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A NOTE ON RELATIVE GENERALIZED COHEN-MACAULAY MODULES

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Abstract. Let $\mathfrak{a}$ be a proper ideal of a ring R. A finitely generated R-module M is said to be $\mathfrak{a}$-relative generalized Cohen-Macaulay if $f_{\mathfrak{a}}(M) = \mathrm{cd}(\mathfrak{a}, M)$. In this note, we introduce a suitable notion of length of a module to characterize the above mentioned modules. Also certain syzygy modules over a relative Cohen-Macaulay ring are studied.

1. Introduction

Throughout this note, R is a commutative Noetherian ring with identity and $\mathfrak{a}$ is a proper ideal of R.

Suppose, for a moment, that $(R, \mathfrak{m})$ is local and M is a finitely generated R-module of dimension $d > 0$. Then M is said to be a generalized Cohen-Macaulay module if $l\left(\mathrm{H}^i_{\mathfrak{m}}(M)\right) < \infty$ for $i = 0, \dots, d-1$, where l denotes the length and $\mathrm{H}^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module of M with respect to $\mathfrak{m}$.

Clearly, the class of generalized Cohen-Macaulay modules contains the class of Cohen-Macaulay modules. Indeed generalized Cohen-Macaulay modules enjoy many interesting properties similar to the ones of Cohen-Macaulay modules. As a generalization of the notion of Cohen-Macaulay modules, relative Cohen-Macaulay modules were introduced by Rahro Zargar and Zakeri in [11] and studied in [7], [8], [9], [10]. It

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could be of interest to establish a theory of relative generalized Cohen-Macaulay modules. Indeed this is done already in [4].

In this note, we continue the study of $\mathfrak{a}$-relative generalized Cohen-Macaulay modules and $\mathfrak{a}$-relative Cohen-Macaulay modules. First, we provide a characterization of relative generalized Cohen-Macaulay modules in terms of a suitable notion of length of a module which will be given in Section 2. Next, in Section 3, some properties of syzygy modules of a finitely generated module are established. Finally, the relative Cohen-Macaulayness of certain syzygy modules over a relative Cohen-Macaulay ring are presented.

2. Relative generalized Cohen-Macauly modules

Definitions and Remark 2.1. *Let M be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an ideal of R.*

(i) *Cohomological dimension of M with respect to $\mathfrak{a}$ is defined as*

$$\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} : H^i_{\mathfrak{a}}(M) \neq 0\}.$$

(ii) *If $M \neq \mathfrak{a}M$, then M is said to be $\mathfrak{a}$-relative Cohen-Macaulay, $\mathfrak{a}$-RCM, if* $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$.

We say that M is maximal $\mathfrak{a}$-RCM if M is $\mathfrak{a}$-RCM and $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, R)$.

(iii) *Following* [1, Definition 9.1.3], *the $\mathfrak{a}$-finiteness dimension of M, $f_{\mathfrak{a}}(M)$, is defined by*

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\}$$
$$\left(\overset{\dagger}{=} \inf\{i \in \mathbb{N} \mid \mathfrak{a} \nsubseteq \operatorname{Rad}\left(\operatorname{Ann}_R\left(H^i_{\mathfrak{a}}(M)\right)\right)\}\right).$$

(The equality † holds by Faltings' Local-global Principle Theorem [6, Satz 1]*.)*

(iv) *If $c := \operatorname{cd}(\mathfrak{a}, M) > 0$, then by* [3, Corollary 3.3(i)], *the R-module $H^c_{\mathfrak{a}}(M)$ is not finitely generated. So in this case, one has $f_{\mathfrak{a}}(M) \leq \operatorname{cd}(\mathfrak{a}, M)$.*

Definition 2.2. Let $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module, we say that M is $\mathfrak{a}$-relative generalized Cohen-Macaulay if $\operatorname{cd}(\mathfrak{a}, M) \leq 0$; or $\operatorname{cd}(\mathfrak{a}, M) = f_{\mathfrak{a}}(M)$.

