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VOLUNTARY GE-FILTERS AND FURTHER RESULTS OF GE-FILTERS IN GE-ALGEBRAS

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ABSTRACT. Further properties on (belligerent) GE-filters are discussed, and the quotient GE-algebra via a GE-filter is established. Conditions for the set $\vec{c}$ to be a belligerent GE-filter are provided. The extension property of belligerent GE-filter is composed. The notions of a balanced element, a balanced GE-filter, an antisymmetric GE-algebra and a voluntary GE-filter are introduced, and its properties are examined. The relationship between a GE-subalgebra and a GE-filter is established. Conditions for every element in a GE-algebra to be a balanced element are provided. The conditions necessary for a GE-filter to be a voluntary GE-filter are considered. The GE-filter generated by a given subset is established, and its shape is identified.

1. Introduction

The notion of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other nonclassical logics. In mathematics, Hilbert algebras occur in the theory of von Neumann algebras in: Commutation theorem and Tomita–Takesaki theory. Hilbert algebras are an important tool for certain investigations in algebraic logic since they

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can be considered as fragments of any propositional logic containing a logical connective implication ($\rightarrow$) and the constant 1 which is considered as the logical value "true". As can be seen in references [3, 4, 5, 6, 7, 8, 10, 11, 12], some researchers have studied various things about Hilbert algebra. The study of generalization on a given algebraic structure is also an important research process in algebra. As a generalization of Hilbert algebras, R.K. Bandaru et al. [2] introduce the notion of GE-algebras. They studied the various properties and filter theory of Hilbert algebras. In [1], R.K. Bandaru et al. introduced the concept of belligerent GE-filter of a GE-algebra and investigate its properties. They studied the relation between GE-filter and belligerent GE-filter of a GE-algebra.

In this paper, we consider further properties on (belligerent) GE-filters. We make the quotient GE-algebra via a GE-filter. We take a special set $\vec{c} := \{x \in X \mid c \leq x\}$ for an element c in a GE-algebra X, and provide conditions for the set $\vec{c}$ to be a belligerent GE-filter of X. We discuss the extension property of belligerent GE-filter. We define a balanced element and show that the set of all balanced elements forms a GE-filter. We introduce the notion of balanced GE-filters and examine its properties. We look into the relationship between a GE-subalgebra and a GE-filter, and explore what conditions are necessary for a GE-subalgebra to be a GE-filter. We introduce the concept of antisymmetric GE-algebras and investigate related properties. We provide conditions for every element in a GE-algebra X to be a balanced element. We construct a subset $F^\circ := \bigcap\limits_{c\in F} c^\circ$ where $c^\circ = \{x \in X \mid x \mathring{+} c = 1\}$ and F is a subset of X, and we look at the several properties of this set. We find the condition under which F° is a GE-filter for any subset F of a GE-algebra. We introduce the voluntary GE-filter and further investigate the related properties. We discuss the conditions necessary for a GE-filter to be a voluntary GE-filter. We construct a GE-filter generated by a given subset, and identify its shape.

2. Preliminaries

Definition 2.1 ([2])**.** A *GE-algebra* is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $u * u = 1$,
(GE2) $1 * u = u$,
(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

In a GE-algebra X, an order relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{2.1}$$

Definition 2.2 ([2, 1])**.** A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.2}$$

- *commutative* if it satisfies:

$$(\forall x, y \in X)\,((x * y) * y = (y * x) * x)\,. \tag{2.3}$$

- *left exchangeable* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))\,. \tag{2.4}$$

- *belligerent* if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * z) = (x * y) * (x * z))\,. \tag{2.5}$$

Note that every self-distributive BE-algebra is a GE-algebra (see [2]), and every left exchangeable GE-algebra is a BE-algebra (see [9]).

Proposition 2.3 ([2])**.** *Every GE-algebra X satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.6}$$

$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.7}$$

$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.8}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{2.9}$$

$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.10}$$

Definition 2.4 ([2])**.** A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.11}$$

$$(\forall x, y \in X)(x * y \in F,\; x \in F \;\Rightarrow\; y \in F). \tag{2.12}$$

Lemma 2.5 ([2])**.** *In a GE-algebra X, every filter F of X satisfies:*

$$(\forall x, y \in X)\,(x \leq y,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{2.13}$$

Definition 2.6 ([1])**.** A subset F of a GE-algebra X is called a *belligerent GE-filter* of X if it satisfies (2.11) and

$$(\forall x, y, z \in X)(x * (y * z) \in F,\; x * y \in F \;\Rightarrow\; x * z \in F). \tag{2.14}$$

Lemma 2.7 ([1])**.** *If a GE-filter F of a GE-algebra X satisfies:*

$$(\forall x, y, z \in X)\,(x * (y * z) \in F \;\Rightarrow\; (x * y) * (x * z) \in F)\,, \tag{2.15}$$

then F is a belligerent GE-filter of X.

3. Further properties on GE-filters

In this section, we first consider the quotient GE-algebra X/F of a GE-algebra X by a GE-filter F.

