and F a nonempty subset of A. Then F is said to be a *filter* of A if it satisfies

(1) If $x, y \in F$, then $x \otimes y \in F$;
(2) If $x \in F$ and $x \leq y$, then $y \in F$.

It is easy to see that for any filter F, $0 \in F$ and for every $x \in A$ we have $x \in F$ if and only if $x^{--} \in F$.

Definition 2.6. A pseudo-BL algebra A is called a *pseudo-Gödel algebra* if for all $x \in L, x \otimes x = x$.

Definition 2.7. Let A be a pseudo-BL algebra. Then, a function $f : A \longrightarrow A$ is called *isoton*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in A$.

Definition 2.8. Let X, Y be two pseudo-BL algebras. A map $f : X \longrightarrow Y$ is called a pseudo-BL *homomorphism* if for all $x, y \in X$:

(1) $f(x \otimes y) = f(x) \otimes f(y)$;
(2) $f(x \to y) = f(x) \to f(y)$;
(3) $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$;
(4) $f(0) = 0$.

An element $a \in A$ is called *complemented* if there is an element $b \in A$ such that $a \vee b = 1$, $a \wedge b = 0$, and if the element b exists it is called the complement of a. For any pseudo-BL algebra A, we shall denote by $B(A)$ the Boolean algebra of complemented elements in the lattice of A and it is called the *Boolean center* of A. It has been proved in [7] that $B(A) = \{x \in A : x \otimes x = x, x = (x^{\sim})^{-} = (x^{-})^{\sim}\}$. The elements of $B(A)$ are called Boolean elements of A. Clearly, $0, 1 \in B(A)$. Also, it is straightforward that $B(A)$ is a subalgebra of the pseudo-BL algebra.

Proposition 2.9. [7, Lemma 2.3] *If A is a pseudo-BL algebra and $a, b \in A$ such that $a \otimes a = a$, then*

(1) $a \otimes b = a \wedge b = b \otimes a$,
(2) $a \wedge a^{\sim} = 0 = a \wedge a^{-}$,
(3) $a \rightsquigarrow b = a \to b$,
(4) $a^{\sim} = a^{-}$.

3. Derivations of a pseudo-BL algebra

In this section, five different types of derivations on a pseudo-BL algebra are introduced. The first three are referenced as type 1, 2 and 3 which are defined as $(\otimes, \vee)$-derivation, $(\ominus, \otimes)$-derivation and $(©, \otimes)$-derivation, respectively. The remaining two are described as implicative derivation, defined by $(\to, \vee)$ and $(\rightsquigarrow, \vee)$ and we investigate their properties.

Definition 3.1. Let $(A, \vee, \wedge, \otimes, \to, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra. Then the map $D : A \to A$ is called

(1) a *derivation of type 1*, if $D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y))$ for all $x, y \in A$;
(2) a *derivation of type 2*, if $D(x \ominus y) = (D(x) \ominus y) \otimes (x \ominus D(y))$ for all $x, y \in A$;

(3) a *derivation of type 3*, if $D(x \circledcirc y) = (D(x) \circledcirc y) \otimes (x \circledcirc D(y))$ for all $x, y \in A$.

For a pseudo-BL algebra A, for convenience, we denote by D_1, D_2 and D_3 the derivations of types 1, 2 and 3, respectively.

Definition 3.2. Let $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra. Then the map $D : A \rightarrow A$ is an *implicative derivation* and called

(4) a $(\rightarrow, \vee)$-*derivation* if $D(x \rightarrow y) = (Dx \rightarrow y) \vee (x \rightarrow Dy)$ for all $x, y \in A$;

(5) a $(\rightsquigarrow, \vee)$-*derivation* if $D(x \rightsquigarrow y) = (Dx \rightsquigarrow y) \vee (x \rightsquigarrow Dy)$ for all $x, y \in A$.

The abbreviation $\overrightarrow{D}$ and $\overset{\rightsquigarrow}{D}$ are used for $(\rightarrow, \vee)$-derivation and $(\rightsquigarrow, \vee)$-derivation in above definition.

Example 3.3. Let A be a pseudo-BL algebra. Consider $1(x) = 1$, $D(x) = 0$ and $I(x) = x$. It can be easily shown which of these functions can be applied to these derivations considering conditions and give us Table1.

TABLE 1.

	D_1	D_2	D_3	$\overrightarrow{D}$	$\overset{\rightsquigarrow}{D}$
$1(x) = 1$	−	−	−	+	+
$D(x) = 0$	+	+	+	−	−
$I(x) = x$	+	−	−	+	+

The conditions which $I(x) = x$ can be the types 2 and 3 derivations are shown below.

Proposition 3.4. *Let I be the identity function on pseudo-BL algebra A. If A is a pseudo-Gödel algebra, then I is a derivation of types 2 and 3 on A.*

Proof. Let $x \otimes x = x$ for all $x \in L$. Then $I(x \ominus y) = (x \ominus y) = (x \ominus y) \otimes (x \ominus y) = (I(x) \ominus y) \otimes (x \ominus I(y))$. Thus, I is a derivation of type 2 on A. For all $x \in A$, we have: $I(x \circledcirc y) = (x \circledcirc y) = (x \circledcirc y) \otimes (x \circledcirc y) = (I(x) \circledcirc y) \otimes (x \circledcirc I(y))$. Hence, I is a derivation of type 3 on A. □

Theorem 3.5. *Let $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra and D_i be a derivation of type i on A, $1 \leq i \leq 3$. Then for all $1 \leq i \leq 3$, we have*

(1) $D_i(0) = 0$;

(2) $D_i(x) = D_i(x) \otimes x$ *then* $D_i(x) \leq x$, *for* $i = 2, 3$ *and all* $x \in A$;

(3) $D_i(x^{\sim}) \leq (D_i(x))^{\sim}$ *for* $i = 2, 3$;
(4) $D_i(x^{-}) \leq (D_i(x))^{-}$ *and moreover* $x \in B(A)$ *implies that* $D_1(x) \leq x$;
(5) $D_1(x) = 1$ *implies that* $x^{-} = 0$, *and for* $i = 2, 3$ *and* $D_i(x) = 1$ *implies that* $x = 1$.

Proof. (1) We have

$$D_1(0) = D_1(0 \otimes 0) = (D_1(0) \otimes 0) \vee (0 \vee D_1(0)) = 0.$$
$$D_2(0) = D_2(x \ominus 1) = (D_2(x) \ominus 1) \otimes (x \ominus D_2(1)) = 0, \text{for all } x \in A.$$
$$D_3(0) = D_3(0 \copyright 0) = (D_3(0) \copyright 0) \otimes (0 \copyright D_3(0)) = 0.$$

(2) We can write

$$\begin{aligned} D_3(x) &= D_3(1 \otimes x) = D_3(0^{\sim} \otimes x) = D_3(x \copyright 0) \\ &= (D_3(x) \copyright 0) \otimes (x \copyright D_3(0)) \\ &= 0^{\sim} \otimes D_3(x) \otimes 0^{\sim} \otimes x = D_3(x) \otimes x \leq x. \end{aligned}$$

Also, we have

$$\begin{aligned} D_2(x) &= D_2(x \ominus 0) = (D_2(x) \ominus 0) \otimes (x \ominus D_2(0)) \\ &= D_2(x) \otimes 0^{\sim} \otimes x \otimes 0^{\sim} = D_2(x) \otimes x \leq x. \end{aligned}$$

(3) For $i = 2, 3$, we have $D_i(x) \leq x$, and so $D_i(x^{\sim}) \leq x^{\sim}$ and $x^{\sim} \leq (D_i(x))^{\sim}$. Hence, we conclude that $D_i(x^{\sim}) \leq (D_i(x))^{\sim}$.
(4) For $i = 2, 3$, we have $D_i(x) \leq x$, and so $D_i(x^{-}) \leq x^{-}$ and $x^{-} \leq (D_i(x))^{-}$. Thus, we obtain $D_i(x^{-}) \leq (D_i(x))^{-}$ and for $i = 1$, by Proposition 2.4, we have $x^{-} \otimes x = 0$ and

$$0 = D_1(0) = D_1(x^{-} \otimes x) = (D_1(x^{-}) \otimes x) \vee (x^{-} \otimes D_1(x)).$$

Hence, we obtain $(D_1(x^{-}) \otimes x) = 0, (x^{-} \otimes D_1(x)) = 0$. This yields that $D_1(x^{-}) \leq x^{-}$, $x^{-} \leq (D_1(x))^{-}$. Thus $D_1(x^{-}) \leq (D_1(x))^{-}$. Also, if $x \in B(A)$ then $D_1(x) \leq x$.
(5) $D_1(x) = 1$, by (4), $D_1(x^{-}) \leq x^{-} \leq (D_1(x))^{-}$, and so $x^{-} \leq 1^{-} = 0$. This implies that $x^{-} = 0$. For $i = 2, 3$ we have $D_i(x) \leq x$, $1 \leq x$, and consequently $x = 1$.

