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ON THE m_c-TOPOLOGY ON THE FUNCTIONALLY COUNTABLE SUBALGEBRA OF $C(X)$

A. VEISI

Abstract. In this paper, we consider the m_c-topology on $C_c(X)$, the functionally countable subalgebra of $C(X)$. We show that a Tychonoff space X is countably pseudocompact if and only if the m_c-topology and the u_c-topology on $C_c(X)$ coincide. It is shown that whenever X is a zero-dimensional space, then $C_c(X)$ is first countable if and only if $C(X)$ with the m-topology is first countable. Also, the set of all zero-divisors of $C_c(X)$ is closed if and only if X is an almost P-space. We show that if X is a strongly zero-dimensional space and U is the set of all units of $C_c(X)$, then the maximal ring of quotients of $C_c(U)$ and $C_c(C_c(X))$ are isomorphic.

1. Introduction

As usual, all topological spaces in this article are infinite Hausdorff completely regular (i.e., infinite Tychonoff) spaces. We denote by $C(X)$ ($C^*(X)$) the ring of all real-valued, continuous (bounded) functions on a space X. The subring of $C(X)$ consisting of those functions with countable (resp. finite) image, denoted by $C_c(X)$ (resp. $C^F(X)$), is called the functionally countable subalgebra of $C(X)$. The subring $C_c^*(X)$ of $C_c(X)$ consists of bounded elements of $C_c(X)$. In fact, we have $C_c^*(X) = C_c(X) \cap C^*(X)$. The rings $C_c(X)$ and $C^F(X)$ are introduced and fully investigated in [6], [7] and [10]. For each $f \in C(X)$, the *zero-set* of f, denoted by $Z(f)$, is the set of zeros of

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f and $\operatorname{coz}(f) = X \setminus Z(f)$ is the *cozero-set* of f and the set of all zero sets in X is denoted by $Z(X)$. An ideal I in $C(X)$ is called a *z-ideal* if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$.

We recall that a *zero-dimensional space* is a Hausdorff space with a base consisting of clopen (closed-open) sets. In [6], a Hausdorff space X is called *countably completely regular* (briefly, *c-completely regular*) if whenever $F \subseteq X$ is a closed set and $x \notin F$, then there exists $f \in C_c(X)$ such that $f(F) = \{0\}$ and $f(x) = 1$. In [6, Proposition 4.4], it is shown that X is zero-dimensional if and only if for each closed set F and $x \notin F$ there is some $f \in C_c(X)$ (or equivalently $f \in C^F(X)$) such that $f(x) = 1$ and $f(F) = \{0\}$. Therefore a completely regular space is zero-dimensional if and only if it is c-completely regular. It is shown in [6, Theorem 4.6] that for any topological space X (not necessarily completely regular), there exists a zero-dimensional space Y which is a continuous image of X with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$. Also, in [6, Remark 7.5], it is shown that there is a topological space X such that there is no space Y with $C_c(X) \cong C(Y)$.

Let us put $Z_c(X) = \{Z(f) : f \in C_c(X)\}$ and

$$Z_c^*(X) = \{Z(g) : g \in C_c^*(X)\}.$$

These two latter sets are in fact equal since $Z(f) = Z(\frac{f}{1+|f|})$, where $f \in C_c(X)$. A Tychonoff space X is called *strongly zero-dimensional* if for every finite cover $\{U_i\}_{i=1}^k$ of X by cozero-sets there exists a finite refinement $\{V_i\}_{i=1}^m$ of mutually disjoint open sets, see [3]. Two subsets A and B of a space X are said to be *completely separated* (from one another) in X if there exists a function $f \in C^*(X)$ such that $0 \leq f \leq 1$, $f(A) = \{0\}$ and $f(B) = \{1\}$, see [8, 1.15].