For a given GE-filter F of a transitive GE-algebra X, let $\sim_F$ be a binary relation on X defined by

$$(\forall x, y \in X)\,(x \sim_F y \;\Leftrightarrow\; x * y \in F,\; y * x \in F)\,. \tag{3.1}$$

Then $\sim_F$ is an equivalent relation on X. In fact, it is clear that $\sim_F$ is reflexive and symmetry. Let $x, y, z \in X$ be such that $x \sim_F y$ and $y \sim_F z$. Then $x * y \in F$, $y * x \in F$, $y * z \in F$ and $z * y \in F$. Since $z * y \leq (y * x) * (z * x)$ and $x * y \leq (y * z) * (x * z)$ by (2.10), it follows from Lemma 2.5 that $(y * x) * (z * x) \in F$ and $(y * z) * (x * z) \in F$. Since F is a GE-filter of X, we have $x * z \in F$ and $z * x \in F$, and so $x \sim_F z$. Therefore, X can be decomposed by the equivalence relation $\sim_F$. The equivalence class of a in X under $\sim_F$ is denoted by F_a, that is,

$$F_a = \{x \in X \mid a \sim_F x\}.$$

The collection of all such equivalence classes is denoted by X/F, that is,

$$X/\!\sim_F = \{F_a \mid a \in X\},$$

which is called the *quotient set* of X by $\sim_F$.

Theorem 3.1. *Let F be a GE-filter of a transitive GE-algebra X. Given an element $a \in X$, let F_a be the equivalence relation on X which is defined by (3.1). Then $F = F_1$ and $(X/\!\sim_F, *_F, F_1)$ is a GE-algebra where $*_F$ is defined by $F_x *_F F_y = F_{x*y}$ for every $F_x, F_y \in X/\!\sim_F$. Moreover it is transitive.*

We say that $(X/\!\sim_F, *_F, F_1)$ is the *quotient GE-algebra* via F, and it is also denoted by $(X/F, *_F, F_1)$ because the relation $\sim_F$ was defined using a given GE-filter F.

Proof. If $x \in F$, then $1 * x = x \in F$ and $x * 1 = 1 \in F$. Thus $1 \sim_F x$, i.e., $x \in F_1$. If $x \in F_1$, then $1 * x \in F$ and so $x \in F$. Hence $F = F_1$. Let

$F_x, F_y, F_z \in X/{\sim_F}$ Then $F_x *_F F_x = F_{x*x} = F_1$, $F_1 *_F F_x = F_{1*x} = F_x$ and

$$\begin{aligned} F_x *_F (F_y *_F F_z) &= F_x *_F F_{y*z} = F_{x*(y*z)} \\ &= F_{x*(y*(x*z))} = F_x *_F F_{y*(x*z)} \\ &= F_x *_F (F_y *_F F_{x*z}) \\ &= F_x *_F (F_y *_F (F_x *_F F_z)). \end{aligned}$$

Hene $(X/{\sim_F}, *_F, F_1)$ is a GE-algebra. Now we will prove:

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; F_x \leq_F F_y) \tag{3.2}$$

where $F_x \leq_F F_y$ means $F_x *_F F_y = F_1$. If $x \leq y$ for all $x, y \in X$, then $x * y = 1$ and hence $F_x *_F F_y = F_{x*y} = F_1$, that is, $F_x \leq_F F_y$. If X is transitive, then $x * y \leq (z * x) * (z * y)$ for all $x, y, z \in X$. It follows from (3.2) that

$$F_x *_F F_y = F_{x*y} \leq_F F_{(z*x)*(z*y)} = F_{z*x} *_F F_{z*y} = (F_z *_F F_x) *_F (F_z *_F F_y).$$

Therefore $(X/{\sim_F}, *_F, F_1)$ is a transitive GE-algebra. □

The following example illustrates Theorem 3.1.

Example 3.2. Let $X = \{1, a, b, c, d\}$ be a set with a binary operation $*$ in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	d	1	1	d
c	1	a	1	1	a
d	1	1	b	b	1

Then $(X, *, 1)$ is a transitive GE-algebra and $F = \{1, a, d\}$ is a GE-filter of X. Let

$$\begin{aligned} \sim_F\ = \{&(1,1), (a,a), (b,b), (c,c), (d,d), (1,a), (1,d), (a,1), (a,d), \\ &(b,c), (c,b), (d,1), (d,a)\}. \end{aligned}$$

Then $\sim_F$ is an equivalence relation on X. Also,

$$F_1 = \{1, a, d\} = F_a = F_d$$

and $F_b = \{b, c\} = F_c$. Hence $F_1 = F$ and $X/{\sim_F} = \{F_1, F_b\}$. Then $(X/{\sim_F}, *_F, F_1)$ is a transitive GE-algebra where $*_F$ is given in the following table:

$*_F$	F_1	F_b
F_1	F_1	F_b
F_b	F_1	F_1

Theorem 3.3. *Let F be a GE-filter of a transitive GE-algebra X. If X is commutative, then so is the quotient GE-algebra.*

Proof. Straightforward. □

Given an element c of X, consider the set $\vec{c} := \{x \in X \mid c \leq x\}$. In general, the set $\vec{c}$ is not a GE-filter of X (see [1]), and so it is not a belligerent GE-filter of X.