Definition 2.3. Let $\mathfrak{a}$ be an ideal of R and M be an R-module. We say that the length of M with respect to $\mathfrak{a}$ is finite, if there is a chain of submodules of M as

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \quad (*)$$

such that M_i/M_{i-1} is a homomorphic image of $R/\mathfrak{a}$ for all $i = 1, \dots, n$. Set

$$l(\mathfrak{a}, M) := \inf\{n \in \mathbb{N}_0 | \text{ there is a chain of length n as in (*) }\}.$$

We call $l(\mathfrak{a}, M)$, $\mathfrak{a}$-relative lenght of M. Clearly, $l(\mathfrak{a}, M)$ is nonnegative or $+\infty$.

Remark 2.4. Let $l(\mathfrak{a}, M) = n$, then there is a chain of submodules of M as

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

such that M_i/M_{i-1} is a homomorphic image of $R/\mathfrak{a}$ for all $i = 1, \dots, n$. Thus for $i = 1, \dots, n$, M_i/M_{i-1} is a finitely generated R-module. By using the exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$$

for all $i = 1, \dots, n$, we see that if $l(\mathfrak{a}, M) < \infty$ then M is a finitely generated R-module.

Lemma 2.5. *Let L be a submodule of an R-module M. Then*

(i) $l(\mathfrak{a}, M) \leq l(\mathfrak{a}, L) + l(\mathfrak{a}, M/L)$.
(ii) $l(\mathfrak{a}, M/L) \leq l(\mathfrak{a}, M)$.

Proof. (i) Obviously, we may and do assume that $t := l(\mathfrak{a}, L) < \infty$ and $k := l(\mathfrak{a}, M/L) < \infty$. Then there is a chain of submodules of L as

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_t = L$$

such that for all $i = 1, \dots, t$, L_i/L_{i-1} is a homomorphic image of $R/\mathfrak{a}$. Also, there is a chain of submodules of M/L as

$$L/L = N_0 \subseteq N_1 = M_1/L \subseteq \dots \subseteq N_k = M_k/L = M/L$$

such that for all $i = 1, \dots, k$, N_i/N_{i-1} is a homomorphic image of $R/\mathfrak{a}$. Now, using the above two chains yield the chain

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_t = L \subseteq M_1 \subseteq \dots \subseteq M_k = M$$

and hence $l(\mathfrak{a}, M) \leq t + k$.

(ii) Obviously, we may and do assume that $n := l(\mathfrak{a}, M) < \infty$. Then there is a chain of submodules of M as

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

such that for all $i = 1, \dots, n$, M_i/M_{i-1} is a homomorphic image of $R/\mathfrak{a}$. Above chain yields the chain

$$0 \subseteq \frac{M_1 + L}{L} \subseteq \dots \subseteq \frac{M_n + L}{L} = \frac{M}{L}.$$

Since $\frac{M_i+L}{M_{i-1}+L}$ is a homomorphic image of M_i/M_{i-1}, it follows that $\frac{M_i+L}{M_{i-1}+L}$ is a homomorphic image of $R/\mathfrak{a}$. Thus $l(\mathfrak{a}, M/L) \leq n$. □

Lemma 2.6. *Let $\mathfrak{a}$ be an ideal of R and M be an R-module. Consider the following statements:*

(i) *There is $t \in \mathbb{N}$ such that $\mathfrak{a}^t M = 0$.*
(ii) $l(\mathfrak{a}, M) < \infty$.
(iii) $\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R M) = \operatorname{Rad}(\operatorname{Ann}_R M)$.

Then (iii) $\Longleftrightarrow$ (i) and (ii) $\Longrightarrow$ (i). Furthermore, if M is finitely generated, then (i) $\Longrightarrow$ (ii).

Proof. (i) $\Longrightarrow$ (ii) Let $t = 1$. If $0 \neq x \in M$, then $\mathfrak{a}x = 0$ and there is an epimorphism

$$R/\mathfrak{a} \longrightarrow R/\operatorname{Ann}_R(x) \cong Rx.$$

Set $M_1 := Rx$. Since $\mathfrak{a}(M/M_1) = 0$, there is a submodule M_2/M_1 of M/M_1 and an epimorphism

$$R/\mathfrak{a} \longrightarrow M_2/M_1.$$

Proceeding in this way, we get the following chain of submodules of M

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n \subseteq \ldots$$

such that the map $R/\mathfrak{a} \longrightarrow M_i/M_{i-1}$ is surjective for all $i = 1, \ldots, n$. Since M is Noetherian, the above chain stops somewhere.