We remind that X is *pseudocompact* if each element of $C(X)$ is bounded, i.e., $C(X) = C^*(X)$. A topological space X is called *countably pseudocompact* (in brief, *c-pseudocompact*) whenever $C_c(X) = C_c^*(X)$. For instance, all spaces in Example 2.5 are c-pseudocompact. Clearly, if X is pseudocompact then it is c-pseudocompact. But the converse may be false. To see this, take $X = (0, 1)$ and notice that $C_c(X) = C_c^*(X) = \mathbb{R}$, while $f(x) = \frac{1}{x} \in C(X) \setminus C^*(X)$.

In Section 2, unless otherwise mentioned, each Tychonoff space X is assumed to be zero-dimensional and we concentrate on the m_c-topology on $C_c(X)$, the functionally countable subalgebra of $C(X)$, and record the counterparts of some of the important results in the context of $C(X)$ in [8, 2N]. We show that X is countably pseudocompact if and only if the m_c-topology and the u_c-topology on $C_c(X)$ coincide. It is also shown that when X is a zero-dimensional space, then the set of all zero-divisors of $C_c(X)$ is closed if and only if X is an almost

P-space. We show that if X is strongly zero-dimensional (so it is zero-dimensional) and U is the set of all units of $C_c(X)$, then the maximal ring of quotients of $C_c(U)$ and $C_c(C_c(X))$ are isomorphic.

2. Main results

The m-topology on $C(X)$ as a generalizing of a work of E.H. Moore was first introduced [9] by Hewitt. In his article, he demonstrated that certain classes of topological spaces X can be characterized by topological properties of $C(X)$ with the m-topology. For example, he showed that X is pseudocompact if and only if $C(X)$ with the m-topology is first countable. Several authors have investigated the topological properties of X via properties of $C(X)$. For more information, see [12].
In what follows, we establish the countable analogue of some of the important and well-known facts in the context of $C(X)$. First, we introduce the m_c-topology and the u_c-topology on $C_c(X)$. Before it, it is recalled that $u \in C_c(X)$ is a unit if and only if $Z(u) = \varnothing$ and $u \in C_c^*(X)$ is a unit if and only if u is bounded away from zero, i.e., $|u| \geq r$, for some $r > 0$. The *m_c-topology* (in brief, m_c) on $C_c(X)$ is determined by considering the sets of the form

$$B(f,u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\},$$

as a base for the neighborhood system at f, for each $f \in C_c(X)$ and each positive unit u of $C_c(X)$. The uniform topology on $C_c(X)$ which is called the *u_c-topology* (in brief, u_c), is defined by taking the sets of the form

$$B(f,\varepsilon) = \{g \in C_c(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in X\},$$

as a base for the neighborhood system at f, for each $f \in C_c(X)$ and each $\varepsilon > 0$. Equivalently, a base at f is given by all sets

$$B(f,u) = \{g \in C_c(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\},$$

where u is a positive unit of $C_c^*(X)$.
Now since every positive unit of $C_c^*(X)$ is a unit of $C_c(X)$, we infer that the m_c-topology is finer than the u_c-topology, i.e., $u_c \subseteq m_c$.

Remark 2.1. In [10, Theorem 6.3], it is shown that if X is a zero-dimensional space, then X is pseudocompact if and only if $C_c(X) = C_c^*(X)$, i.e., a zero-dimensional space is pseudocompact if and only if it is c-pseudocompact.

Proposition 2.2. *A Tychonoff space X is c-pseudocompact if and only if the m_c-topology and the u_c-topology on $C_c(X)$ coincide.*

Proof. ($\Rightarrow$). It is evident.
($\Leftarrow$). Suppose that $f \in C_c(X)$. So $\frac{1}{f^2+1}$ is a unit of $C_c(X)$. Since

$u_c = m_c$, it follows that there is some $\varepsilon > 0$ such that $B(0,\varepsilon) \subseteq B(0, \frac{1}{f^2+1})$. Set $\varepsilon_1 = \frac{\varepsilon}{2}$, then $\varepsilon_1 \in B(0,\varepsilon)$ and thus $\varepsilon_1 \in B(0, \frac{1}{f^2+1})$. Hence $\varepsilon_1 < \frac{1}{f^2+1}$, so $f^2 + 1 < \frac{1}{\varepsilon_1}$, and, therefore $f^2 + 1$ is bounded. It shows that f is bounded. Consequently, $f \in C^*_c(X)$ and thus X is c-pseudocompact. □