□

Proposition 3.6. *Let A be a pseudo-BL algebra. If D is an isotone derivation of type 1 on A such that $D(x) \leq x$ and $D(x) = D(x) \otimes D(x)$, for all $x \in A$, then for all $x, y \in A$ the following hold:*
(1) $D(x) = D(1) \otimes x = x \otimes D(1)$;
(2) $D(x \otimes y) = D(x) \otimes D(y)$;
(3) $D(x \ominus y) \leq D(x) \ominus D(y)$, $D(x \copyright y) \leq D(x) \copyright D(y)$;
(4) $D(x \vee y) = D(x) \vee D(y)$;

(5) $D(x \wedge y) = D(x) \wedge D(y)$;
(6) $D(D(x)) = D(x)$;
(7) $D(x \rightsquigarrow y) \le D(x) \rightsquigarrow D(y), D(x \to y) \le D(x) \to D(y)$.

Proof. (1) Suppose that $x \in A$. We have $D(x) = D(1 \otimes x) = (D(1) \otimes x) \vee (1 \otimes D(x))$ also $D(x) = D(x \otimes 1) = (D(x) \otimes 1) \vee (x \otimes D(1))$. Then $D(1) \otimes x \le D(x)$ and $x \otimes D(1) \le D(x)$. Since $D(1) \otimes x \le D(x) \otimes D(x) = D(x) \le D(1) \otimes x$. Therefore $D(x) = D(1) \otimes x = x \otimes D(1)$.

(2) By (1), we have $D(x \otimes y) = D(1) \otimes (x \otimes y) = D(1) \otimes D(1) \otimes x \otimes y = x \otimes D(1) \otimes y \otimes D(1) = D(x) \otimes D(y)$.

(3) By (2) and Theorem 3.5 (4), we obtain $D(x \ominus y) = D(x) \otimes D(y^-) \le D(x) \otimes D(y)^- = D(x) \ominus D(y)$. Similarly we can prove $D(x \copyright y) \le D(x) \copyright D(y)$.
It is proved in Theorem 3.5 (3) that $D(x^\sim) \le (D(x))^\sim$. We have $D(x \copyright y) = D(y^\sim \otimes x) = D(y^\sim) \otimes D(x) \le (D(y))^\sim \otimes D(x) = D(x) \copyright D(y)$.

(3) The result follows from (2) and Theorem 3.5 (4).

(4) We use Proposition 2.4 (15) to get $D(x \vee y) = D(1) \otimes (x \vee y) = (D(1) \otimes x) \vee (D(1) \otimes y) = D(x) \vee D(y)$.

(5) By using Proposition 2.4 (16), the proof is similar to (4).

(6) By (1), $D(D(x)) = D(1) \otimes D(x) = D(1 \otimes x) = D(x)$.

(7) By (2), (PBL-3) and (PBL-4), we have $D(x) \otimes D(x \rightsquigarrow y) = D(x \otimes (x \rightsquigarrow y)) = D(x \wedge y) = D(x) \wedge D(y) \le D(y)$.
Thus $D(x \rightsquigarrow y) \le D(x) \rightsquigarrow D(y)$. Similarly we have $D(x \to y) \le D(x) \to D(y)$.

□

Example 3.7. Consider the pseudo-BL algebra A, defined in Example 2.3. We will show below that every derivation of type 3 on A should be written in the form:

$$D_3(x) = \begin{cases} \left(\dfrac{1}{2}, 0\right) & \text{if } x \neq (1,0) \\ (a,b) & \text{if } x = (1,0) \end{cases}$$

for any $(a,b) \in A$

Consider $A_1 := \{\left(\dfrac{1}{2}, b\right) \in \mathbb{R}^2 : b \ge 0\}, A_2 := \{(a,b) \in \mathbb{R}^2 : \dfrac{1}{2} < a < 1, b \in \mathbb{R}\}$ and $A_3 := \{(1,b) \in \mathbb{R}^2 : b \le 0\}$, such that $A = \bigcup_{i=1}^{3} A_i, A_i \cap A_j = \emptyset$ for $i,j \in \{1,2,3\}, i \neq j$.

In Table 2, we present the result of calculating the $x^\sim$ and x^- in A_1, A_2 and A_3

Let $x \in A_1$. $x^- = (\frac{1}{2}, b)^- = (\frac{1}{2}, 0) \vee [(1, -2b) \wedge (1, 0)] = (1, -2b)$,

$x^\sim = (\frac{1}{2}, b)^\sim = (\frac{1}{2}, 0) \vee [(1, -b) \wedge (1, 0)] = (1, -b)$.

Let $x \in A_2$. $x^- = (a, b)^- = (\frac{1}{2}, 0) \vee [(\frac{1}{2a}, \frac{-b}{a}) \wedge (1, 0)] = (\frac{1}{2a}, \frac{-b}{a})$,

$x^\sim = (a, b)^\sim = (\frac{1}{2}, 0) \vee [(\frac{1}{2a}, \frac{-b}{2a}) \wedge (1, 0)] = (\frac{1}{2a}, \frac{-b}{a})$.

Let $x \in A_3$. $x^- = (1, b)^- = (\frac{1}{2}, 0) \vee [(\frac{1}{2}, -b) \wedge (1, 0)] = (\frac{1}{2}, -b)$,

$x^\sim = (1, b)^\sim = (\frac{1}{2}, 0) \vee [(\frac{1}{2}, \frac{-b}{2}) \wedge (1, 0)] = (\frac{1}{2}, \frac{-b}{2})$.

TABLE 2.

A_i	x	x^-	$x^\sim$
A_1	$\left(\frac{1}{2}, b\right)$	$(1, -2b)$	$(1, -b)$
A_2	(a, b)	$\left(\frac{1}{2a}, \frac{-b}{a}\right)$	$\left(\frac{1}{2a}, \frac{-b}{2a}\right)$
A_3	$(1, b)$	$\left(\frac{1}{2}, -b\right)$	$\left(\frac{1}{2}, \frac{-b}{2}\right)$

Now, we calculate $(a, b)©(c, d) = (c, d)^\sim \otimes (a, b)$.

$$
\begin{aligned}
(c,d) \in A_1, \quad (a,b)©(c,d) = (c,d)^\sim \otimes (a,b) \quad &= (1, -d) \otimes (a, b) \\
&= \left(\frac{1}{2}, 0\right) \vee (a, -ad + b). \\
(c,d) \in A_2, \quad (a,b)©(c,d) = (c,d)^\sim \otimes (a,b) \quad &= \left(\frac{1}{2c}, \frac{-d}{2c}\right) \otimes (a, b) \\
&= \left(\frac{1}{2}, 0\right) \vee \left(\frac{a}{2c}, \frac{-ad}{2c} + b\right). \\
(c,d) \in A_3, \quad (a,b)©(c,d) = (c,d)^\sim \otimes (a,b) \quad &= \left(\frac{1}{2}, \frac{-d}{2}\right) \otimes (a, b) \\
&= \left(\frac{1}{2}, 0\right) \vee \left(\frac{a}{2}, \frac{-ad}{2} + b\right).
\end{aligned}
$$

Let $D_3 : A \longrightarrow A$ be defined by $D_3(a,b) = (x,y)$. We notice that $D_3(x) = D_3(x) \otimes x$.

(1) Let $(a,b) \in A_1$. $D_3(\frac{1}{2},b) = (x,y) \otimes (\frac{1}{2},b) = (\frac{1}{2},0) \vee (\frac{x}{2},\frac{y}{2} + b) = (x,y)$. Then, $x = 0, y = 2b$. Hence $(\frac{1}{2},0) \vee (0,2b) = (0,2b)$, which is a contradiction. Consequently, we have $(x,y) = (\frac{1}{2},0) = 0_A$.

(2) Let $(a,b) \in A_2$. $D_3(a,b) = (x,y) \otimes (a,b) = (\frac{1}{2},0) \vee (xa, ya+b) = (x,y)$. If $(x,y) = (xa, ya+b)$ and $\frac{1}{2} < a < 1$, then $x = 0$ and $(x,y) = (\frac{1}{2},0) = 0_A$.