Let $t > 1$ and assume that the result has been proved for $t - 1$. Since $\mathfrak{a}^{t-1}(\mathfrak{a}M) = 0$ and $\mathfrak{a}(M/\mathfrak{a}M) = 0$, it follows from the inductive hypothesis and Lemma 2.5(i) that $l(\mathfrak{a}, M) < \infty$.

(ii) $\Longrightarrow$ (i) Let $l(\mathfrak{a}, M) = n$. Then there is a chain of submodules of M as

$$0 = M_1 \subseteq M_2 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M,$$

such that for all $i = 1, \ldots, n$, M_i/M_{i-1} is a homomorphic image of $R/\mathfrak{a}$.

Since M_1 is a homomorphic image of $R/\mathfrak{a}$, one has $\mathfrak{a}M_1 = 0$. Using the epimorphism $R/\mathfrak{a} \longrightarrow M_i/M_{i-1}$ we get

$$0 = \mathfrak{a}(\frac{M_2}{M_1}) = \frac{\mathfrak{a}M_2 + M_1}{M_1}.$$

Thus $\mathfrak{a}M_2 \subseteq M_1$. So $\mathfrak{a}^2 M_2 = 0$. continuing this way, yields that $\mathfrak{a}^n M = \mathfrak{a}^n M_n = 0$.

$(i) \Longrightarrow (iii)$ Since $\mathfrak{a}^t M = 0$, we have $\mathfrak{a}^t \subseteq \operatorname{Ann}_R M$. The following display

$$\begin{aligned}\operatorname{Rad}(\operatorname{Ann}_R M) &\subseteq \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R M)\\ &= \operatorname{Rad}(\operatorname{Rad}(\mathfrak{a}) + \operatorname{Rad}(\operatorname{Ann}_R M))\\ &= \operatorname{Rad}\left(\operatorname{Rad}(\mathfrak{a}^t) + \operatorname{Rad}(\operatorname{Ann}_R M)\right)\\ &= \operatorname{Rad}\left(\mathfrak{a}^t + \operatorname{Ann}_R M\right)\\ &= \operatorname{Rad}(\operatorname{Ann}_R M),\end{aligned}$$

shows that $\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R M) = \operatorname{Rad}(\operatorname{Ann}_R M)$.

$(iii) \Longrightarrow (i)$ It is clear. □

Corollary 2.7. *Let L be a submodule of an R-module M. If $l(\mathfrak{a}, M) < \infty$, then $l(\mathfrak{a}, L) < \infty$.*

Proof. Remark 2.4 yields that M is finitely generated. Since $l(\mathfrak{a}, M) < \infty$, by Lemma 2.6, there is $t \in \mathbb{N}$ such that $\mathfrak{a}^t M = 0$. Since $L \subseteq M$, we have $\mathfrak{a}^t L = 0$. So by Lemma 2.6, $l(\mathfrak{a}, L) < \infty$. □

Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of R and M be a finitely generated R-module with $c := \operatorname{cd}(\mathfrak{a}, M) > 0$. Then the following are equivalent:*

(i) *M is $\mathfrak{a}$-relative generalized Cohen-Macaulay.*
(ii) *$l\left(\mathfrak{a}, H^i_{\mathfrak{a}}(M)\right) < \infty$ for all $i < c$.*

Proof. $(i) \Longrightarrow (ii)$ By assumption $f_{\mathfrak{a}}(M) = c$. Hence

$$\mathfrak{a} \subseteq \operatorname{Rad}\left(\operatorname{Ann}_R\left(\mathrm{H}^i_{\mathfrak{a}}(M)\right)\right)$$

for all $i < c$. So there is $n \in \mathbb{N}$ such that $\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i < c$. By Lemma 2.6, $l\left(\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M)\right) < \infty$ for all $i < c$.

$(ii) \Longrightarrow (i)$ By Lemma 2.6, there is $n \in \mathbb{N}$ such that $\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i < c$. Hence, $f_{\mathfrak{a}}(M) \geq c$. As $f_{\mathfrak{a}}(M) \leq c$, we deduce that $f_{\mathfrak{a}}(M) = c$. □

3. Special relative generalized Cohen-Macaulay modules

Let

$$F_\bullet : \ldots F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \xrightarrow{\varphi_{-1}} 0$$

be a free resolution of M and $\Omega^i_R(M) := \ker \varphi_{i-1}$ be the i-th syzygy module of M for all $i \in \mathbb{N}_0$.