Proposition 2.3. *Let X be a Tychonoff space. Then $C^*_c(X)$ is a clopen subset of $C_c(X)$ with u_c and m_c-topologies.*

Proof. Since we always have $u_c \subseteq m_c$, we provide the proof for the case in which the topology on $C_c(X)$ is u_c. If $C_c(X) = C^*_c(X)$, the assertion is evident. Suppose that $f \notin C^*_c(X)$. For each $g \in B(f,1)$, we have $|g| \geq |f| - |f - g| > |f| - 1$ and thus $g \notin C^*_c(X)$. Hence $B(f,1) \cap C^*_c(X) = \varnothing$. It gives $B(f,1) \subseteq C_c(X) \setminus C^*_c(X)$ and thus $C_c(X) \setminus C^*_c(X)$ is open, i.e., $C^*_c(X)$ is closed. Now suppose that $f \in C^*_c(X)$. For each $g \in B(f,1)$, we have $|g| \leq |f| + |f - g| < |f| + 1$ and thus $g \in C^*_c(X)$. Hence $B(f,1) \subseteq C^*_c(X)$. This shows that $C^*_c(X)$ is open. □

Corollary 2.4. *If X is not c-pseudocompact, then $C_c(X)$ is disconnected with u_c and m_c-topologies.*

In the next example, we give examples of topological spaces X, where one is zero-dimensional, another is connected, and the third is neither zero-dimensional nor connected, but $C_c(X)$ is connected.

Example 2.5.
- (i) Let X be a finite discrete space with n elements. Then $C(X) = C_c(X) = \mathbb{R}^n$.
- (ii) Let X be a connected space. Then $C_c(X) = \mathbb{R}$. To see this, suppose that $f \in C_c(X)$ and $a, b \in f(X)$ such that $a \neq b$. We claim that $[a,b] \subseteq f(X)$. Otherwise, there is $c \in (a,b)$ such that $c \notin f(X)$. So $X = f^{-1}(-\infty, c) \cup f^{-1}(c, +\infty)$, i.e., X is disconnected which is a contradiction. Hence $[a,b] \subseteq f(X)$, and, therefore $f(X)$ is uncountable which is impossible, since $f \in C_c(X)$. So $a = b$ and hence f is constant.
- (iii) Let $X = \mathbb{R} \setminus \{0\}$ with the usual topology. Then X is neither zero-dimensional nor connected, but $C_c(X) = \mathbb{R}^2$ is connected.

Theorem 2.6. *Let X be zero-dimensional and the topology on $C_c(X)$ be the m_c-topology. Then $C_c(X)$ is first countable if and only if X is pseudocompact.*

Proof. ($\Rightarrow$). Suppose that X is not pseudocompact, then $C^*(X) \subsetneqq C(X)$ and thus $C^*_c(X) \subsetneqq C_c(X)$(see Remark 2.1). Let $f \in C_c(X) \setminus C^*_c(X)$. Since $f^2 + 1$ is unbounded, we can assume at first that $f \geq 1$. We consider the constant function 0 and show that for any countable family $\mathcal{F}$ consisting of open neighborhoods of 0, there exists an open neighborhood of 0 which does not contain any element of $\mathcal{F}$. Take a sequence $\{x_n : n \in \mathbb{N}\} \subseteq X$ and a strictly increasing sequence $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ such that $f(x_n) = a_n$ (note, the sequence a_n is divergent). Each a_n can be inserted in a bounded open interval, V_n say, such that $V_n \cap V_m = \varnothing$, where $n \neq m$. Furthermore, since $f(X)$ is countable, each element of $f(X) \setminus \bigcup_{n=1}^{\infty} V_n$ is contained in a bounded open interval, W_n say, such that $W_n \cap V_m = \varnothing$ for each m. Set

$$T = \big(\bigcup_{n=1}^{\infty} V_n\big) \cup \big(\bigcup_{n=1}^{\infty} W_n\big).$$