(3) Let $(a,b) \in A_3$. $D_3(1,b) = (x,y) \otimes (1,b) = (\frac{1}{2},0) \vee (x, y+b) = (x,y)$. If $b < 0$, then $D_3(1,b) = (\frac{1}{2},0) = 0_A$ and if $b = 0$, then $D_3(1,0) = (x,y) \otimes (1,0) = (x,y)$, for every $(a,b) \in A$.

$$D_3(a,b) = \begin{cases} 0_A & (a,b) \neq (1,0) \\ (x,y) & (a,b) = (1,0) \end{cases}$$

In Example 3.7, none of the functions is derivation of type 2. This result will now be derived computationally. For D_2 since $D_2(x) = D_2(x) \otimes x$, we have

$$D_2(x,y) = \begin{cases} 0_A & (x,y) \neq (1,0) \\ (a,b) & (x,y) = (1,0) \end{cases}$$

Let $Y = (\frac{1}{2}, n)$, $0 < n < \frac{b}{2}$ and $X = 1$. $D_2(1 \ominus Y) = D_2(Y^-) = D_2(1,-2n) = (\frac{1}{2},0) = 0_A$.
On the other hand, we can write

$$\begin{aligned} (D_2(1) \ominus Y) \otimes (1 \ominus D_2(Y)) &= (a,b) \otimes Y^- \otimes 1 \otimes (D_2(Y))^- \\ &= (a,b) \otimes Y^- \otimes 0^- \\ &= (a,b) \otimes (1,-2n) = \left(\frac{1}{2},0\right) \vee (a, b-2n). \end{aligned}$$

If $D_2(X \ominus Y) = (D_2(X) \ominus Y) \otimes (X \ominus D_2(Y))$, then $(a, b-2n) < (\frac{1}{2},0)$, which is impossible. Thus, the derivation condition does not hold for D_2.

Proposition 3.8. *Let D be a derivation of type 1 on the pseudo-Gödel algebra A. Then for every $x, y \in A$ the following hold :*

1. *$D(x) \leq x$,*
2. *If $x \leq D(1)$, then $D(x) = x$, and $D(D(x)) = D(x)$,*
3. *If $x \geq D(1)$, then $D(1) \leq D(x)$,*
4. *If $x \leq y$, then $D(x) = x$ or $D(y) \leq D(x)$.*

Proof. (1) If $x \in A$, then $D(x) = D(x \otimes x) = (D(x) \otimes x) \vee (x \otimes D(x)) = x \otimes D(x) \leq x$.

(2) If $x \leq D(1)$, then $D(x) = D(x \otimes 1) = D(x) \vee (x \otimes D(1)) = x$.

(3) If $x \geq D(1)$, then similar to the proof of (2), we obtain $D(1) \leq D(x)$. Suppose that $x \geq D(1)$. Then $D(x) = D(x \otimes 1) = D(x) \vee (x \otimes D(1))$. By Proposition 2.9 we get that $D(x) = D(x) \vee (x \wedge D(1))$, and so $D(x) = D(x) \vee D(1)$. Therefore, we deduce that $D(1) \leq D(x)$.

(4) If $x \leq y$, then by (1) and Proposition 2.9, we have $D(x) \leq x$. This yields that $D(x) = D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y)) = (D(x) \wedge y) \vee (x \wedge D(y)) = D(x) \vee (x \wedge D(y))$. Now, we have two cases: (i) If $x \leq D(y)$, then $D(x) = D(x) \vee x$, therefore $D(x) = x$. (ii) If $D(y) \leq x$, then $D(x) = D(x) \vee D(y)$ and so $D(y) \leq D(x)$.

□

Proposition 3.9. *Let A be a pseudo-Gödel algebra. The map D given by*

$$D(x) = \begin{cases} a & \text{if } x > a \\ x & \text{if } x \leq a \end{cases}$$

is a derivation of type 1 on A.

Proof. For $x, y \in A$, we have four cases:

(1) If $x, y \leq a$, then $D(x) = x, D(y) = y, x \otimes y \leq a$,

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y)) = (x \otimes y) \vee (x \otimes y) = x \otimes y.$$

(2) If $x, y > a$, then $D(x) = D(y) = a$, by Proposition 2.4 (4), $x \otimes y > a \otimes y > a \otimes a = a$,

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y)) = (a \otimes y) \vee (x \otimes a) = (a \wedge y) \vee (x \wedge a) = a.$$

(3) If $x \leq a$ and $y > a$, then $D(x) = x, D(y) = a$, $x \otimes a < x \otimes y \leq a$,

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y)) = (x \otimes y) \vee (x \otimes a) = x \otimes y.$$

(4) If $x > a$ and $y \leq a$, then $D(x) = a, D(y) = y, x \otimes y \leq a$,

$$D(x \otimes y) = (D(x) \otimes y) \vee (x \otimes D(y)) = (a \otimes y) \vee (x \otimes y) = x \otimes y.$$

□

Proposition 3.10. *Let D be a derivation of type 3 on a pseudo-BL algebra A. Then for all $x, y \in A$, $D(x \copyright y) \leq D(x) \copyright D(y)$.*

Proof. We have

$$\begin{aligned} D(x \copyright y) &= (D(x) \copyright y) \otimes (x \copyright D(y)) \\ &\leq D(x) \copyright y = y^{\sim} \otimes D(x) \\ &\leq D(y)^{\sim} \otimes D(x) = D(x) \copyright D(y). \end{aligned}$$

□

Theorem 3.11. *Let D be an implicative derivation on pseudo-BL algebra A. For all $x, y \in A$ the following conditions hold:*

1. $\overset{\rightsquigarrow}{D}(1) = 1$ *and* $\overrightarrow{D}(1) = 1$;
2. *If* $x \leq y$ *then* $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = 1$ *and* $\overrightarrow{D}(x \rightarrow y) = 1$,
3. $\overset{\rightsquigarrow}{D}(x) = x \vee \overset{\rightsquigarrow}{D}(x)$ *and then* $\overset{\rightsquigarrow}{D}(x) \geq x$,
   $\overrightarrow{D}(x) = x \vee \overrightarrow{D}(x)$ *and then* $\overrightarrow{D}(x) \geq x$;
4. $(\overset{\rightsquigarrow}{D}x)^{\sim} \leq \overset{\rightsquigarrow}{D}(x^{\sim}), (\overrightarrow{D}x)^{-} \leq \overrightarrow{D}(x^{-})$;
5. $y \leq \overset{\rightsquigarrow}{D}(x \rightsquigarrow y)$, $y \leq \overrightarrow{D}(x \rightarrow y)$;
6. $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = x \rightsquigarrow \overset{\rightsquigarrow}{D}y, \overrightarrow{D}(x \rightarrow y) = x \rightarrow \overrightarrow{D}y$.

Proof. (1) $\overset{\rightsquigarrow}{D}(1) = \overset{\rightsquigarrow}{D}(1 \rightsquigarrow 1) = (\overset{\rightsquigarrow}{D}(1) \rightsquigarrow 1) \vee (1 \rightsquigarrow \overset{\rightsquigarrow}{D}(1) = 1$, By (PBL-5).

(2) By Proposition 2.4 (2), $x \leq y$ implies that $x \rightsquigarrow y = 1, x \rightarrow y = 1$ and by (1) it is done.

(3) $\overset{\rightsquigarrow}{D}x = \overset{\rightsquigarrow}{D}(1 \rightsquigarrow x) = (\overset{\rightsquigarrow}{D}(1) \rightsquigarrow x) \vee (1 \rightsquigarrow \overset{\rightsquigarrow}{D}x) = x \vee \overset{\rightsquigarrow}{D}x$.

(4) $\overset{\rightsquigarrow}{D}(x^{\sim}) = \overset{\rightsquigarrow}{D}(x \rightsquigarrow 0) = (\overset{\rightsquigarrow}{D}(x) \rightsquigarrow 0) \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}(0)) = (\overset{\rightsquigarrow}{D}x)^{\sim} \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}(0)) \geq (\overset{\rightsquigarrow}{D}x)^{\sim}$.

(5) $y \leq \overset{\rightsquigarrow}{D}x \rightsquigarrow y \leq (\overset{\rightsquigarrow}{D}x \rightsquigarrow y) \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}y) = \overset{\rightsquigarrow}{D}(x \rightsquigarrow y)$.

(6) From (1), we obtain $y \leq \overset{\rightsquigarrow}{D}y$, $x \leq \overset{\rightsquigarrow}{D}x$. According to Proposition 2.4 (5), $x \rightsquigarrow y \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}y$ and $\overset{\rightsquigarrow}{D}x \rightsquigarrow y \leq x \rightsquigarrow y$ which gives $\overset{\rightsquigarrow}{D}x \rightsquigarrow y \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}y$.

