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GRADED SEMIPRIME SUBMODULES OVER NON-COMMUTATIVE GRADED RINGS

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Abstract. Let G be a group with identity e, R be an associative graded ring and M be a G-graded R-module. In this article, we introduce the concept of graded semiprime submodules over non-commutative graded rings. First, we study graded prime R-modules over non-commutative graded rings and we get some properties of such graded modules. Second, we study graded semiprime and graded radical submodules of graded R-modules. For example, we give some equivalent conditions for a graded module to have zero graded radical submodule.

1. Introduction

The study of graded rings arises naturally out of the study of affine schemes and allows them to formalize and unify arguments by induction. However, this is not just an algebraic trick. The concept of grading in algebra, in particular graded modules is essential in the study of homological aspect of rings. Much of the modern development of the commutative algebra emphasizes graded rings. Graded rings play a central role in algebraic geometry and commutative algebra. Gradings appear in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become increasingly important and consequently, the graded analogues of different concepts are widely studied (see [1], [4], [9], [12] and

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[13]). Graded primary ideals of a commutative graded ring have been introduced and studied in [14]. Graded prime submodules of graded modules over commutative graded rings have been studied in [2], [3], [8] and [10]. Also graded prime submodules over non-commutative graded rings have been introduced and studied by R. Abu-Dawwas and M. Bataineh in [1]. Then graded 2-absorbing submodules over non-commutative graded rings have been studied in [6]. Here we introduce and study the concept of graded semiprime submodules of graded modules over non-commutative graded rings and give a number of its properties. First, we recall some basic properties of graded rings and graded modules which will be used in the sequel. Let G be a group with identity e and R be a ring. Then R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are homogeneous of degree g. Consider $supp(R, G) = \{g \in G \mid R_g \neq 0\}$. For simplicity, we will denote the graded ring (R, G) by R. If $r \in R$, then r can be written as $\sum_{g\in G} r_g$, where r_g is the component r in R_g. Moreover, R_e is a subring of R and $1_R \in R_e$. Furthermore, $h(R) = \bigcup_{g\in G} R_g$.

Let I be a left ideal of a graded ring R. Then I is said to be a graded left ideal of R, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. The following example from [1] shows that a left ideal of a graded ring need not to be graded.

Example 1.1. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbb{Z}_4$ (the group of integers modulo 4). Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \ \textit{and} \ R_1 = R_3 = \{0\}.$$

Consider the left ideal $I = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$ of R. Note that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in I$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If I is a graded left ideal of R, then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I$ which is a contradiction. So I is not a graded left ideal of R.

A proper graded ideal I of a graded ring R is said to be graded prime if whenever J and K are graded ideals of R such that $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$ (see [1]). If R is commutative, this definition is equivalent to: a proper graded ideal I of a graded ring R is said to be graded prime if whenever $r_gs_h \in I$ for some $r_g, s_h \in h(R)$, then

$r_g \in I$ or $s_h \in I$ [14]. Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G \mid M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $m \in N$, $m = \sum_{g\in G} m_g$, where $m_g \in N$ for all $g \in G$. The following example shows that an R-submodule of a graded R-module need not be graded (see [1]).

Example 1.2. Consider $R = M = M_2(K)$ and $G = \mathbb{Z}_4$. Then R and M are G-graded as in Example 1.1 and similarly, $N = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$ is an R-submodule of M which is not graded.

A proper graded submodule N of a graded R-module M is said to be graded prime, if $IK \subseteq N$ for some graded ideal I of R and graded submodule K of M, then $K \subseteq N$ or $I \subseteq (N : M)$. It is easy to show that this definition is equivalent to: a proper graded submodule N of a graded R-module M is graded prime, if $r_g R m_h \subseteq N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. If N is a graded prime submodule of M, then $(N : M) = P$ is a graded prime ideal of R. In this case, we say that N is a P-graded prime submodule of M. If R is commutative, this definition is equivalent to: a proper graded submodule N of a graded R-module M is said to be graded prime, if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. A graded R-module M is called graded prime, if the zero graded submodule is graded prime in M. The graded radical of a graded submodule N of a graded R-module M, denoted by $Grad(N)$, is defined to be the intersection of all graded prime submodules of M containing of N. If there is no such graded prime submodule of M, then we write $Grad(M) = M$. For graded submodule N of M, if $Grad(N) = N$, then we say that N is a graded radical submodule of M. A graded R-module M is called graded Noetherian, if it satisfies the ascending chain condition on graded submodules.

In the second section of this article, we discuss direct sums of graded prime submodules and give some of their properties. In the third section, we study on graded semiprime submodules. We investigate when a graded semiprime submodule of a particular graded module is a graded radical submodule. In the last section of our article, we consider graded modules over commutative graded rings and give a characterization for graded semiprime submodules of graded Noetherian modules to be graded radical by using the notion of graded primary decomposition.

Throughout this work, all rings are assumed to be associative graded rings with identity, and all modules are unitary graded left R-modules.

2. Graded prime R-modules

In this section, we study graded prime modules over non-commutative graded rings.