So $f(X) \subseteq T$. Now let $\mathcal{F} = \{B(0, u_n) : n \in \mathbb{N}\}$ be a countable family of open neighborhoods of 0, where u_n is a positive unit of $C_c(X)$ and let $b_n = \frac{1}{3}u_n(x_n)$. Define $g : T \to \mathbb{R}$ by

$$g(t) = \begin{cases} \frac{1}{b_n} & t \in \bigcup_{n=1}^{\infty} V_n, \\ 1 & \text{otherwise.} \end{cases}$$

By [8, 1A(2)], g is continuous and it is obvious that the range of g is countable, so $g \in C_c(T)$. Letting $\alpha(x) = \frac{1}{g(f(x))}$ yields α is a positive unit of $C_c(X)$. We now claim that $B(0, \alpha)$ does not contain any element of $\mathcal{F}$. Otherwise, for some n, we have $B(0, u_n) \subseteq B(0, \alpha)$. Let us put $h(x) = \frac{1}{2}u_n(x)$. Then obviously $h \in B(0, u_n)$, while $h \notin B(0, \alpha)$ which is a contradiction (note, $h(x_n) = \frac{1}{2}u_n(x_n) \not< b_n = \alpha(x_n)$). This is the desired conclusion.

($\Leftarrow$). Assume that X is pseudocompact, so $C_c(X) = C_c^*(X)$. By Proposition 2.2, we have $m_c = u_c$. Since d, the function below

$$d(f, g) = \|f - g\| = \sup\{|f(x) - g(x)| : x \in X\},$$

turns $C_c(X)$ into a metric space, the proof is now complete by this fact, the fact that every metric space is first countable. □

Corollary 2.7. *Let X be zero-dimensional. Then $C_c(X)$ with the m_c-topology is first countable if and only if $C(X)$ with the m-topology is first countable.*

Proof. It follows from Theorem 2.6 and [9, Theorem 3]. □

Proposition 2.8. *Let X be a zero-dimensional space and $C_c(X)$ is equipped with the m_c-topology. Then the following statements hold.*

(a) *Let $x \in X$ be fixed and $e_x : C_c(X) \to \mathbb{R}$ be a map defined by $e_x(f) = f(x)$. Then e_x is continuous, i.e, $e_x \in C(C_c(X))$.*

(b) *Suppose that the topology on $C(C_c(X))$ is the m-topology and let $e : X \to C(C_c(X))$ defined by $e(x) = e_x$. Then e is continuous if and only if X is discrete.*

Proof. (a). Let $a, b \in \mathbb{R}$. We claim that

$$e_x^{-1}((a,b)) = \{f \in C_c(X) : a < f(x) < b\}$$

is an open set in $C_c(X)$. To see this, let $f_0 \in e_x^{-1}((a,b))$. Then $a < e_x(f_0) = f_0(x) < b$. A real number $r > 0$ can be taken such that $a < f_0(x) - r < f_0(x) < f_0(x) + r < b$. Now for each $g \in B(f_0, r)$, we have $|g - f_0| < r$, hence $a < f_0(x) - r < g(x) < f_0(x) + r < b$. It implies that $B(f_0, r) \subseteq e_x^{-1}((a,b))$. This gives $e_x^{-1}((a,b))$ is an open set in $C_c(X)$. So e_x is continuous.
$(b)(\Rightarrow)$. Let $x \in X$. By our assumption, $e^{-1}(B(e_x, \frac{1}{2}))$ is an open set containing x. Now we claim that $e^{-1}(B(e_x, \frac{1}{2})) = \{x\}$. On the contrary, suppose that $e^{-1}(B(e_x, \frac{1}{2}))$ contains an element y distinct from x. By [6, Proposition 4.4], there is some $f \in C_c(X)$ such that $f(x) = 0$ and $f(y) = 1$, since $y \in e^{-1}(B(e_x, \frac{1}{2}))$, it follows that

$$1 = |f(x) - f(y)| = |e_x(f) - e_y(f)| = |e(x) - e(y)| < \tfrac{1}{2}$$

which is a contradiction.
$(\Leftarrow)$. It is evident. □

In sequel, whenever $C_c(X)$ (resp. $C_c^*(X)$) is referred to as a topological space, its topology is the m_c-topology (resp. the relative m_c-topology).