In the following theorem, we provide conditions for the set $\vec{c}$ to be a belligerent GE-filter of X.

Theorem 3.4. *Given an element c in a GE-algebra X, the following are equivalent.*

(i) *The set $\vec{c} := \{x \in X \mid c \leq x\}$ is a belligerent GE-filter of X.*
(ii) *X satisfies:*

$$(\forall x, y, z \in X)\,(c \leq x * (y * z),\ c \leq x * y \ \Rightarrow\ c \leq x * z)\,. \tag{3.3}$$

Proof. Assume that $\vec{c} := \{x \in X \mid c \leq x\}$ is a belligerent GE-filter of X. Let $x, y, z \in X$ be such that $c \leq x * (y * z)$ and $c \leq x * y$. Then $x * (y * z) \in \vec{c}$ and $x * y \in \vec{c}$. Since $\vec{c}$ is a belligerent GE-filter of X, it follows from (2.14) that $x * z \in \vec{c}$. Hence $c \leq x * z$.

Conversely, suppose that X satisfies (3.3). Clearly $1 \in \vec{c}$. Let $x, y, z \in X$ be such that $x*(y*z) \in \vec{c}$ and $x*y \in \vec{c}$. Then $c \leq x*(y*z)$ and $c \leq x * y$, which imply from (3.3) that $c \leq x * z$. Thus $x * z \in \vec{c}$, and therefore $\vec{c}$ is a belligerent GE-filter of X. □

Theorem 3.5. (Extension property for the belligerent GE-filter) *Let F and G be GE-filters of a left exchangeable GE-algebra X such that $F \subseteq G$. If F is a belligerent GE-filter of X, then so is G.*

Proof. Let $x, y, z \in X$ be such that $x * (y * z) \in G$. Using (2.4) and (GE1) induces

$$x * (y * ((x * (y * z)) * z)) = (x * (y * z)) * (x * (y * z)) = 1 \in F.$$

Since F is a belligerent GE-filter of X, it follows from (2.4) and Lemma 2.7 that

$$\begin{aligned}(x * (y * z)) * ((x * y) * (x * z)) &= (x * y) * ((x * (y * z)) * (x * z)) \\ &= (x * y) * (x * ((x * (y * z)) * z)) \in F \subseteq G.\end{aligned}$$

Since G is a GE-filter of X, we have $(x * y) * (x * z) \in G$. Therefore G is a belligerent GE-filter of X by Lemma 2.7. □

Corollary 3.6. *In a left exchangeable GE-algebra, the trivial GE-filter* $\{1\}$ *is a belligerent GE-filter if and only if every GE-filter is a belligerent GE-filter.*

Definition 3.7. An element b of a GE-algebra X is said to be *balanced* if it satisfies:

$$(\forall x \in X)\,(b \neq x \;\Rightarrow\; b * x = x,\; x * b = b)\,. \tag{3.4}$$

Let $\mathbb{B}(X)$ denote the set of all balanced elements of a GE-algebra X, and it is called the *balanced part* of X.

It is obvious that $1 \in \mathbb{B}(X)$ for every GE-algebra X.

Example 3.8. Consider a GE-algebra $(X, *, 1)$, where

$$X = \{1, a, b, c, d, e\}$$

and the binary operation $*$ is given by the following Cayley table.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	c	e
b	1	a	1	d	d	e
c	1	1	b	1	1	e
d	1	1	1	1	1	e
e	1	a	b	c	d	1

It is routine to calculate that 1 and e are balanced elements of X.

Theorem 3.9. *The balanced part of a GE-algebra X is a GE-filter of X.*

Proof. Let $\mathbb{B}(X)$ be the balanced part of a GE-algebra X. Clearly, $1 \in \mathbb{B}(X)$ as mentioned in the above. Let $x, y \in X$ be such that $x * y \in \mathbb{B}(X)$ and $x \in \mathbb{B}(X)$. If $y \neq x$, then $y = x * y$ since $x \in \mathbb{B}(X)$, and so $y \in \mathbb{B}(X)$. If $y = x$, then $y = x \in \mathbb{B}(X)$. Hence $\mathbb{B}(X)$ is a GE-filter of X. □

Definition 3.10. A GE-algebra X is said to be *balanced* if its balanced part is X itself.

Example 3.11. Let $X = \{1, 2, a, b, c\}$ be a set with the following Cayley table:

$*$	1	2	a	b	c
1	1	2	a	b	c
2	1	1	a	b	c
a	1	2	1	b	c
b	1	2	a	1	c
c	1	2	a	b	1

It is routine to verify that X is a balanced GE-algebra.

Definition 3.12. A (belligerent) GE-filter F of a GE-algebra X is said to be *balanced* if F contains the balanced part of X.