Therefore $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = (\overset{\rightsquigarrow}{D}x \rightsquigarrow y) \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}y) = x \rightsquigarrow \overset{\rightsquigarrow}{D}y$. □

Theorem 3.12. *Let D be an implicative derivation on the pseudo-BL algebra A. For all $x, y \in A$ the following conditions hold:*

(1) $\overset{\rightsquigarrow}{D}(x^{\sim}) = x \rightsquigarrow \overset{\rightsquigarrow}{D}(0)$ *so if* $\overset{\rightsquigarrow}{D}(0) = 0$ *then* $\overset{\rightsquigarrow}{D}(x^{\sim}) = x^{\sim}$ *and* $\overset{\rightsquigarrow}{D}(x^{-}) = x \rightarrow \vec{D}(0)$ *so if* $\overset{\rightsquigarrow}{D}(0) = 0$ *then* $\overset{\rightsquigarrow}{D}(x^{-}) = x^{-}$.

(2) $\overset{\rightsquigarrow}{D}(0) \leq \overset{\rightsquigarrow}{D}(x^{\sim})$, $\vec{D}(0) \leq \vec{D}(x^{-})$.

(3) $\overset{\rightsquigarrow}{D}(x) \otimes \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x) \wedge \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x \rightsquigarrow y)$.
$\vec{D}(x) \otimes \vec{D}(y) \leq \vec{D}(x) \wedge \vec{D}(y) \leq \vec{D}(x) \rightarrow \vec{D}(y) \leq \vec{D}(x \rightarrow y)$.

(4) $\overset{\rightsquigarrow}{D}(x) \rightsquigarrow y \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}(y)$, $\vec{D}(x) \rightarrow y \leq x \rightarrow \vec{D}(y)$.

(5) $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) \vee \overset{\rightsquigarrow}{D}(y \rightsquigarrow x) = 1$, $\vec{D}(x \rightarrow y) \vee \vec{D}(y \rightarrow x) = 1$.

(6) *If F is a filter of A, then* $\overset{\rightsquigarrow}{D}(F) \subseteq F, \vec{D}(F) \subseteq F$.

(7) $\overset{\rightsquigarrow}{D}_a(x) \rightsquigarrow (y) = \overset{\rightsquigarrow}{D}_{a \rightsquigarrow x}(y) \leq \overset{\rightsquigarrow}{D}_a(x \rightsquigarrow y) = \overset{\rightsquigarrow}{D}_a(\overset{\rightsquigarrow}{D}_x(y))$ *and* $\vec{D}_a(x) \rightarrow (y) = \vec{D}_{a \rightarrow x}(y)) \leq \vec{D}_a(x \rightarrow y) = \vec{D}_a(\vec{D}_x(y)$.

Proof. (1) In order to see this, it is enough to consider $y = 0$ in Theorem 3.11 (5).

(2) For all x, $x \leq 1$. Hence $1 \rightsquigarrow \overset{\rightsquigarrow}{D}(0)) \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}(0)$ then $\overset{\rightsquigarrow}{D}(0) \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}(0) = \overset{\rightsquigarrow}{D}(x^{\sim})$.

(3) We should prove the last inequality. By Theorem 3.11 and Proposition 2.4, $\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y) \leq x \rightsquigarrow \overset{\rightsquigarrow}{D}(y)$.

(4) See the proof of Theorem 3.11 (6)

(5) Applying (PBL-5) and (4) gives $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) \vee \overset{\rightsquigarrow}{D}(y \rightsquigarrow x) = (x \rightsquigarrow \overset{\rightsquigarrow}{D}(y)) \vee (y \rightsquigarrow \overset{\rightsquigarrow}{D}(x)) \geq (\overset{\rightsquigarrow}{D}(x) \rightsquigarrow y) \vee y \rightsquigarrow \overset{\rightsquigarrow}{D}(x) = 1$.

(6) If $x \in F$ then $\overset{\rightsquigarrow}{D}(x) \in \overset{\rightsquigarrow}{D}(F)$. Since $x \leq \overset{\rightsquigarrow}{D}(x)$ then $\overset{\rightsquigarrow}{D}(x) \in F$.

(7) We have $x \otimes a \leq a \wedge x \leq a \rightsquigarrow x$. Then $(a \rightsquigarrow x) \rightsquigarrow y \leq (x \otimes a) \rightsquigarrow y = a \rightsquigarrow (x \rightsquigarrow y)$.

□

Lemma 3.13. *Let* $\overset{\rightsquigarrow}{D}$, $\vec{D}$ *are implicative derivation on a pseudo-BL algebra A. Then*

(1) *If* $\overset{\rightsquigarrow}{D}$ *is isotone,* $\overset{\rightsquigarrow}{D}(x) \geq \overset{\rightsquigarrow}{D}(0) \vee x$. *If* $\vec{D}$ *is isotone,* $\vec{D}(x) \geq \vec{D}(0) \vee x$.

(2) *If* $\overset{\rightsquigarrow}{D}(x) = \overset{\rightsquigarrow}{D}(0) \vee x$ *then* $\overset{\rightsquigarrow}{D}$ *is isotone. If* $\overset{\rightsquigarrow}{D}(x) = \overset{\rightsquigarrow}{D}(0) \vee x$ *then* $\overset{\rightsquigarrow}{D}$ *is isotone.*

Proof. (1) For all $x \in A, x \geq 0$. Therefore $\overset{\rightsquigarrow}{D}(x) \geq \overset{\rightsquigarrow}{D}(0)$ and by Theorem 3.11 (3), $\overset{\rightsquigarrow}{D}(x) \geq \overset{\rightsquigarrow}{D}(0) \vee x$.

(2) If $x \leq y$, then $x \vee \overset{\rightsquigarrow}{D}(0) \leq y \vee \overset{\rightsquigarrow}{D}(0)$. Hence $\overset{\rightsquigarrow}{D}(x) \leq \overset{\rightsquigarrow}{D}(y)$.

□

Theorem 3.14. *Let $\overset{\rightsquigarrow}{D}$ and $\overrightarrow{D}$ be implicative derivations on a pseudo-BL algebra A. Then, $\overset{\rightsquigarrow}{D}$ is an isotone derivation if and only if* $\overset{\rightsquigarrow}{D}(x \wedge y) \leq \overset{\rightsquigarrow}{D}(x) \wedge \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x) \vee \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x \vee y)$ *and $\overrightarrow{D}$ is an isotone derivation if and only if* $\overrightarrow{D}(x \wedge y) \leq \overrightarrow{D}(x) \wedge \overrightarrow{D}(y) \leq \overrightarrow{D}(x) \vee \overrightarrow{D}(y) \leq \overrightarrow{D}(x \vee y)$.

Proof. $(\Rightarrow)$: We have $x \wedge y \leq x, y \leq x \vee y$. Then $\overset{\rightsquigarrow}{D}(x \wedge y) \leq \overset{\rightsquigarrow}{D}(x) \wedge \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x) \vee \overset{\rightsquigarrow}{D}(y)$ also $\overset{\rightsquigarrow}{D}(x), \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x \vee y)$ therefore $\overset{\rightsquigarrow}{D}(x) \vee \overset{\rightsquigarrow}{D}(y) \leq \overset{\rightsquigarrow}{D}(x \vee y)$.

$(\Leftarrow)$: Suppose that $x \leq y$. Then, we obtain $x \wedge y = x, x \vee y = y$. The remain is straightforward.

□

Proposition 3.15. *Let $\overset{\rightsquigarrow}{D}_1, \overset{\rightsquigarrow}{D}_2, ..., \overset{\rightsquigarrow}{D}_n$ be $(\rightsquigarrow, \vee)$-derivations on the pseudo-BL algebra A. Then $\overset{\rightsquigarrow}{D}_1 o \overset{\rightsquigarrow}{D}_2 o ... o \overset{\rightsquigarrow}{D}_n$ is a $(\rightsquigarrow, \vee)$-derivation on A.*

Proof. $\overset{\rightsquigarrow}{D}_1 \text{o} \overset{\rightsquigarrow}{D}_2 \text{o} ... \text{o} \overset{\rightsquigarrow}{D}_n(x \rightsquigarrow y) = \overset{\rightsquigarrow}{D}_1 \text{o} \overset{\rightsquigarrow}{D}_2 \text{o} ... \text{o} \overset{\rightsquigarrow}{D}_{n-1}(x \rightsquigarrow \overset{\rightsquigarrow}{D}_n(y)) = \overset{\rightsquigarrow}{D}_1 \text{o} \overset{\rightsquigarrow}{D}_2 \text{o} ... \text{o} \overset{\rightsquigarrow}{D}_{n-2}(x \rightsquigarrow \overset{\rightsquigarrow}{D}_{n-1}(\overset{\rightsquigarrow}{D}_n(y))) = \overset{\rightsquigarrow}{D}_1(x \rightsquigarrow \overset{\rightsquigarrow}{D}_2(\overset{\rightsquigarrow}{D}_3(...(\overset{\rightsquigarrow}{D}_n(y)))))$ $= x \rightsquigarrow \overset{\rightsquigarrow}{D}_1 \text{o} \overset{\rightsquigarrow}{D}_2 \text{o} ... \text{o} \overset{\rightsquigarrow}{D}_n(y)$. □

Corollary 3.16. *Let $\overrightarrow{D}_1, \overrightarrow{D}_2, \ldots, \overrightarrow{D}_n$ be $(\rightarrow, \vee)$-derivations on the pseudo-BL algebra A. Then $\overrightarrow{D}_1 o \overrightarrow{D}_2 o ... o \overrightarrow{D}_n$ is a $(\rightarrow, \vee)$-derivation on A.*

Corollary 3.17. $\overset{\rightsquigarrow}{D^n}(x \rightsquigarrow y) = x \rightsquigarrow \overset{\rightsquigarrow}{D^n}(y)$ *and* $\overrightarrow{D^n}(x \rightarrow y) = x \rightarrow \overrightarrow{D^n}(y)$.