We need the following lemma proved in [4, 14].

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals of R.*
(ii) *If I is a graded ideal of R, N is a graded submodule of M and $x \in h(M)$, then Rx, IN are graded submodules of M.*
(iii) *If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.*
(iv) *Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Lemma 2.2. *Let R be a graded ring and M be a graded R-module. Let $\{N_i\}_{i\in I}$ be a family of graded prime R-modules contained in M with distinct annihilators in R. Then the sum $\sum_{i\in I} N_i$ is direct.*

Proof. It is enough to show that for any finite collection $\{N_1, N_2, \ldots, N_n\}$ of graded prime R-modules in M with distinct annihilators the sum $N_1 + \cdots + N_n$ is direct. To see this, we use induction on n. If $n = 2$, and $N_1 \cap N_2 \neq 0$, then $ann(N_1) = ann(N_1 \cap N_2) = ann(N_2)$, a contradiction. Now assume that $n > 2$ and the sum of graded prime modules in any proper subset of $\{N_1, N_2, \ldots, N_n\}$ is direct. We show that $N = N_1 \cap (N_2 \oplus \cdots \oplus N_n) = 0$, and this will complete the proof. Suppose that $N_1 \cap (N_2 \oplus \cdots \oplus N_n) \neq 0$. By induction hypothesis, $N_1 \cap (N_3 \oplus \cdots \oplus N_n) = 0$, and hence N can be embedded in N_2. Thus N is graded prime R-module with $ann(N) = ann(N_2)$. On the other hand, since N is a non-zero graded submodule of N_1, we also have $ann(N) = ann(N_1)$, a contradiction. □

Lemma 2.3. *Let M be a graded module over a graded ring R, P a graded prime ideal of R, and $\{N_\alpha\}_{\alpha\in I}$ a family of graded prime R-modules contained in M such that $ann(N_\alpha) = P$ for any $\alpha \in I$. If* *the sum* $\sum_{\alpha\in I} N_\alpha$ *is direct, then it is a graded prime* R*-module with annihilator* P*.*

Proof. Let $N = \bigoplus_{\alpha\in I} N_\alpha$. It is clear that N is a graded R-module since for any α, N_α is a graded R-module and also, we get $ann(N) = P$. Assume that there exist $r_g \in h(R) - P$ and $0 \neq n_h \in h(N)$ such that $r_g R n_h = 0$. Now, we have $n_h \in N_{\alpha_1} \oplus \cdots \oplus N_{\alpha_k}$ for some $\alpha_j \in I$ $(1 \leq j \leq k)$. Since $r_g \in h(R) - P$ and $r_g(N_{\alpha_j} \cap Rn_h) = 0$ $(1 \leq j \leq k)$, then $N_{\alpha_j} \cap Rn_h = 0$ for $1 \leq j \leq k$. In this case, Rn_h can be embedded in N_{α_j} for some $1 \leq j \leq k$. This gives $ann(Rn_h) = P$, a contradiction. □

Lemma 2.4. *Let* R *be a graded ring and* $\{N_\alpha\}_{\alpha\in I}$ *be a family of graded prime* R*-modules. If* K *is a graded prime* R*-module contained in* $\bigoplus_{\alpha\in I} N_\alpha$*, then* $ann(K) = ann(N_\beta)$ *for some* $\beta \in I$*.*

Proof. Say $N = \bigoplus_{\alpha\in I} N_\alpha$. By Lemma 2.3, the direct sum N can be reduced to one whose terms are all graded prime R-modules with distinct annihilators. Since $N \cap K = K \neq 0$, by Lemma 2.2,

$$ann(K) = ann(N \cap K) = ann(N) = \bigcap_{\alpha\in I} ann(N_\alpha).$$

Since $ann(K)$ is a graded prime ideal of R, so $ann(K) = ann(N_\beta)$ for some $\beta \in I$. □

Theorem 2.5. *Let* R *be a graded ring and let* $\{N_\alpha\}_{\alpha\in I}$ *be a family of graded prime* R*-modules with distinct annihilators . If* $\{N'_\beta\}_{\beta\in J}$ *be another family of graded prime* R*-modules with distinct annihilators such that*

$$M = \bigoplus_{\alpha\in I} N_\alpha = \bigoplus_{\beta\in J} N'_\beta,$$

then there is a one-to-one correspondence between these two families such that corresponding graded modules are isomorphic.