Example 3.13. Consider the GE-algebra X in Example 3.8. Then $F := \{1, a, b, e\}$ is a GE-filter of X and $\mathbb{B}(X) = \{1, e\} \subseteq F$. Thus F is a balanced GE-filter of X.

By Definitions 3.10 and 3.12, we know that there does not exist a proper balanced GE-filter in a balanced GE-algebra.

Theorem 3.14. *The intersection of balanced (belligerent) GE-filters of a GE-algebra is also a balanced (belligerent) GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a family of balanced (belligerent) GE-filters of a GE-algebra X where Λ is any index set. Then $\mathbb{B}(X) \subseteq F_i$ for all $i \in \Lambda$, and so $\mathbb{B}(X) \subseteq \bigcap_{i\in\Lambda} F_i$. Obviously $1 \in \bigcap_{i\in\Lambda} F_i$. Let $x, y \in X$ be such that $x * y \in \bigcap_{i\in\Lambda} F_i$ and $x \in \bigcap_{i\in\Lambda} F_i$. Then $x * y \in F_i$ and $x \in F_i$ for all $i \in \Lambda$, which imply from (2.12) that $y \in F_i$ for all $i \in \Lambda$. Hence $y \in \bigcap_{i\in\Lambda} F_i$, and therefore $\bigcap_{i\in\Lambda} F_i$ is a balanced GE-filter of X. Now, let $x, y, z \in X$ be such that $x * (y * z) \in \bigcap_{i\in\Lambda} F_i$ and $x * y \in \bigcap_{i\in\Lambda} F_i$. Then $x * (y * z) \in F_i$ and $x * y \in F_i$ for all $i \in \Lambda$. Since F_i is belligerent, it follows that $x * z \in F_i$ for all $i \in \Lambda$. Thus $x * z \in \bigcap_{i\in\Lambda} F_i$ and hence $\bigcap_{i\in\Lambda} F_i$ is a balanced belligerent GE-filter of X. □

The following example shows that the union of balanced GE-filters of a GE-algebra is not a balanced GE-filter.

Example 3.15. Let $X = \{1, a, b, c, d, e, f, g\}$ and the binary operation $*$ is given by the following Cayley table.

$*$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	1	c	e	e	1	1
b	1	a	1	d	d	d	g	g
c	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	1	1
e	1	1	1	1	1	1	1	1
f	1	a	1	e	e	e	1	1
g	1	a	b	d	d	d	b	1

Then $(X, *, 1)$ is a GE-algebra. Let $F_1 = \{1, b\}$ and $F_2 = \{1, g\}$. Then F_1 and F_2 are GE-filters of X and $B(X) = \{1\}$. Hence F_1 and F_2 are balanced GE-filters of X. But $F_1 \cup F_2 = \{1, b, g\}$ is not a balanced GE-filter of X since $g * f = b \in F_1 \cup F_2$ and $g \in F_1 \cup F_2$ but $f \notin F_1 \cup F_2$.

In the following example, we know that any GE-subalgebra F of a GE-algebra X, that is, $x * y \in F$ for all $x, y \in F$, is not a GE-filter of X.

Example 3.16. Let $X = \{1, a, b, c, d\}$ and the binary operation $*$ is given by the following Cayley table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	1	1	1	1

Then $(X, *, 1)$ is a GE-algebra and $F = \{1, b, d\}$ is a GE-subalgebra of X. But F is not a GE-filter of X since $b * c = d \in F$ and $b \in F$ but $c \notin F$.

We provide conditions for a GE-subalgebra to be a GE-filter.

Theorem 3.17. *Every GE-subalgebra of a balanced GE-algebra is a GE-filter.*

Proof. Let F be a GE-subalgebra of a balanced GE-algebra X. Then $1 = x * x \in F$ for all $x \in F$. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Since x is balanced, we have $y = x * y \in F$. Therefore F is a GE-filter of X. □

We define a binary operation "$\mathring{+}$" on a GE-algebra X as follows:

$$\mathring{+} : X \times X \to X,\ (x, y) \mapsto (x * y) * y. \tag{3.5}$$

For every subset F of a GE-algebra X, we define a new subset F° of X as follows:

$$F^\circ := \bigcap_{c \in F} c^\circ \tag{3.6}$$

where $c^\circ := \{x \in X \mid x \mathring{+} c = 1\}$.

It is clear that $1 \in F^\circ$, $1^\circ = X$, $X^\circ = \{1\}$ and $1, x \in x^\circ$ for all $x \in X$.

Definition 3.18. A GE-algebra X is said to be *antisymmetric* if the binary relation "$\leq$" is antisymmetric.

Example 3.19. Let $X = \{1, a, b, c\}$ be a set with binary operation $*$ given in the next Cayley table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	a	1	1

It is routine verify that $(X, *, 1)$ is an antisymmetric GE-algebra.