Theorem 3.18. *Let A be a pseudo-BL algebra, $a \in A$ and suppose that $\overset{\rightsquigarrow}{D}_a$ and $\overrightarrow{D}_a$ are functions $\overset{\rightsquigarrow}{D}_a : A \longrightarrow A, \overrightarrow{D}_a : A \longrightarrow A$ such that $\overset{\rightsquigarrow}{D}_a(x) = a \rightsquigarrow x, \overrightarrow{D}_a(x) = a \rightarrow x$. Then the following conditions hold:*

1. *If for all $x \in A, x \otimes a = a \otimes x$ then $\overset{\rightsquigarrow}{D}_a$ is $(\rightsquigarrow, \vee)$-derivation and $\overrightarrow{D}_a$ is $(\rightarrow, \vee)$-derivation;*
2. *$\overset{\rightsquigarrow}{D}_a, \overrightarrow{D}_a$ are isotone;*
3. *$\overset{\rightsquigarrow}{D}_1(x), \overrightarrow{D}_1(x)$ are the identity function. In addition, $\overset{\rightsquigarrow}{D}_0(x)$, $\overset{\rightsquigarrow}{D}_x(x)$, $\overrightarrow{D}_0(x)$ and $\overrightarrow{D}_x(x)$ are constant.*

Proof. (1) We should prove: $\overset{\rightsquigarrow}{D}_a(x \rightsquigarrow y) = (\overset{\rightsquigarrow}{D}_a(x) \rightsquigarrow y) \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}_a(y))$.

(LHS:) We have $\overset{\rightsquigarrow}{D}_a(x \rightsquigarrow y) = a \rightsquigarrow (x \rightsquigarrow y) = (x \otimes a) \rightsquigarrow y = \overset{\rightsquigarrow}{D}_{x\otimes a}(y)$.
(RHS:) We have

$$\begin{aligned}(\overset{\rightsquigarrow}{D}_a(x) \rightsquigarrow y) \vee (x \rightsquigarrow \overset{\rightsquigarrow}{D}_a(y)) &= ((a \rightsquigarrow x) \rightsquigarrow y) \vee (x \rightsquigarrow (a \rightsquigarrow y)) \\ &= ((a \rightsquigarrow x) \rightsquigarrow y) \vee ((a \otimes x) \rightsquigarrow y) \\ &= ((a \otimes x) \rightsquigarrow y) = \overset{\rightsquigarrow}{D}_{x\otimes a}(y).\end{aligned}$$

Since $a \otimes x \leq a \rightsquigarrow x$, it follows that $(a \rightsquigarrow x) \rightsquigarrow y \leq (a \otimes x) \rightsquigarrow y$.

(2) If $x \leq y$, then $a \rightsquigarrow x \leq a \rightsquigarrow y$.

(3) It is straightforward.

□

Corollary 3.19. *Let A be a pseudo-BL algebra, $a \in A$ and suppose that $\overset{\rightsquigarrow}{D}_a, \vec{D}_a$ are functions $\overset{\rightsquigarrow}{D}_a : A \longrightarrow A, \vec{D}_a : A \longrightarrow A$ such that $\overset{\rightsquigarrow}{D}_a(x) = a \rightsquigarrow x, \vec{D}_a(x) = a \rightarrow x$. Then the following conditions hold:*

(1) $\overset{\rightsquigarrow}{D}_x(\overset{\rightsquigarrow}{D}_y(z)) = \overset{\rightsquigarrow}{D}_{y\otimes x}(z) = \overset{\rightsquigarrow}{D}_{y\wedge x}(z) = \overset{\rightsquigarrow}{D}_x(y \rightsquigarrow z)$.

(2) $\overset{\rightsquigarrow}{D}_a(b) \leq \overset{\rightsquigarrow}{D}_{c\rightsquigarrow a}(c \rightsquigarrow b) = \overset{\rightsquigarrow}{D}_c(a) \rightsquigarrow \overset{\rightsquigarrow}{D}_c(b) = \overset{\rightsquigarrow}{D}_{c\wedge a}(b)$.

(3) $\overset{\rightsquigarrow}{D}_a(b) \otimes \overset{\rightsquigarrow}{D}_{a'}(b') \leq \overset{\rightsquigarrow}{D}_{a\vee a'}(b \vee b'), \overset{\rightsquigarrow}{D}_{a\wedge a'}(b \wedge b')$.

(4) $\overset{\rightsquigarrow}{D}_{a_1}(a_2) \otimes \overset{\rightsquigarrow}{D}_{a_2}(a_3) \otimes ... \otimes \overset{\rightsquigarrow}{D}_{a_{n-1}}(a_n) \leq \overset{\rightsquigarrow}{D}_{a_1}(a_n)$.

(5) $\overset{\rightsquigarrow}{D}_a(b) \leq \overset{\rightsquigarrow}{D}_{b^\sim}(a^\sim)$.

(6) $\vec{D}_a(b^\sim) = \overset{\rightsquigarrow}{D}_b(a^-)$.

(7) $\overset{\rightsquigarrow}{D}_a(b) \leq \overset{\rightsquigarrow}{D}_{c\otimes a}(c \otimes b)$.

4. (φ, ψ)-DERIVATIONS ON PSEUDO-BL ALGEBRAS

In this section, we have generalized the notion of derivation on a pseudo-BL algebra A to (φ, ψ)-derivations on A by using two functions φ and ψ of A into itself. These derivations are extended by introducing the notions of (φ, ψ)-derivations of type 1, 2, 3, $\overset{\rightsquigarrow}{(\varphi, \psi)}$-derivation, $\overset{\rightarrow}{(\varphi, \psi)}$-derivation and also investigate some related properties.

Definition 4.1. Let $(A, \vee, \wedge, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra. Then for all $x, y \in A$ the map $D : A \rightarrow A$ is called

(1) a (φ, ψ)-*derivation of type 1*, if $D(x \otimes y) = (D(x) \otimes \varphi(y)) \vee (\psi(x) \otimes D(y))$;

(2) a (φ, ψ)-*derivation of type 2*, if $D(x \ominus y) = (D(x) \ominus \varphi(y)) \otimes (\psi(x) \ominus D(y))$;

(3) a (φ,ψ)-*derivation of type 3*, if $D(x© y) = (D(x)© \varphi(y)) \otimes (\psi(x)© D(y))$.

If a pseudo-BL algebra A is BL-algebra, then every derivation of type 3 on A coincides with derivation of type 2 on A.

Definition 4.2. Let $(A, \vee, \wedge, \otimes, \to, \rightsquigarrow, 0, 1)$ be a pseudo-BL algebra. Then the map $D : A \to A$ is a (φ,ψ)-implicative derivation and called

(4) a $\overrightarrow{(\varphi,\psi)}$-*derivation* if $D(x \to y) = (Dx \to \varphi(y)) \vee (\psi(x) \to Dy)$ for all $x, y \in A$;

(5) a $\overset{\rightsquigarrow}{(\varphi,\psi)}$-*derivation* if $D(x \rightsquigarrow y) = (Dx \rightsquigarrow \varphi(y)) \vee (\psi(x) \rightsquigarrow Dy)$ for all $x, y \in A$.

Theorem 4.3. *Let A be a pseudo-BL algebra and D be a (φ,ψ)-derivation of type 1 on A. Then the following conditions hold*

(1) $D(0) = 0$;

(2) *If $x \le y$ then $D(x) \le \varphi(y)^{\sim\sim}$ and $\psi(x) \le D(y^{\sim})^{\sim}$;*

(3) $D(x) \le \varphi(x)^{\sim\sim}, \psi(x) \le D(x^{\sim})^{\sim}$ *and moreover $x \in B(A)$ implies that $D(x) \le \varphi(x)$;*

(4) $D(x) = (D(1) \otimes \varphi(x)) \vee D(x)$;

(5) $D(x^{\sim}) \le D(x)^{\sim}$;

(6) $D(x^{\sim}) \le \varphi(x^{\sim})$.