Proposition 2.9. *Let X be zero-dimensional and U (resp. U^*) be the set of all units of $C_c(X)$ (resp. $C_c^*(X)$). Then*

(a) *U (resp. U^*) is an open set in $C_c(X)$ (resp. $C_c^*(X)$).*
(b) *The mapping $\psi : f \mapsto f^{-1}$ is a homeomorphism from U (resp. U^*) onto itself.*

Proof. (a). Suppose that $f \in U$ (resp. $f \in U^*$). Then $\varnothing = Z(f) = Z(\frac{|f|}{2})$ (resp. $\frac{|f|}{2} \geq r$, for some $r > 0$). Hence $\frac{|f|}{2} \in U$ (resp. $\frac{|f|}{2} \in U^*$). Now for any $g \in B(f, \frac{|f|}{2})$ we have

$$|g| = |f - (f - g)| \geq \big||f| - |f - g|\big| \geq |f| - |f - g| > |f| - \tfrac{|f|}{2} = \tfrac{|f|}{2}.$$

This yields $Z(g) \subseteq Z(\frac{|f|}{2}) = \varnothing$ (resp. $|g| > r$), and, therefore g is a unit, i.e., $g \in U$ (resp. $g \in U^*$). So $B(f, \frac{|f|}{2}) \subseteq U$ (resp. $B(f, \frac{|f|}{2}) \subseteq U^*$), and we are done.
(b). Clearly, ψ is a bijection. Let $0 < v \in U$, then $u = \frac{vf^2}{2} \wedge \frac{|f|}{2} > 0$.

For each $g \in U$, $|f-g| < u$ implies that $|f-g| < \frac{|f|}{2}$, and $|f-g| < \frac{vf^2}{2}$. Thus
$$|g| \geq |f| - |f-g| > |f| - \frac{|f|}{2} = \frac{|f|}{2}.$$
Hence
$$\Big|\frac{1}{f} - \frac{1}{g}\Big| = \frac{\big|f-g\big|}{\big|fg\big|} < \frac{vf^2}{2} \times \frac{1}{|f|} \times \frac{2}{|f|} = v.$$
This shows that $\psi(B(f,u)) \subseteq B(f,v)$, and, therefore ψ is continuous. Since $\psi^{-1} = \psi$, it follows that ψ^{-1} is also continuous (note, if we at first put $u = \frac{f^2 v}{2(1+|f|v)}$, then by the chosen u, we can show that $\psi(B(f,u)) \subseteq B(f,v)$). □

We have observed that $U \cap C_c^*(X)$ is an open set in $C_c^*(X)$ but its elements need not be units of $C_c^*(X)$. For example, the inverse of the identity $j = i^{-1}$ is a unit of $C(\mathbb{N}) = C_c(\mathbb{N})$, while it is not a unit of $C^*(\mathbb{N}) = C_c^*(\mathbb{N})$. Hence U^* may be contained in $U \cap C_c^*(X)$ properly.

An element a of a commutative ring R is called a zero-divisor, if for some $0 \neq b \in R$, we have $ab = 0$. In a commutative ring with identity R, the set of zero-divisors and units are disjoint. This is also easily seen in the context of $C(X)$ by these facts: $f \in C(X)$ is a zero-divisor if and only if $\text{int}_X Z(f) \neq \varnothing$. Also, $f \in C(X)$ is a unit if and only if $Z(f) = \varnothing$.