Proposition 3.20. *Every GE-algebra X satisfies:*

$$(\forall x \in X)\left(1\mathring{+}x = x\mathring{+}1 = 1,\ x\mathring{+}x = x\right). \tag{3.7}$$

$$(\forall x, y \in X)\left(x\mathring{+}(x * y) = 1,\ x\mathring{+}(y * x) = y * x\right). \tag{3.8}$$

$$(\forall x, y, z \in X)(x \le y \ \Rightarrow\ x\mathring{+}z \le y\mathring{+}z). \tag{3.9}$$

If X is antisymmetric, then

$$(\forall a, b \in X)(a \le b \ \Rightarrow\ \vec{b} \subseteq \vec{a}). \tag{3.10}$$

If X is commutative, then

$$(\forall x, y \in X)(x\mathring{+}y = y\mathring{+}x). \tag{3.11}$$

If X is a left exchangeable, then

$$(\forall x, y \in X) \tag{3.12}$$
$$\left(x * (x\mathring{+}y) = x * (y\mathring{+}x) = 1,\ x\mathring{+}(x\mathring{+}y) = x\mathring{+}y,\ x\mathring{+}(y\mathring{+}x) = y\mathring{+}x\right).$$

Proof. Using (GE1), (GE2) and (2.8) induces the results (3.7) and (3.8). Let $x, y, z \in X$ be such that $x \le y$. Then $y * z \le x * z$ by (2.9), and so $x\mathring{+}z = (x * z) * z \le (y * z) * z = y\mathring{+}z$. Thus (3.9) is valid. Assume that X is antisymmetric and $a \le b$ for $a, b \in X$. Let $x \in \vec{b}$. Then $1 = b * x \le a * x$ by (2.9), and thus $a * x = 1$, that is, $a \le x$. Hence $x \in \vec{a}$, and therefore $\vec{b} \subseteq \vec{a}$ which proves (3.10). (3.11) is straightforward. (3.12) can be induced by using (GE1), (2.4) and (2.6). □

Proposition 3.21. *If X is commutative and antisymmetric GE-algebra, then every element $a \in c^\circ$ with $a \ne c$ is a balanced element of X.*

Proof. Let a be an element of c° that is $a \ne c$. Then $1 = a\mathring{+}c = (a*c)*c$, i.e., $a * c \le c$. Since X is antisymmetric, it follows fom (2.8) that $a * c = c$. Since X is commutative, we have $c * a = a$. Hence a is a balanced element of X. □

In a GE-algebra X, the set F° is not a GE-filter of X for any subset F of X as shown in the following example.

Example 3.22. Consider the GE-algebra X in Example 3.16 and let $F = \{1, a, b\}$. Then $\mathring{b} = \{1, a\}$ is a GE-filter of X, but $\mathring{a} = \{1, b, c\}$ is not a GE-filter of X since $c \in \mathring{a}$ and $c * d = 1 \in \mathring{a}$ while $d \notin \mathring{a}$. Also, if we take $F = \{1, a\}$, then $\mathring{F} = \mathring{1} \cap \mathring{a} = X \cap \{1, b, c\} = \{1, b, c\}$, which is not a GE-filter of X.

We find the condition under which F° is a GE-filter for any subset F of a GE-algebra.

Theorem 3.23. *For every subset F of an antisymmetric transitive GE-algebra X, the set F° is a GE-filter of X.*

Proof. Let F be a subset of an antisymmetric transitive GE-algebra X. Obviously, $1 \in F^\circ$. We first show that

$$x \in F^\circ \;\Leftrightarrow\; (\forall c \in F)(x * c = c). \tag{3.13}$$

If $x \in F^\circ$, then $x \in c^\circ$, i.e., $(x * c) * c = x \mathring{+} c = 1$ for all $c \in F$. Hence $x * c \leq c$. Also, $c \leq x * c$ by (2.8). Since X is antisymmetric, it follows that $x * c = c$ for all $c \in F$. Conversely, if $x * c = c$ for all $c \in F$, then $x \mathring{+} c = (x * c) * c = 1$, that is, $x \in c^\circ$ for all $c \in F$. Therefore $x \in \bigcap_{c \in F} c^\circ = F^\circ$. Let $x, y \in X$ be such that $x * y \in F^\circ$ and $x \in F^\circ$. Then $(x * y) * c = c$ and $x * c = c$ for all $c \in F$ by (3.13). Hence $y * c \leq (x * y) * (x * c) = (x * y) * c = c$. It follows from (2.8) and the antisymmetry of X that $y * c = c$ for all $c \in F$. Thus $y \in F^\circ$, and therefore F° is a GE-filter of X. □

The following example illustrates Theorem 3.23.

Example 3.24. Consider the GE-algebra X in Example 3.19. It is routine to verify that X is an antisymmetric transitive GE-algebra. Then $1^\circ = \{1, a, b, c\}$, $a^\circ = \{1, b, c\}$, $b^\circ = \{1\}$ and $c^\circ = \{1, b\}$. It can be easily observed that F° is a GE-filter of X for every subset F of X.