Proof. (1) Since φ and ψ are homomorphisms, it follows that $D(0) = D(0 \otimes 0) = (D(0) \otimes \varphi(0)) \vee (\psi(0) \otimes D(0))$. Consequently, we obtain $D(0) = 0$.

(2) Suppose that $x \le y$. Then, we get $x \otimes y^{\sim} = 0$, and so $0 = D(0) = D(x \otimes y^{\sim}) = (D(x) \otimes \varphi(y^{\sim})) \vee (\psi(x) \otimes D(y^{\sim}))$. Hence, we obtain $D(x) \otimes \varphi(y^{\sim}) = \psi(x) \otimes D(y^{\sim}) = 0$. Now, by Proposition 2.4, we have $D(x) \le \varphi(y)^{\sim\sim}, \psi(x) \le D(y^{\sim})^{\sim}$.

(3) Take $x = y$ in (2).

(4) Let $x \in A$. We have $D(x) = D(1 \otimes x) = (D(1) \otimes \varphi(x)) \vee (\psi(1) \otimes D(x))$. By (As) ψ is homomorphism, $\psi(1) = 1, D(x) = (D(1) \otimes \varphi(x)) \vee D(x)$.

(5) By (3) and Proposition 2.4, $D(x) \le \varphi(x)^{\sim\sim}, D(x^{\sim}) \le \varphi(x)^{\sim\sim\sim}$ and so $\varphi(x)^{\sim\sim\sim} \le D(x)^{\sim}$. Hence $D(x^{\sim}) \le D(x)^{\sim}$.

(6) For every $x \in A$, we have $D(x^{\sim}) \le \varphi(x)^{\sim\sim\sim} = \varphi(x)^{\sim} = \varphi(x^{\sim})$. □

Theorem 4.4. *Let D be a (φ,ψ)-derivation of type1 on A and assume that $D(1) = 1$. Then the following conditions hold:*

(1) $\varphi(x) \le D(x)$ *and* $\psi(x) \le D(x)$ *for all* $x \in A$.

(2) $D(B(A)) = \varphi(B(A))$.

(3) *D is an isotone on A.*

Proof. (1) Let $D(1) = 1$. Then, by Theorem 4.3, for all $x \in A$ we have $\varphi(x) = D(1) \otimes \varphi(x) \leq D(x)$. Similarly we can conclude $\psi(x) \leq D(x)$.

(2) Let $x \in B(A)$. From Theorem 4.3 we have $D(x) \leq \varphi(x)$ and by (1), we get that $D(B(A)) = \varphi(B(A))$.

(3) Let $x \leq y$. By (PBL-4) and (1), we get $D(x) = D(y \wedge x) = D(y \otimes (y \rightsquigarrow x)) = (D(y) \otimes \varphi(y \rightsquigarrow x)) \vee (\psi(y) \otimes D(y \rightsquigarrow x)) \leq D(y) \vee \psi(y) = D(y)$.

□

Theorem 4.5. *Let D be a (φ, ψ)-derivation of type 1 on the pseudo-BL algebra A. If $D(x \vee y) = D(x) \vee D(y)$ or $D(x \wedge y) = D(x) \wedge D(y)$ for all $x, y \in A$, then D is an isotone on A.*

Proof. Let $x, y \in A$ and $x \leq y$. Then $D(x) \leq D(x) \vee D(y) = D(x \vee y) = D(y)$ or $D(x) = D(x \wedge y) = D(x) \wedge D(y) \leq D(y)$. This shows that $D(x) \leq D(y)$. □

Theorem 4.6. *Let D be a (φ, ψ)-derivation of type 1 on the pseudo-BL algebra A. Then for all $x, y \in A$ we have*

(1) *$D(x \otimes y) \leq D(x) \vee D(y)$.*

(2) *$KerD = \{x \in A : D(x) = 0\}$ is closed under $\otimes$.*

Proof. (1) By Definition 4.1 and Proposition 2.4, we have $D(x \otimes y) = (D(x) \otimes \varphi(y)) \vee (\psi(x) \otimes D(y)) \leq D(x) \vee D(y)$.

(2) Suppose that x and y are arbitrary elements in A. Then $D(x) = D(y) = 0$. From (1) we have $D(x \otimes y) \leq D(x) \vee D(y) = 0 \vee 0 = 0$. So, we obtain $D(x \otimes y) = 0$. This yields that $x \otimes y \in \mathrm{Ker}A$.

□

Lemma 4.7. *If D is a (φ, ψ)-derivation of type 1 on the Boolean center $B(A)$ then D is a lattice (φ, ψ)-derivation.*

Proof. Let $x, y \in B(A)$. Since φ, ψ are homomorphisms, it follows that $D(x \wedge y) = D(x \otimes y) = (D(x) \otimes \varphi(y)) \vee (\psi(x) \otimes D(y)) = (D(x) \wedge \varphi(y)) \vee (\psi(x) \wedge D(y))$.

□

Theorem 4.8. *Let D be a (φ, ψ)-derivation of type 1 on pseudo-BL algebra A and assume that $A = B(A)$. Then for all $x, y \in A$ the following hold:*

(1) *If $y \leq x$ and $D(x) = \varphi(x)$ then $D(y) = \varphi(y)$.*

(2) *Let $Fix_D(A) = \{x \in A : D(x) = \varphi(x)\}$. If D is a homomorphism, then $Fix_D(x)$ is an ideal of A.*

(3) *If* $x \in Fix_D(A)$ *and* $D(1) = 1$ *then* $x^{\sim} \in Fix_D(A)$.
(4) $D(1) = 1$ *if and only if* $Fix_D(A) = A$.

Proof. This has already been proved in [1] Theorem 4.6. □

Theorem 4.9. *Let* $D : A \longrightarrow A$ *be defined by* $D(x) = \varphi(x) \otimes a$ *for all* $a \in B(A)$ *and* $x \in A$ *such that* φ *is a homomorphism on* A*. Then the following conditions hold:*

(1) D *is a* φ*-derivation of type 1 on* A.
(2) D *is an isotone on* A.
(3) $D(x \vee y) = D(x) \vee D(y)$ *and* $D(x \wedge y) = D(x) \wedge D(y)$ *for every* $x, y \in A$.

Proof. (1) From (PBL-2) and Proposition 2.4 we get

$$\begin{aligned} D(x \otimes y) = \varphi(x \otimes y) \otimes a &= (\varphi(x \otimes y) \vee \varphi(x \otimes y)) \otimes a \\ &= (\varphi(x \otimes y) \otimes a) \vee (\varphi(x \otimes y) \otimes a) \\ &= (\varphi(x) \otimes \varphi(y) \otimes a) \vee (\varphi(x)\varphi(y) \otimes a) \\ &= (\varphi(x) \otimes a \otimes \varphi(y)) \vee (\varphi(x) \otimes D(y)) \\ &= (D(x) \otimes \varphi(y)) \vee (\varphi(x) \otimes D(y)). \end{aligned}$$

(2) Let $x \leq y$. Then $D(x) = \varphi(x) \otimes a = D(x \wedge y) = \varphi(x \wedge y) \otimes a = (\varphi(x) \otimes a) \wedge (\varphi(y) \otimes a) = D(x) \wedge D(y) \leq D(y)$. Hence D is an isotone.
(3) We have $D(x \vee y) = \varphi(x \vee y) \otimes a = (\varphi(x) \vee \varphi(y) \otimes a) = D(x) \vee D(y)$. Similarly, we obtain $D(x \wedge y) = D(x) \wedge D(y)$. □

Theorem 4.10. *Let* D *be a* φ*-derivation of type 1 on* A *and assume that* $D(1) \in B(A)$*. Then the following are equivalent for all* $x, y \in B(A)$*:*

(1) D *is an isotone;*
(2) $D(x) \leq D(1)$*;*
(3) $D(x) = \varphi(x) \otimes D(1)$*;*
(4) $D(x \wedge y) = D(x) \wedge D(y)$*;*
(5) $D(x \vee y) = D(x) \vee D(y)$*;*
(6) $D(x \otimes y) = D(x) \otimes D(y)$.