Proposition 2.10. *Let X be a zero-dimensional space containing more than one point and S be a subring of $C(X)$ containing $C^F(X)$ (the subring of $C(X)$ consisting of elements with finite image). Then S is not an integral domain.*

Proof. It is enough to show that $C^F(X)$ is not an integral domain, in other words, it has a zero-divisor element. Let $x_1, y_1 \in X$ and $x_1 \neq y_1$. According to [6, Proposition 4.4] there is some $f \in C^F(X)$ such that $x_1 \in \text{int}_X Z(f)$ and $f(y_1) = 1$. So x_1 does not belong to the closed set $F := X \setminus \text{int}_X Z(f)$. Therefore for some $g \in C^F(X)$ we have $g(x_1) = 1$ and $g(F) = \{0\}$. So $g \neq 0$. Moreover, $fg = 0$, since $f(x) = 0$ if $x \in Z(f)$ and $g(x) = 0$ if $x \notin Z(f)$. Hence f is a zero-divisor, or equivalently, $C^F(X)$ is not an integral domain. □

In what follows, cl$D$ means that $\text{cl}_{C_c(X)} D$.

Lemma 2.11. *Let X be zero-dimensional and D (resp. U) be the set of all zero-divisors (resp. units) of $C_c(X)$. Then* cl$D = C_c(X) \setminus U$.

Proof. Since $D \subseteq C_c(X) \setminus U$, by Proposition 2.9($a$), it follows that cl$D \subseteq C_c(X) \setminus U$. For the reverse inclusion, let $g \in C_c(X) \setminus U$ and consider an open neighborhood $B(g,u)$ of g, where u is a positive unit of $C_c(X)$. Now we define a function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x) - \frac{u(x)}{2} & \text{where } g(x) \geq \frac{u(x)}{2}, \\ 0 & \text{where } |g(x)| \leq \frac{u(x)}{2}, \\ g(x) + \frac{u(x)}{2} & \text{where } g(x) \leq -\frac{u(x)}{2}. \end{cases}$$

From the continuity of f on the three closed sets $(g - \frac{u}{2})^{-1}([0,\infty))$, $(g + \frac{u}{2})^{-1}([0,\infty)) \cap (g - \frac{u}{2})^{-1}((-\infty, 0])$, and $(g + \frac{u}{2})^{-1}((-\infty, 0])$, which whose union is X, we infer that $f \in C(X)$. Moreover, since the ranges of g and u are countable, the range of f is also countable, i.e., $f \in C_c(X)$. Also, it is easy to see that $f \in B(g,u)$. Since g is not a unit of $C_c(X)$, it follows that $Z(g) \neq \varnothing$. Let $G = \{x \in X : |g(x)| < \frac{u(x)}{2}\}$. Since $\varnothing \neq Z(g) \subseteq G \subseteq Z(f)$ and G is open, it follows that $\text{int}_X Z(f) \neq \varnothing$, in other words, f is a zero-divisor. Therefore $f \in B(g,u) \cap D$. So $g \in \text{cl}D$, and the proof is complete. □

In [6, Definition 5.1], a topological space X (not necessarily zero-dimensional) is called a CP-space if $C_c(X)$ is a regular ring (Von-Neumann). Also, it is shown in [6, Theorem 5.5] that X is a CP-space if and only if $Z(f)$ is open for each $f \in C_c(X)$. Evidently, every P-space (see [8, 4J]) is a CP-space. The converse may be false (see [6, Example 5.2]). It is shown in [6, Corollary 5.7] that every zero-dimensional CP-space is a P-space.

Proposition 2.12. *Every P-space is strongly zero-dimensional.*

Proof. Let X be a P-space and $A, B \subseteq X$ be completely separated. Then for two disjoint zero-sets Z_1, Z_2 of X, we have $A \subseteq Z_1$ and $B \subseteq Z_2$, see [8, Theorem 1.15]. By the assumption, Z_1 is a clopen set and $A \subseteq Z_1 \subseteq X \setminus B$. Now it follows from [3, Theorem 6.2.4]. □

Proposition 2.13. *Let X be strongly zero-dimensional and U be the set of all units of $C_c(X)$. Then U is a dense subset of $C_c(X)$.*