Proposition 3.25. *For two subsets F and G of an antisymmetric transitive GE-algebra X, we have*

(i) *If $F \subseteq G$, then $G^\circ \subseteq F^\circ$.*
(ii) *$F \subseteq (F^\circ)^\circ$.*
(iii) *$F = ((F^\circ)^\circ)^\circ$.*
(iv) *$(F \cup G)^\circ = F^\circ \cap G^\circ$.*
(v) *If F is a GE-filter of X, then $F \cap F^\circ = \{1\}$.*
(vi) *If F and G are GE-filters of X, then $F \cap G = \{1\}$ if and only if $F \subseteq G^\circ$.*

Proof. (i) If $x \in G^\circ$, then $x \in d^\circ$ and so $x \mathring{+} d = 1$ for all $d \in G$. Since $F \subseteq G$, it follows that $x \mathring{+} d = 1$ for all $d \in F$. Hence $x \in d^\circ$ for all $d \in F$. Thus $x \in \bigcap_{d \in F} d^\circ = F^\circ$. Therefore $G^\circ \subseteq F^\circ$.

(ii) First, (3.6) is equivalent to

$$F^\circ = \{x \in X \mid x \mathring{+} c = 1 \text{ for all } c \in F\}.$$

Let $a \in F$. Then $x \mathring{+} a = 1$ for all $x \in F^\circ$, and so $a \in (F^\circ)^\circ$ which shows that (ii) is valid.

(iii) This is straightforward by (i) and (ii).

(iv) Using (i), we have $(F \cup G)^\circ \subseteq F^\circ$ and $(F \cup G)^\circ \subseteq G^\circ$. Hence $(F \cup G)^\circ \subseteq F^\circ \cap G^\circ$. Let $x \in F^\circ \cap G^\circ$. Then $x \in F^\circ$ and $x \in G^\circ$ which imply that $x \mathring{+} c = 1$ and $x \mathring{+} d = 1$ for all $c \in F$ and $d \in G$. It follows that $x \mathring{+} e = 1$ for all $e \in F \cup G$, that is, $x \in (F \cup G)^\circ$. Hence $F^\circ \cap G^\circ \subseteq (F \cup G)^\circ$. This shows that (iv) is valid.

(v) Suppose that F is a GE-filter of X and let $x \in F \cap F^\circ$. Then $x \in F$ and $x \in F^\circ$. Hence $x = 1 * x = (x * x) * x = x \mathring{+} x = 1$, and therefore $F \cap F^\circ = \{1\}$.

(vi) Assume that F and G are GE-filters of X. If $F \cap G = \{1\}$, then $F \subseteq G^\circ$ by the definition of G°. If $F \subseteq G^\circ$, then $F \cap G \subseteq G^\circ \cap G = \{1\}$ and thus $F \cap G = \{1\}$. □

Question 3.26. In an antisymmetric transitive GE-algebra X, if F is a GE-filter of X, is $F = (F^\circ)^\circ$?

The following example shows that the answer to Question 3.26 is negative.

Example 3.27. Consider the GE-algebra X in Example 3.19. It is routine to verify that X is an antisymmetric transitive GE-algebra and $F := \{1, b\}$ is a GE-filter of X. Then $F^\circ = \{1\}$ and $(F^\circ)^\circ = \{1, a, b, c\}$. Hence $F \neq (F^\circ)^\circ$.

Definition 3.28. Let X be an antisymmetric transitive GE-algebra. A GE-filter F of X is said to be *voluntary* if $F = (F^\circ)^\circ$.

Example 3.29. (1) $\{1\}$ and X are voluntary GE-filters in every antisymmetric transitive GE-algebra X.

(2) Let $X = \{1, a, b, c, d\}$ and the binary operation $*$ is given by the following Cayley table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	1	1
c	1	a	b	1	d
d	1	a	b	c	1

It is routine to verify that X is an antisymmetric transitive GE-algebra and $F = \{1, a\}$ is a GE-filter of X. Then $F^\circ = \{1, b, c, d\}$ and $(F^\circ)^\circ = \{1, a\}$. Hence $F = F^{\circ\circ}$, and so F is a proper voluntary GE-filter of X.

In Example 3.27, we examined that any GE-filter may not be a voluntary GE-filter. So we need to find conditions in which a GE-filter can be a voluntary GE-filter.

For every elements x and a in a GE-algebra X, we denote

$$a^n * x := a * (\cdots * (a * (a * x)) \cdots) \tag{3.14}$$

where a occurs n times.

We discuss the conditions necessary for a GE-filter to be a voluntary GE-filter.

Theorem 3.30. *Let X be an antisymmetric commutative and transitive GE-algebra in which if*

$$x \leq a * x \leq a^2 * x \leq \cdots \leq a^{n-a} * x \leq a^n * x \leq \cdots,$$

then there exists a natural number k such that $a^{k-1} * x = a^k * x$. Then every GE-filter is a voluntary GE-filter.