Lemma 3.1. *Let M be a finitely R-module, n a positive integer and $\Omega^n_R(M)$ the n-th syzygy of M. Then*

$$\operatorname{grade}(\mathfrak{a}, \Omega^n_R(M)) \geq \min\{n, \operatorname{grade}(\mathfrak{a}, R)\}.$$

Proof. We do induction on n. If $n = 0$, it is trivial. If $n = 1$, consider the exact sequence

$$0 \to \Omega_R^1(M) \to F_0 \to M \to 0.$$

By [2, Proposition 1.2.9],

$$\begin{aligned} \operatorname{grade}(\mathfrak{a}, \Omega_R^1(M)) &\geq \min\{\operatorname{grade}(\mathfrak{a}, F_0), \operatorname{grade}(\mathfrak{a}, M) + 1\} \\ &\geq \min\{\operatorname{grade}(\mathfrak{a}, R), 0 + 1\}. \end{aligned}$$

Next, assume that the result has been proved for $n - 1$. Consider the exact sequence

$$0 \to \Omega_R^n(M) \to F_{n-1} \to \Omega_R^{n-1}(M) \to 0.$$

By [2, Proposition 1.2.9] and induction hypothesis, one has

$$\begin{aligned} \operatorname{grade}(\mathfrak{a}, \Omega_R^n(M)) &\geq \min\{\operatorname{grade}(\mathfrak{a}, F_{n-1}), \operatorname{grade}(\mathfrak{a}, \Omega_R^{n-1}(M)) + 1\} \\ &\geq \min\{\operatorname{grade}(\mathfrak{a}, R), \min\{\operatorname{grade}(\mathfrak{a}, R), n - 1\} + 1\}. \end{aligned}$$

Case1: If $\operatorname{grade}(\mathfrak{a}, R) \geq n - 1$, then

$$\min\{\operatorname{grade}(\mathfrak{a}, R), \min\{\operatorname{grade}(\mathfrak{a}, R), n - 1\} + 1\} = \min\{\operatorname{grade}(\mathfrak{a}, R), n\}.$$

Case 2: If $\operatorname{grade}(a, R) < n - 1$, then

$$\begin{aligned} \min\{\operatorname{grade}(\mathfrak{a}, R), \min\{\operatorname{grade}(\mathfrak{a}, R), n - 1\} + 1\} &= \operatorname{grade}(\mathfrak{a}, R) \\ &= \min\{\operatorname{grade}(\mathfrak{a}, R), n\}. \end{aligned}$$

This completes the inductive step. □

Let $\mathfrak{a}$ be an ideal of R and M, N be two finitely generated R-modules such that $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. Then, by [5, Theorem 2.2], $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$. In particular if $\operatorname{Supp}_R N = \operatorname{Supp}_R M$, then $\operatorname{cd}(\mathfrak{a}, N) = \operatorname{cd}(\mathfrak{a}, M)$. In the rest of the paper, we shall use this several times without any further comment.

Lemma 3.2. *Let R be an $\mathfrak{a}$-RCM ring with $\operatorname{cd}(\mathfrak{a}, R) = c$ and M be a finitely generated R-module. Then for every $i \geq c$, either $\Omega_R^i(M) = \mathfrak{a}\Omega_R^i(M)$ or $\Omega_R^i(M)$ is maximal $\mathfrak{a}$-RCM.*

Proof. Let $i \geq c$ and assume that $\Omega_R^i(M) \neq \mathfrak{a}\Omega_R^i(M)$. Then,

$$\operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)) \leq \operatorname{cd}(\mathfrak{a}, \Omega_R^i(M)).$$

By Lemma 3.1,

$$\begin{aligned}\operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)) & \\ &\geq \min\{i, \operatorname{grade}(\mathfrak{a}, R)\} \\ &= \operatorname{cd}(\mathfrak{a}, R) \\ &\geq \operatorname{cd}(\mathfrak{a}, \Omega_R^i(M)) \\ &\geq \operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)).\end{aligned}$$

Thus $\operatorname{cd}(\mathfrak{a}, R) = \operatorname{cd}(\mathfrak{a}, \Omega_R^i(M)) = \operatorname{grade}(\mathfrak{a}, \Omega_R^i(M))$. □