Proof. Let $g \in C_c(X)$ and u be a positive unit of $C_c(X)$. We claim that $B(g,u) \cap U \neq \varnothing$. Suppose that

$$A = \{x \in X : g(x) \geq \tfrac{1}{3}u(x)\}, \text{ and } B = \{x \in X : g(x) \leq -\tfrac{1}{3}u(x)\}.$$

Since A and B are completely separated zero-sets and X is strongly zero-dimensional, there exists a clopen set C in X such that $A \subseteq C \subseteq X \setminus B$, see [3, Theorem 6.2.4]. Define a function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x) + \frac{1}{3}u(x) & x \in C, \\ g(x) - \frac{1}{3}u(x) & x \notin C. \end{cases}$$

It is easy to show that $f \in C_c(X)$ and $|f - g| < u$, i.e., $f \in B(g, u)$. Clearly, $Z(f) = \varnothing$ and it yields that f is a unit. So $f \in B(g, u) \cap U$. This means that U is dense in $C_c(X)$. □

We remind that the above proposition is the counterpart of this fact: If X is strongly zero-dimensional and U is the set of all units of $C(X)$ with the m-topology, then U is dense in $C(X)$.
We need the following definition in which a commutative ring B and A as its subring have the same identity.

Definition 2.14. ([5, Definition 1.4]) $B \supseteq A$ is a ring of quotients of A, provided that for every $b \in B$, the ideal $b^{-1}A := \{a \in A : ab \in A\}$ is dense in B, that is to say, for each $0 \neq b' \in B$ there exists $a \in A$ such that $ba \in A$ and $b'a \neq 0$.

Proposition 2.15. *Suppose that X is strongly zero-dimensional and $C_c(X)$ with the m_c-topology is zero-dimensional. Let U be the set of all units of $C_c(X)$. Then $C_c(U)$ is a ring of quotients of $C_c(C_c(X))$. Moreover, the maximal ring of quotients of $C_c(U)$ and $C_c(C_c(X))$ are isomorphic.*

Proof. By Propositions 2.9(a) and 2.13, U is open and dense in $C_c(X)$. Now use [14, Proposition 2.2(5) and Lemma 2.1] to obtain the result. □

An almost P-space is a Tychonoff space in which every nonempty G_δ-set has a nonempty interior. This is equivalent to say that every nonempty zero-set has a nonempty interior. Any discrete space as well as the one-point compactification $D^* := D \cup \{x\}$ ($x \notin D$) of an uncountable discrete space D is an almost P-space, since any nonempty G_δ-set of D^* contains an isolated point of the space and further $\{x\}$ is not a G_δ-set, otherwise, $\{x\} = \bigcap_{n=1}^{\infty} V_n$, where each $D^* \setminus V_n$ is finite, so D is countable which is a contradiction. Clearly, every P-space is an almost P-space.

Definition 2.16. A Tychonoff space X is called an almost CP-space if $\varnothing \neq Z \in Z_c(X)$, then we have $\mathrm{int}_X Z \neq \varnothing$. Also, a point $p \in X$ is called an almost CP-point if $\mathrm{int}_X Z \neq \varnothing$, for each $Z \in Z_c(X)$ containing p.

It is clear that X is an almost CP-space if and only if every element of X is an almost CP-point. Any almost P-space is an almost CP-space. The converse does not hold in general. For instance, let $X = \mathbb{R}^n$ with the usual topology. Then $C_c(X) = \mathbb{R}$ (see Example 2.5(ii)), so the only nonempty element of $Z_c(X)$ is X while for $f(x) = \|x\| \in C(X)$, where $x = (x_1, x_2, \ldots, x_n)$ and $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, we have $Z(f) = \{0\}$

and hence $\text{int}_X Z(f) = \varnothing$.
The next result states that every zero-dimensional almost CP-space is an almost P-space.