Proof. Let F be a GE-filter of X. Then $F \subseteq (F^\circ)^\circ$ by Proposition 3.25(ii). Let $x \in X$ be such that $x \notin F$. If there exists $a \in F$ such that $a * x \in F$, then $x \in F$ since F is a GE-filter of X. This is a contradiction, and so $a * x \notin F$ for all $a \in F$. If there exists $a \in F$ such that $a^2 * x \in F$, then $x \in F$ since F is a GE-filter of X. Hence $a^2 * x \notin F$ for all $a \in F$. By repeating this process, we obtain a sequence $\{a^n * x\}$ with $a^n * x \notin F$. Then

$$x \leq a * x \leq a^2 * x \leq \cdots \leq a^{n-a} * x \leq a^n * x \leq \cdots,$$

and so $a^{k-1} * x = a^k * x$ for some natural number k. It follows that $a * (a^{k-1} * x) = a^k * x = a^{k-1} * x$ for all $a \in F$. Hence $a^{k-1} * x \in F^\circ$. Since $F^\circ \cap (F^\circ)^\circ = \{1\}$, we know that $a * (a^{k-2} * x) = a^{k-1} * x \notin (F^\circ)^\circ$. Since $a \in F \subseteq (F^\circ)^\circ$ and $(F^\circ)^\circ$ is a GE-filter of X, we get $a^{k-2} * x \notin (F^\circ)^\circ$.

Repeated implementation of this process induces $x \notin (F^{\circ})^{\circ}$. Hence $(F^{\circ})^{\circ} \subseteq F$, and therefore F is a voluntary GE-filter of X. □

Definition 3.31. Let F be a subset of a GE-algebra X. The *GE-filter* of X generated by F is denoted by $\langle F\rangle$ and is defined to be the intersection of all GE-filters of X containing F.

Example 3.32. Let $X = \{1, a, b, c, d, e, f\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	c	e	e	1
b	1	a	1	d	d	d	f
c	1	1	b	1	1	1	1
d	1	a	1	1	1	1	f
e	1	a	b	1	1	1	1
f	1	a	b	e	d	e	1

Then $(X, *, 1)$ is a GE-algebra. If we take a subset $G = \{1, a\}$ of X, then the GE-filter of X generated by G is $\langle G\rangle = \{1, a, b, f\}$.

The following theorem shows how $\langle F\rangle$ is constructed.

Theorem 3.33. *If F is a non-empty subset of an antisymmetric left exchangeable GE-algebra X, then $\langle F\rangle$ consists of x's that satisfy the following condition:*

$$(\exists a_1, a_2, \cdots, a_n \in F)\,(a_n * (\cdots * (a_2 * (a_1 * x)) \cdots) = 1)\,, \tag{3.15}$$

that is,
$\langle F\rangle = \{x \in X \mid a_n*(\cdots*(a_2*(a_1*x))\cdots) = 1 \text{ for some } a_1, a_2, \cdots, a_n \in F\}$.

Proof. Let
$G := \{x \in X \mid a_n*(\cdots*(a_2*(a_1*x))\cdots) = 1 \text{ for some } a_1, a_2, \cdots, a_n \in F\}$.

Obviously, $1 \in G$. Let $x, y \in X$ be such that $x * y \in G$ and $x \in G$. Then there exist $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n \in F$ such that

$$a_m * (\cdots * (a_2 * (a_1 * (x * y))) \cdots) = 1, \tag{3.16}$$

$$b_n * (\cdots * (b_2 * (b_1 * x)) \cdots) = 1. \tag{3.17}$$

If (2.4) is used repeatedly, (3.16) changes as follows.

$$x * (a_m * (\cdots * (a_2 * (a_1 * y)) \cdots)) = 1,$$

that is,

$$x \leq a_m * (\cdots * (a_2 * (a_1 * y)) \cdots). \tag{3.18}$$

If we multiply both sides of (3.18) by b_1 from the left, then

$$b_1 * x \leq b_1 * (a_m * (\cdots * (a_2 * (a_1 * y)) \cdots)). \tag{3.19}$$

If we repeat this process n times, we have

$$\begin{aligned} 1 &= b_n * (\cdots * (b_2 * (b_1 * x)) \cdots) \\ &\leq b_n * (\cdots * (b_1 * (a_m * (\cdots * (a_2 * (a_1 * y)) \cdots))) \cdots). \end{aligned}$$

Since X is antisymmetric and $x \leq 1$ for all $x \in X$, it follows that

$$b_n * (\cdots * (b_1 * (a_m * (\cdots * (a_2 * (a_1 * y)) \cdots))) \cdots) = 1.$$

Hence $y \in G$, and therefore G is a GE-filter of X. It is clear that $F \subseteq G$. Let H be a GE-filter of X containing F. If $x \in G$, then $c_n * (\cdots * (c_2 * (c_1 * x)) \cdots) = 1 \in H$ for some $c_1, c_2, \cdots , c_n \in F \subseteq H$. It follows that $x \in H$. Hence $G \subseteq H$. This shows that $G = \langle F \rangle$. □

For any GE-filters F and G of an antisymmetric transitive GE-algebra X, $F^\circ \cup G^\circ$ may not be a GE-filter of X as shown in the next example.

Example 3.34. Let $X = \{1, a, b, c, d, e\}$ and the binary operation $*$ is given by the following Cayley table.