Remark 3.3. Let N be an $\mathfrak{a}$-RCM R-module and M a finitely generated R-module. If $M \neq \mathfrak{a}M$ and $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$, then

$$\operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{grade}(\mathfrak{a}, N).$$

Proof. One has

$$\operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, N) = \operatorname{grade}(\mathfrak{a}, N).$$

□

Proposition 3.4. *Let $\mathfrak{a}$ be a proper ideal of R and M be a non-zero finitely generated R-module. If $r = \operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{grade}(\mathfrak{a}, R) = s$, then $\operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)) = r + i$ for all $0 \leq i \leq s - r$. In particular, $\operatorname{pd}_R M \geq \operatorname{grade}(\mathfrak{a}, R) - \operatorname{grade}(\mathfrak{a}, M)$.*

Proof. The exact sequence $0 \longrightarrow \Omega_R^{i+1}(M) \longrightarrow F_i \longrightarrow \Omega_R^i(M) \longrightarrow 0$ implies the following exact sequences:

$$0 \longrightarrow \operatorname{Ext}_R^{s-1}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) \longrightarrow \operatorname{Ext}_R^s(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)), \tag{1}$$

and

$$0 \longrightarrow \operatorname{Ext}_R^{j-1}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) \longrightarrow \operatorname{Ext}_R^j(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)) \longrightarrow 0 \quad (j < s). \tag{2}$$

We use induction on i. If $i = 0$, the claim is trivial because $\Omega_R^0(M) = M$. Assume that $0 < i + 1 \leq s - r$ and the result has been proved for i. If $j < r + i + 1 \leq s$, then $j - 1 < r + i$, and so by the induction hypothesis, $\operatorname{Ext}_R^{j-1}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) = 0$. Thus the exact sequence (2) implies that $\operatorname{Ext}_R^j(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)) = 0$.

Now, we prove that $\operatorname{Ext}_R^{r+i+1}(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)) \neq 0$. If $r + i + 1 < s$, by the induction hypothesis $\operatorname{Ext}_R^{r+i}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) \neq 0$. So the exact sequence (2) implies that $\operatorname{Ext}_R^{r+i+1}(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)) \neq 0$. If $r+i+1 = s$, then by the induction hypothesis $\operatorname{Ext}_R^{r+i}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) = \operatorname{Ext}_R^{s-1}(\frac{R}{\mathfrak{a}}, \Omega_R^i(M)) \neq 0$. So the exact sequence (1) implies that $\operatorname{Ext}_R^{r+i+1}(\frac{R}{\mathfrak{a}}, \Omega_R^{i+1}(M)) \neq 0$. Hence $\operatorname{grade}(\mathfrak{a}, \Omega_R^{i+1}(M)) = r + i + 1$. □

Corollary 3.5. *Let M be a non-zero finitely generated R-module. If M is $\mathfrak{a}$-torsion, then* $\operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)) = i$ *for all* $0 \leq i \leq \operatorname{grade}(\mathfrak{a}, R)$. *In particular,* $\operatorname{pd}_R M \geq \operatorname{grade}(\mathfrak{a}, R)$.

Proof. Note that $0 = \operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{grade}(\mathfrak{a}, R)$, so the claim follows by Proposition 3.4. □

Theorem 3.6. *Let R be an $\mathfrak{a}$-RCM ring and M an $\mathfrak{a}$-torsion R-module. Assume that* $c := \operatorname{cd}(\mathfrak{a}, R) > 0$ *and*

$$F_\bullet : \ldots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M, then

(i) *For every $i < c$, one has*

$$H_{\mathfrak{a}}^i(\Omega_R^j(M)) = \begin{cases} M & (\text{if } i = j) \\ 0 & (\text{if } i \neq j) \end{cases}$$

(ii) *For every $1 \leq j \leq c-1$, the sequence*

$$0 \to H_{\mathfrak{a}}^c(\Omega_R^j(M)) \to H_{\mathfrak{a}}^c(F_{j-1}) \to H_{\mathfrak{a}}^c(\Omega_R^{j-1}(M)) \to 0$$

is exact. Also the sequence

$$0 \longrightarrow M \to H_{\mathfrak{a}}^c(\Omega_R^c(M)) \to H_{\mathfrak{a}}^c(F_{c-1}) \to H_{\mathfrak{a}}^c(\Omega_R^{c-1}(M)) \to 0$$

is exact.