Proposition 2.17. *Let X be a zero-dimensional space. Then the following statements are equivalent.*

(a) X is an almost P-space.
(b) If $\varnothing \neq G$ is a G_δ-set in X, then $\text{int}_X G \neq \varnothing$.
(c) If $\varnothing \neq Z \in Z_c(X)$, then $\text{int}_X Z \neq \varnothing$.
(d) X is an almost CP-space.

Proof. The equivalences $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ follow from definitions. It remains to show that $(b) \Leftrightarrow (c)$.
$(b) \Rightarrow (c)$. Let $\varnothing \neq Z = Z(f)$ for some $f \in C_c(X)$. Then we have $Z = \bigcap_{n=1}^{\infty} f^{-1}((\frac{-1}{n}, \frac{1}{n}))$, i.e., it is a G_δ-set, so $\text{int}_X Z \neq \varnothing$, and we are done.
$(c) \Rightarrow (b)$. Let $p \in G$. Since $\{p\}$ is compact and G is a G_δ-set, there exists some $Z \in Z_c(X)$ such that $p \in Z \subseteq G$, see [6, Corollary 5.7]. The result now holds by the assumption. $\square$

Corollary 2.18. *Let X be zero-dimensional. Then X is an almost P-space if and only if $\text{int}_X Z \neq \varnothing$ for each $\varnothing \neq Z \in Z_c(X)$.*

Theorem 2.19. *Let X be zero-dimensional and D be the set of all zero-divisors of $C_c(X)$. Then D is closed if and only if X is an almost P-space.*

Proof. $(\Rightarrow)$. By the assumption and Lemma 2.11, we have $D = \text{cl}D = C_c(X) \setminus U$. Let $Z(f)$ be a nonempty zero-set of X, for some $f \in C_c(X)$. Then f is not unit, so $f \in C_c(X) \setminus U = D$ and thus $\text{int}_X Z(f) \neq \varnothing$. This implies that X is an almost P-space, by Corollary 2.18.
$(\Leftarrow)$. Let $f \in \text{cl}D = C_c(X) \setminus U$. Since $f \notin U$, we have that $Z(f) \neq \varnothing$, and, therefore by the hypothesis, $\text{int}_X Z(f) \neq \varnothing$. This gives f is a zero-divisor, i.e., $f \in D$. So $\text{cl}D \subseteq D$ and hence $\text{cl}D = D$. $\square$

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Amir Veisi

Faculty of Petroleum and Gas, Yasouj University, Gachsaran, Iran.

Email:aveisi@yu.ac.ir

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ON THE m_c-TOPOLOGY ON THE FUNCTIONALLY COUNTABLE SUBALGEBRA OF $C(X)$

A. VEISI

m_c-توپولوژی روی زیرجبر شمارا تابعیِ $C(X)$

امیر ویسی

دانشکده‌ی نفت و گاز، دانشگاه یاسوج، گچساران، ایران

در این مقاله، m_c-توپولوژی را روی $C_c(X)$، زیرجبر شمارا تابعیِ $C(X)$ در نظر می‌گیریم. نشان می‌دهیم هر فضای تیخونوف X شمارا فشرده است اگر و تنها اگر m_c-توپولوژی و u_c-توپولوژی روی $C_c(X)$ معادل باشند. نشان داده می‌شود اگر X یک فضای صفر بعدی باشد، آنگاه $C_c(X)$ با m_c-توپولوژی شمارای نوع اول است اگر و تنها اگر $C(X)$ با m-توپولوژی شمارای نوع اول باشد. هم‌چنین، نشان می‌دهیم مجموعه‌ی تمام عناصر مقسوم علیه صفر در $C_c(X)$ بسته است اگر و تنها اگر X تقریبا P-فضا باشد. ثابت می‌کنیم اگر X یک فضای صفر بعدی قوی و U مجموعه‌ی تمام عناصر یکه در $C_c(X)$ باشد، آنگاه بزرگترین حلقه‌های خارج قسمتی $C_c(U)$ و $C_c(C_c(X))$ یک‌ریخت هستند.

کلمات کلیدی: زیرجبر شمارا تابعی، m_c-توپولوژی، فضای شبه فشرده، فضای صفر بعدی.