Proof. ($1 \Rightarrow 2$) For all $x \in A$ we always have $x \leq 1$. Since D is isotone then $D(x) \leq D(1)$.
($2 \Rightarrow 3$) Suppose that $D(x) \leq D(1)$. By Theorem 4.3 $D(x) \leq \varphi(x)$ and also by Definition 4.1, we have $D(x) = D(x \otimes 1) = (D(x) \otimes \varphi(1)) \vee (\varphi(x) \otimes D(1))$. Therefore $\varphi(x) \otimes D(1) \leq D(x) \leq \varphi(x) \wedge D(1) = \varphi(x) \otimes D(1)$ That proves $D(x) = \varphi(x) \otimes D(1)$.
($3 \Rightarrow 1$) Let $x \leq y$. Then $D(x) = \varphi(x) \otimes D(1) \leq \varphi(y) \otimes D(1) = D(y)$.

$(3 \Rightarrow 4)$ Setting $a = D(1)$ in Theorem 4.9 yields the assertion.
$(4 \Rightarrow 1)$,$(5 \Rightarrow 1)$ Follows from Theorem 4.5.
$(3 \Rightarrow 6)$ For all $x, y \in A$ (Let $x, y \in A$) $D(x \otimes y) = \varphi(x \otimes y) \otimes D(1) = (\varphi(x) \otimes \varphi(y)) \otimes (D(1) \otimes D(1)) = (\varphi(x) \otimes D(1)) \otimes (\varphi(y) \otimes D(1)) = D(x) \otimes D(y)$.
$(6 \Rightarrow 2)$ We have $D(x) = D(x \otimes 1) = D(x) \otimes D(1) \leq D(1)$. Hence $D(x) \leq D(1)$.

□

The remainder of this section will be devoted to the derivation of types 2, 3 and the following is about (φ, ψ)-implicative derivation. In some similar theorems about types 2 or 3 we prove theorem for type 3 and for type 2 can be proved in much the same way. Similarly for implicative derivation we only prove theorem for $(\rightsquigarrow, \vee)$-derivation.

Theorem 4.11. *Let A be a pseudo-BL algebra and D be a (φ, ψ)-derivation of type 2 or 3 on A. Then for all $x \in A$ the following conditions hold:*

(1) $D(0) = 0$;
(2) $D(x) = D(x) \otimes \psi(x)$;
(3) $D(x) \leq \psi(x)$;
(4) *If* $D(x) = 1$ *then* $\psi(x) = 1$;
(5) *For type 3,* $D(x^{\sim}) = D(1)©\varphi(x) \otimes D(x^{\sim})$ *and* $D(x^{\sim}) \leq \varphi(x^{\sim}) \wedge (D(x))^{\sim}$. *Also for type 2,* $D(x^{-}) = (D(1) \ominus \varphi(x)) \ominus D(x)$ *and so* $D(x^{-}) \leq (D(x))^{-}$.

Proof. (1) Let $x \in A$. Then $D(0) = D(0©x) = (D(0)©\varphi(x)) \otimes (\psi(0)©D(x) = (\varphi(x))^{\sim} \otimes D(0)) \otimes 0 = 0$.
(2) We have $x = x©0$, by Definition (4.1), $D(x) = D(x© 0) = (D(x)©\varphi(0)) \otimes (\psi(x)©D(0)) = (D(x)©0) \otimes (\psi(x)©0) = D(x) \otimes \psi(x)$.
(3) By (2) and Proposition 2.4 $D(x) = D(x) \otimes \psi(x) \leq \psi(x)$.
(4) If $D(x) = 1$ then $\psi(x) \geq 1$ this gives $\psi(x) = 1$.
(5) For every $x \in A$, $D(x^{\sim}) = D(1©x) = (D(1)©\varphi(x)) \otimes (\psi(1) ©D(x)) = (\varphi(x))^{\sim} \otimes D(1) \otimes (D(x))^{\sim} \leq (D(x))^{\sim} \wedge \varphi(x^{\sim})$.

□

Theorem 4.12. *Let A be a pseudo-BL algebra and D be a (φ, ψ)-derivation of type i on A, i={1,2} . If $x \in B(A)$ then for i=2; $D(x) = (D(1) \otimes \varphi(x)) \ominus D(x^{-})$ and $D(x) \leq \varphi(x)$. If $y \in B(A)$ then $D(x \wedge y) \leq D(x) \ominus (D(y^{-})$. Moreover for i=3; $D(x) = (\varphi(x) \otimes D(1)) \otimes (D(x^{-}))^{\sim}$and $D(x) \leq \varphi(x)$ also if $y \in B(A)$ then $D(x \wedge y) \leq D(y) \otimes (D(x^{-}))^{\sim}$.*

Proof. We prove the theorem for $i = 3$, and for $i = 2$ is similar.
Let $x \in B(A)$. Then, we can write

$$\begin{aligned} D(x) &= D(1 © x^-) = (D(1) © \varphi(x^-)) \otimes (\psi(1) © D(x^-)) \\ &= (D(1) © (\varphi(x))^-) \otimes (D(x^-))^\sim \\ &= (\varphi(x))^{-\sim} \otimes D(1) \otimes (D(x^-))^\sim \\ &= \varphi(x) \otimes D(1) \otimes (D(x^-))^\sim \leq \varphi(x). \end{aligned}$$

□

Corollary 4.13. *Let D be either a φ-derivation of type 2 or 3 on the pseudo-BL algebra A. Then for all $x \in A$, $D(x) = D(x) \otimes \varphi(x)$ and consequently $D(x) \leq \varphi(x)$.*

Theorem 4.14. *Let D be a (φ, ψ)-derivations of type 2 or 3 on the pseudo-BL algebra A. Then the following hold:*

(1) *For type 2, $D(x) \ominus \psi(y) \leq \psi(x) \ominus D(y)$ and for type 3, $D(x) © \psi(y) \leq \psi(x) © D(y)$;*
(2) *For type 2, $D(x \ominus y) \leq D(x) \ominus D(y)$ and for type 3, $D(x © y) \leq D(x) \otimes (D(y))^\sim$.*

Proof. (1) From Theorem 4.11 we have $D(y) \leq \psi(y)$ and $D(x) \leq \psi(x)$. Then by Proposition 2.4, suppose that $(\psi(y))^\sim \leq (D(y))^\sim$ and so $(\psi(y))^\sim \otimes D(x) \leq (D(y))^\sim \otimes \psi(x)$. Consequently, $D(x) © \psi(y) \leq \psi(x) © D(y)$.
(2) Suppose that $x, y \in A$. We have $D(x © y) = (D(x) © \varphi(y)) \otimes (\psi(x) © D(y)) = (\varphi(y^\sim) \otimes D(x)) \otimes ((D(y))^\sim \otimes \psi(x)) \leq D(x) \otimes (D(y))^\sim$.
□

Theorem 4.15. *Let D be a (φ, ψ)-derivations of type 2 or 3 on the pseudo-BL algebra A. Then the following hold:*

(1) *D is an isotone (φ, ψ)-derivation on $B(A)$.*
(2) *If $D(x \wedge y) = D(x) \wedge D(y)$ or $D(x \vee y) = D(x) \vee D(y)$ then D is an isotone on A.*

Proof. (1) Let $x, y \in B(A)$, $x \leq y$ and suppose that D is of type 3. Then, we can write

$$\begin{aligned} D(x) &= D(x \wedge y) = D(x \otimes y) = D(y © x^-) \\ &= (D(y) © \varphi(x^-)) \otimes (\psi(y) © D(x^-)) \\ &= ((\varphi(x^-))^\sim \otimes D(y)) \otimes (\psi(y) © D(x^-)) \leq D(y). \end{aligned}$$

This yields that D is an isotone.

(2) If $x \leq y$, then $x \wedge y = x$, which implies that $D(x) = D(x \wedge y) = D(x) \wedge D(y) \leq D(y)$.