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	e
b	1	a	1	c	d	d
c	1	1	b	1	1	b
d	1	a	b	c	1	b
e	1	a	1	c	1	1

Then $(X, *, 1)$ is an antisymmetric transitive GE-algebra, and we have $1^\circ = X$, $a^\circ = \{1, b, d, e\}$, $b^\circ = \{1, a, c, d\}$, $c^\circ = \{1, a, b, d, e\}$, $d^\circ = \{1, a, b\}$, $e^\circ = \{1, a\}$. Let $F = \{1, b\}$ and $G = \{1, a, c, d\}$. Then F and G are GE-filters of X. Now $F^\circ = \{1, a, c, d\}$ and $G^\circ = \{1, b\}$. But $F^\circ \cup G^\circ = \{1, a, b, c, d\}$ is not a GE-filter of X since $d * e = b \in F^\circ \cup G^\circ$ and $d \in F^\circ \cup G^\circ$ but $e \notin F^\circ \cup G^\circ$.

Theorem 3.35. *Let X be a left exchangeable GE-algebra which is antisymmetric and transitive in which every GE-filter is voluntary. If F and G are GE-filters of X, then $(F \cap G)^\circ = \langle F^\circ \cup G^\circ \rangle$.*

Proof. Since $F \cap G \subseteq F$ and $F \cap G \subseteq G$, we have $F^\circ \subseteq (F \cap G)^\circ$ and $G^\circ \subseteq (F \cap G)^\circ$ by Proposition 3.25(i). Hence $F^\circ \cup G^\circ \subseteq (F \cap G)^\circ$, and so $\langle F^\circ \cup G^\circ \rangle \subseteq (F \cap G)^\circ$. Note that $F^\circ \subseteq F^\circ \cup G^\circ \subseteq \langle F^\circ \cup G^\circ \rangle$ and $G^\circ \subseteq F^\circ \cup G^\circ \subseteq \langle F^\circ \cup G^\circ \rangle$. It follows from Proposition 3.25(i) and assumption that $\langle F^\circ \cup G^\circ \rangle^\circ \subseteq (F^\circ)^\circ = F$ and $\langle F^\circ \cup G^\circ \rangle^\circ \subseteq (G^\circ)^\circ = G$.

Thus $\langle F^\circ \cup G^\circ\rangle^\circ \subseteq F\cap G$, and so $(F\cap G)^\circ \subseteq (\langle F^\circ \cup G^\circ\rangle^\circ)^\circ = \langle F^\circ \cup G^\circ\rangle$ by Proposition 3.25(i) and assumption. Therefore

$$(F \cap G)^\circ = \langle F^\circ \cup G^\circ\rangle.$$

□

Theorem 3.36. *Let X be a left exchangeable GE-algebra which is antisymmetric and transitive in which every GE-filter is voluntary. Then $\langle F\rangle = (F^\circ)^\circ$ for every subset F of X.*

Proof. Let F be a subset of X. Then $F \subseteq (F^\circ)^\circ$ by Proposition 3.25(ii). Since $(F^\circ)^\circ$ is a GE-filter of X, it follows that $\langle F\rangle \subseteq (F^\circ)^\circ$. Since $F \subseteq \langle F\rangle$, we have $\langle F\rangle^\circ \subseteq F^\circ$ by Proposition 3.25(i). It follows from the hypothesis and Proposition 3.25(i) that $(F^\circ)^\circ \subseteq (\langle F\rangle^\circ)^\circ = \langle F\rangle$. Therefore $\langle F\rangle = (F^\circ)^\circ$. □

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VOLUNTARY GE-FILTERS AND FURTHER RESULTS OF GE-FILTERS IN GE-ALGEBRAS

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GE -فیلترهای داوطلب و نتایج بیشتر GE -فیلترها در GE -جبرها

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خواص بیشتری روی GE -فیلترهای جنگجو بررسی و GE -جبر خارج‌قسمتی با استفاده از GE -فیلتر ساخته شده است. همچنین شرایطی برای این‌که یک مجموعه‌ی یک GE -فیلتر بشود، آورده شده است. خاصیت توسیع برای GE -فیلترهای جنگجو تشریح شده است. مفاهیم عنصر متوازن، GE -فیلتر متوازن، GE -جبر پادمتقارن و GE -فیلتر داوطلب معرفی و خواص آن‌ها بررسی شده‌اند و رابطه بین یک GE -زیرجبر و یک GE -فیلتر ارائه شده است. شرایطی برای این‌که یک عنصر در GE -جبر یک عنصر متوازن بشود آورده شده است. شرایط لازم برای این‌که یک GE -فیلتر یک GE -فیلتر داوطلب بشود در نظر گرفته شده است. GE -فیلتر تولید شده توسط یک زیر مجموعه داده شده شرح داده و شکل آن مشخص شده است.

کلمات کلیدی: GE -جبر (جابجایی، تعدی، تعویض‌پذیر چپ، پادمتقارن، متوازن)، GE -جبر خارج‌قسمتی، GE -فیلتر (جنگجو، متوازن، داوطلب)، عنصر متوازن.