(iii) *for every $1 \leq j \leq c-1$ the sequences*

$$0 \to H_{\mathfrak{a}}^c(\Omega_R^j(M)) \to H_{\mathfrak{a}}^c(F_{j-1}) \to \ldots \to H_{\mathfrak{a}}^c(F_0) \to 0$$

and

$$0 \to M \to H_{\mathfrak{a}}^c(\Omega_R^c(M)) \to H_{\mathfrak{a}}^c(F_{c-1}) \to \ldots \to H_{\mathfrak{a}}^c(F_0) \to 0$$

are exact.

Proof. (*i*) Let $i < c$. Note that since R is $\mathfrak{a}$-RCM, $\mathrm{H}_{\mathfrak{a}}^i(F_j) = 0$ for all $j \in \mathbb{N}_0$.

We use induction on j. For $j = 0$, the claim is trivial. Now, Let $j = 1$. The exact sequence

$$0 \to \Omega_R^1(M) \to F_0 \to M \to 0$$

implies exact sequences

$$0 \longrightarrow \mathrm{H}_{\mathfrak{a}}^0\left(\Omega_R^1(M)\right) \longrightarrow \mathrm{H}_{\mathfrak{a}}^0(F_0) = 0$$

$$0 = \mathrm{H}_{\mathfrak{a}}^0(F_0) \longrightarrow \mathrm{H}_{\mathfrak{a}}^0(M) \longrightarrow \mathrm{H}_{\mathfrak{a}}^1(\Omega_R^1(M)) \longrightarrow \mathrm{H}_{\mathfrak{a}}^1(F_0) = 0$$

and

$$0 = \mathrm{H}_{\mathfrak{a}}^{i-1}(M) \longrightarrow \mathrm{H}_{\mathfrak{a}}^i(\Omega_R^1(M)) \longrightarrow \mathrm{H}_{\mathfrak{a}}^i(F_0) = 0$$

for all $1 < i < c$. The above exact sequences show that

$$\mathrm{H}^1_{\mathfrak{a}}\left(\Omega^1_R(M)\right) \cong \mathrm{H}^0_{\mathfrak{a}}(M) = M$$

and for all $1 < i < c$, $\mathrm{H}^i_{\mathfrak{a}}(\Omega^1_R(M)) = 0$.

Let $j > 1$ and the result has been proved for $j-1$. The exact sequence

$$0 \to \Omega^j_R(M) \to F_{j-1} \to \Omega^{j-1}_R(M) \to 0$$

implies the exact sequence

$$0 = \mathrm{H}^{i-1}_{\mathfrak{a}}(F_{j-1}) \longrightarrow \mathrm{H}^{i-1}_{\mathfrak{a}}(\Omega^{j-1}_R(M)) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(\Omega^j_R(M)) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(F_{j-1}) = 0.$$

Hence $\mathrm{H}^{i-1}_{\mathfrak{a}}(\Omega^{j-1}_R(M)) \cong \mathrm{H}^i_{\mathfrak{a}}(\Omega^j_R(M))$. The result follows by induction hypothesis.

(ii) The exact sequence

$$0 \to \Omega^j_R(M) \to F_{j-1} \to \Omega^{j-1}_R(M) \to 0$$

implies the exact sequence

$$\begin{aligned}\mathrm{H}^{c-1}_{\mathfrak{a}}(\Omega^{j-1}_R(M)) &\longrightarrow \mathrm{H}^c_{\mathfrak{a}}(\Omega^j_R(M)) \longrightarrow \mathrm{H}^c_{\mathfrak{a}}(F_{j-1}) \\ &\longrightarrow \mathrm{H}^c_{\mathfrak{a}}(\Omega^{j-1}_R(M)) \longrightarrow \mathrm{H}^{c+1}_{\mathfrak{a}}(\Omega^j_R(M)).\end{aligned}$$

By (i), $\mathrm{H}^{c-1}_{\mathfrak{a}}(\Omega^{j-1}_R(M)) = 0 = \mathrm{H}^{c+1}_{\mathfrak{a}}(\Omega^{j-1}_R(M))$ which yields the assertion.