□

Proposition 4.16. *Let D be a φ-derivation of type 2 or 3 on the pseudo-BL algebra A and φ be a monomorphism. Then for every $x \in Fix_D(A) = \{x \in A : D(x) = \varphi(x)\}$, $x \otimes x = x$.*

Proof. Let $x \in \mathrm{Fix}_D(A)$. From Theorem 4.11 (2), $D(x) = D(x) \otimes \varphi(x)$. We have $\varphi(x) = \varphi(x) \otimes \varphi(x)$. Since φ is monomorphism, it follows that $\varphi(x) = \varphi(x \otimes x)$. Therefore, we conclude that $x = x \otimes x$. □

Theorem 4.17. *Let $D : A \longrightarrow A$ be defined by $D(x) = \varphi(x) \otimes a$ for all $x, a \in A$ such that φ is a homomorphism on the pseudo-BL algebra A and $D(A) \subseteq B(A)$. Then D is a φ-derivation of type 3.*

Proof. Suppose that $x, y \in A$. Since $D(x) \leq \varphi(x)$, by Proposition 2.9, it follows that

$$\begin{aligned}(D(x)©\varphi(y)) \otimes (\varphi(x)©D(y)) &= (\varphi(y)^{\sim} \otimes D(x)) \otimes (D(y)^{\sim} \otimes \varphi(x)) \\ &= (\varphi(y)^{\sim} \otimes D(y)^{\sim}) \otimes (D(x) \otimes \varphi(x)) \\ &= (\varphi(y) \vee D(y))^{\sim} \otimes (D(x) \wedge \varphi(x)) \\ &= \varphi(y^{\sim}) \otimes \varphi(x) \otimes a = \varphi(x©y) \otimes a \\ &= D(x©y).\end{aligned}$$

□

Theorem 4.18. *Let $D : A \longrightarrow A$ be defined by $D(x) = a \otimes \varphi(x)$ for all $x, a \in A$ such that φ is a homomorphism on the pseudo-BL algebra A and $D(A) \subseteq B(A)$. Then D is a φ-derivation of type 2.*

Proof. Similar to the proof of Theorem 4.17. □

Theorem 4.19. *Let D be a φ-derivation of type 2 or 3 on pseudo-BL algebra A. Then for all $x, y \in B(A)$ the following hold:*

(1) *D is an isotone φ-derivation.*
(2) *$D(x) = D(1) \otimes \varphi(x)$.*
(3) *$D(x \wedge y) = D(x) \wedge D(y)$ and $D(x \vee y) = D(x) \vee D(y)$.*
(4) *If $D(1) = D(1) \otimes D(1)$ then $D(x \otimes y) = D(x) \otimes D(y)$.*

Proof. (1) The result follows from Theorem 4.15.

(2) Let $x \in B(A)$. By Theorem 4.11 (6), $\varphi(x)^{\sim} \leq D(x)^{\sim}$. Hence, we obtain

$$\begin{aligned}D(x) &= D(1©x^{-}) = (D(1)©\varphi(x^{-})) \otimes (\varphi(1)©D(x^{-})) \\ &= (\varphi(x^{-}))^{\sim} \otimes D(1) \otimes D(x^{-})^{\sim} = D(1) \otimes [(\varphi(x^{-}))^{\sim} \wedge D(x^{-})^{\sim}] \\ &= D(1) \otimes \varphi(x)^{-\sim} = D(1) \otimes \varphi(x).\end{aligned}$$

For type 2 can be proved in much the same way.

(3) Combining (2) and Proposition 2.4 gives $D(x\wedge y) = D(1)\otimes\varphi(x\wedge y) = D(1)\otimes(\varphi(x)\wedge\varphi(y)) = (D(1)\otimes\varphi(x))\wedge(D(1)\otimes\varphi(y)) = D(x)\wedge D(y)$.

(4) Let $x, y \in A$, $D(x\otimes y) = D(1)\otimes\varphi(x\otimes y) = (D(1)\otimes D(1))\otimes(\varphi(x)\otimes\varphi(y)) = (D(1)\otimes\varphi(x))\otimes(D(1)\otimes\varphi(y)) = D(x)\otimes D(y)$. □

Theorem 4.20. *Let D be an implicative (φ, ψ)-derivation on pseudo-BL algebra A. For all $x, y \in A$ the following hold*

(1) $\overset{\rightsquigarrow}{D}(1) = 1$ *and* $\vec{D}(1) = 1$,

(2) *If* $x \leq y$ *then* $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = 1$ *and* $\vec{D}(x \rightarrow y) = 1$,

(3) $\overset{\rightsquigarrow}{D}(x) = \varphi(x) \vee \overset{\rightsquigarrow}{D}(x)$ *and then* $\overset{\rightsquigarrow}{D}(x) \geq \varphi(x)$;
$\vec{D}(x) = \varphi(x) \vee \vec{D}(x)$ *and then* $\vec{D}(x) \geq \varphi(x)$,

(4) $(\overset{\rightsquigarrow}{D}x)^{\sim} \leq \overset{\rightsquigarrow}{D}(x^{\sim})$ *and* $(\vec{D}x)^{-} \leq \vec{D}(x^{-})$.

Proof. (1) $\overset{\rightsquigarrow}{D}(1) = \overset{\rightsquigarrow}{D}(1 \rightsquigarrow 1) = (\overset{\rightsquigarrow}{D}(1) \rightsquigarrow \varphi(1)) \vee (\psi(1) \rightsquigarrow \overset{\rightsquigarrow}{D}(1) = 1$.

(2) The result follows from (1).

(3) $\overset{\rightsquigarrow}{D}(x) = \overset{\rightsquigarrow}{D}(1 \rightsquigarrow x) = (\overset{\rightsquigarrow}{D}(1) \rightsquigarrow \varphi(1)) \vee (\psi(1) \rightsquigarrow \overset{\rightsquigarrow}{D}(1)) = \varphi(x) \vee \overset{\rightsquigarrow}{D}(x)$.

(4) $\overset{\rightsquigarrow}{D}(x^{\sim}) = \overset{\rightsquigarrow}{D}(x \rightsquigarrow 0) = (\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(0)) \vee (\psi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(0)) \geq (\overset{\rightsquigarrow}{D}x)^{\sim}$. □

Theorem 4.21. *Let D be an implicative φ-derivation on pseudo-BL algebra A. For all $x, y \in A$ the following conditions hold:*

(1) $\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(y) \leq \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y)$ *and* $\vec{D}(x) \rightarrow \varphi(y) \leq \varphi(x) \rightarrow \vec{D}(y)$;

(2) $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}y$ *and* $\vec{D}(x \rightarrow y) = \varphi(x) \rightarrow \vec{D}y$. *Consequently,* $\overset{\rightsquigarrow}{D}(x^{\sim}) = \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(0)$ *and* $\vec{D}(x^{-}) = \varphi(x) \rightarrow \vec{D}(0)$. *So, if* $\overset{\rightsquigarrow}{D}(0) = 0$, *then* $\overset{\rightsquigarrow}{D}(x^{\sim}) = (\varphi(x))^{\sim}$ *and if* $\vec{D}(0) = 0$, *then* $\vec{D}(x^{-}) = \varphi(x)^{-}$;

(3) $\overset{\rightsquigarrow}{D}{}^{n}(x \rightsquigarrow y) = \varphi^{n}(x) \rightsquigarrow \overset{\rightsquigarrow}{D}{}^{n}(y)$ *and* $\vec{D}{}^{n}(x \rightarrow y) = \varphi^{n}(x) \rightarrow \vec{D}{}^{n}(y)$;

(4) $\overset{\rightsquigarrow}{D}(0) \leq \overset{\rightsquigarrow}{D}(x^{\sim})$ *and* $\vec{D}(0) \leq \vec{D}(x^{-})$;

(5) $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) \vee \overset{\rightsquigarrow}{D}(y \rightsquigarrow x) = 1$ *and* $\vec{D}(x \rightarrow y) \vee \vec{D}(y \rightarrow x) = 1$.

Proof. (1) By Theorem 4.20 $\varphi(x) \leq \overset{\rightsquigarrow}{D}(x)$, $\varphi(y) \leq \overset{\rightsquigarrow}{D}(y)$ then $\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(y) \leq \varphi(x) \rightsquigarrow \varphi(y)$ and $\varphi(x) \rightsquigarrow \varphi(y) \leq \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y)$. Then $\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(y) \leq \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y)$.

(2) $\overset{\rightsquigarrow}{D}(x \rightsquigarrow y) = (\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(y)) \vee (\varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(y))$. By (1), the assertion follows.

(3) It results directly from (2).

(4) $\overset{\rightsquigarrow}{D}(0) \leq \varphi(x) \rightsquigarrow \overset{\rightsquigarrow}{D}(0) = \overset{\rightsquigarrow}{D}(x^{\sim})$.

(5) By (1) and (PBL-5), $(\overset{\rightsquigarrow}{D}(x) \rightsquigarrow \varphi(y)) \vee (\varphi(y) \rightsquigarrow \overset{\rightsquigarrow}{D}(x)) = 1$ which is the desired conclusion.

□

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Journal of Algebraic Systems

ON DERIVATIONS OF PSEUDO-BL ALGEBRA

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مشتق‌های شبه -BLجبرها

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شبه BL–جبرها، تعمیمی طبیعی از BL–جبرها و MV–جبرها هستند. در این مقاله پنج نوع مشتق مختلف روی شبه BL–جبرها معرفی شده است که به عنوان تعمیم مشتق‌های یک BL–جبر هستند. علاوه بر این، مفهوم (φ, ψ) –مشتق، به عنوان گسترش این پنج نوع مشتق روی شبه BL–جبر، تعریف می‌شود. در پایان چندین ویژگی مرتبط نیز مورد بحث قرار گرفته است.

کلمات کلیدی: BL–جبر، شبه BL–جبر، مشتق.