The last assertion follows by applying the functor $\mathrm{H}^i_{\mathfrak{a}}(-)$ on the exact sequence

$$0 \to \Omega^c_R(M) \to F_{c-1} \to \Omega^{c-1}_R(M) \to 0.$$

(iii) It follows by (ii). □

Corollary 3.7. *Let R be an $\mathfrak{a}$-RCM ring with $c := \mathrm{cd}(\mathfrak{a}, R) > 0$, and M a non-zero finitely generated $\mathfrak{a}$-torsion R-module. Then for every $i \geq 0$, either $\Omega^i_R(M) = \mathfrak{a}\Omega^i_R(M)$ or $\mathrm{cd}(\mathfrak{a}, \Omega^i_R(M)) = c$.*

Proof. We may and do assume that $\Omega^i_R(M) \neq \mathfrak{a}\Omega^i_R(M)$. If $i \geq c$, then by Lemma 3.2, $\Omega^i_R(M)$ is maximal $\mathfrak{a}$-RCM and so the assertion follows in this case. Therefore we may assume that $0 < i < c$.

By Theorem 3.6, $\mathrm{H}^j_{\mathfrak{a}}(\Omega^i_R(M))$ is finitely generated for $j < c$. So $c \leq f_{\mathfrak{a}}(\Omega^i_R(M))$. By Lemma 3.1 and Definitions and Remark 2.1, we have

$$c \leq f_{\mathfrak{a}}\left(\Omega^i_R(M)\right) \leq \mathrm{cd}\left(\mathfrak{a}, \Omega^i_R(M)\right) \leq c$$

Hence $\mathrm{cd}\left(\mathfrak{a}, \Omega^i_R(M)\right) = c$. □

It is clear that every $\mathfrak{a}$-RCM module is $\mathfrak{a}$-relative generalized Cohen-Macaulay. But the converse is not true.

Example 3.8. Let R be an $\mathfrak{a}$-RCM ring with $c := \operatorname{cd}(a, R) > 0$ and M a non-zero finitely generated $\mathfrak{a}$-torsion R-module. Then

$$\Omega_R^1(M), \Omega_R^2(M), \ldots, \Omega_R^{c-1}(M)$$

are not $\mathfrak{a}$-RCM but they are $\mathfrak{a}$-relative generalized Cohen-Macaulay.

Proof. Let $1 \leq i < c$. By Corollary 3.5, $\operatorname{grade}(\mathfrak{a}, \Omega_R^i(M)) = i$ and by Corollary 3.7 $\operatorname{cd}(\mathfrak{a}, \Omega_R^i(M)) = c$. So $\Omega_R^i(M)$ is not $\mathfrak{a}$-RCM. But by Theorem 3.6, $c \leq f_{\mathfrak{a}}(\Omega_R^i(M)) \leq \operatorname{cd}(\mathfrak{a}, \Omega_R^i(M)) \leq c$. Hence $\Omega_R^i(M)$ is $\mathfrak{a}$-relative generalized Cohen-Macaulay. □

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A NOTE ON RELATIVE GENERALIZED COHEN–MACAULAY MODULES

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یادداشتی درباره مدول‌های کوهن–مکالی تعمیم‌یافته نسبی

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فرض کنیم $\mathfrak{a}$ یك ایده‌آل سره از حلقه نوتری و جابه‌جایی R باشد. R–مدول متناهی‌مولد M را کوهن–مکالی تعمیم‌یافته نسبی می‌نامیم اگر $f_{\mathfrak{a}}(M) = \mathrm{cd}(\mathfrak{a}, M)$، که $\mathrm{cd}(\mathfrak{a}, M)$ و $f_{\mathfrak{a}}(M)$ به ترتیب بیانگر بعدکوهمولوژی و بعدمتناهی هستند. با معرفی مفهوم طول نسبی، مدول‌های کوهن–مکالی تعمیم‌یافته نسبی را مشخص‌سازی می‌کنیم. در ادامه ویژگی‌هایی از مدول‌های سی‌زی‌جی مدول‌های خاصی را روی حلقه کوهن–مکالی نسبی مورد مطالعه قرار می‌دهیم.

کلمات کلیدی: بعدکوهمولوژی، بعدمتناهی، کوهن–مکالی تعمیم‌یافته نسبی.