Proof. Take $\alpha_0 \in I$. By Lemma 2.4, there exists $\beta_0 \in J$ such that $ann(N_{\alpha_0}) = ann(N'_{\beta_0})$. Consider the composition

$$N_{\alpha_0} \xrightarrow{\psi} N'_{\beta_0} \xrightarrow{\phi} N_{\alpha_0}$$

where ψ and ϕ denote restrictions of standard projections. Let $0 \neq x_g \in N_{\alpha_0}$. Then we have $x_g = y_g + y'_g$ for some $y_g \in N'_{\beta_0}$ and $y'_g \in \bigoplus_{\beta\neq\beta_0} N'_\beta$. If $y'_g = 0$, then clearly, $\phi\psi(x_g) = x_g$. Now, let $y'_g \neq 0$. Then there exist finitely many $\beta_1, \ldots, \beta_t \in J - \{\beta_0\}$ such that y'_g belongs to the direct sum $N'_{\beta_1} \bigoplus \cdots \bigoplus N'_{\beta_t}$ where $\{\beta_1, \ldots, \beta_t\}$ is minimal with respect to this property. Let $P'_{\beta_i} = ann(N'_{\beta_i})$ $(1 \leq i \leq t)$ and $P_{\alpha_0} = ann(N_{\alpha_0})$. Then clearly $P_{\alpha_0} \subset P'_{\beta_1} \cap \cdots \cap P'_{\beta_t}$. Let

$$r_h \in (P'_{\beta_1} \cap \cdots \cap P'_{\beta_t}) \cap (h(R) - P_{\alpha_0}).$$

Then $r_h R y'_g = 0 \subseteq \bigoplus_{\alpha \neq \alpha_0} N_\alpha$. Since $\bigoplus_{\alpha \neq \alpha_0} N_\alpha$ is a P_{α_0}-graded prime submodule of M, we must have $y'_g \in \bigoplus_{\alpha \neq \alpha_0} N_\alpha$. Therefore,

$$\phi\psi(x_g) = \phi(\psi(y_g + y'_g)) = \phi(y_g) = \phi(x_g - y'_g) = x_g$$

and so for any $x \in N_{\alpha_0}$, we write $x = x_{g_1} + \cdots + x_{g_n}$ where $x_{g_i} \in h(N_{\alpha_0})$, then

$$\begin{aligned}
\phi\psi(x) &= \phi\psi(x_{g_1} + \cdots + x_{g_n}) \\
&= \phi\psi(x_{g_1}) + \cdots + \phi\psi(x_{g_n}) \\
&= x_{g_1} + \cdots + x_{g_n} \\
&= x.
\end{aligned}$$

Hence $\phi\psi = id_{N_{\alpha_0}}$. Using symmetric arguments, we may also show that $\psi\phi = id_{N'_{\beta_0}}$. Thus $N_{\alpha_0} \cong N'_{\beta_0}$. □

Definition 2.6. A graded submodule N of a graded R-module M is said to be a graded essential submodule of M, if for every graded submodule K of M, $N \cap K = \{0\}$, then $K = \{0\}$.

Definition 2.7. Let M be a graded left R-module. We say that M is a graded uniform R-module, if for every non-zero graded submodules N and K, $N \cap K \neq 0$.

Definition 2.8. Let M be a graded R-module. We say that M has graded finite uniform dimension, if there exists a finite set of graded uniform submodules U_i $(1 \leq i \leq n)$ of M such that $\bigoplus_{i=1}^{n} U_i$ is a graded essential submodule of M.

Lemma 2.9. *Let M be a graded module over a graded ring R. If the set $\{ann(N) \mid 0 \neq N \leq M\}$ has a maximal element, say $ann(N_0)$, then N_0 is a graded prime R-module.*

Proof. Let $IK = \{0\}$ where I is a graded ideal of R and K a graded submodule of N_0 and let $K \neq \{0\}$. So $I \subseteq ann(K) \subseteq ann(N_0)$ since $ann(N_0)$ is a maximal member of $\{ann(N) \mid 0 \neq N \leq M\}$. Hence N_0 is a graded prime R-module. □

Remark 2.10. If either R is a commutative graded ring and M is a graded Noetherian module or R has ascending chain condition ($a.c.c$) on its graded two-sided ideals, then every non-zero graded submodule of M contains a graded prime R-module, and in this case, every maximal independent family of graded prime submodules contained in M has essential sum in M.

We say that a graded ideal P of a graded ring R is an associated graded prime of a graded R-module M if there exists a graded prime submodule N of M such that $ann(N) = P$. The set of all associated graded primes of M in R is denoted by $Gass(M)$.

Proposition 2.11. *Let R be a graded ring with a.c.c. on graded ideals, and let M be a graded module. Then M has graded finite uniform dimension if and only if $|Gass(M)| < \infty$ and every graded prime R-module contained in M has graded finite uniform dimension.*

Proof. If M has finite graded uniform dimension, then by Lemma 2.2, clearly, $|Gass(M)| < \infty$. On the other hand, assume that $|Gass(M)| < \infty$ and every graded prime module M has graded finite uniform dimension. Consider a maximal independent family of graded primes in M. By using Lemma 2.3, and the fact $|Gass(M)| < \infty$, this direct sum can be written as a direct sum of finitely many graded prime modules in M with distinct annihilators. By Remark 2.10, this direct sum is essential in M. This completes the proof. □

Definition 2.12. Let M be a graded R-module and L be a graded submodule of M. We say that a graded prime submodule N of L can be lifted to M if there exists a graded prime submodule K of M such that $N = K \cap L$.

Definition 2.13. Let M be a graded R-module. A graded submodule K of M is called a complement of a graded submodule N in M provided that K is a maximal with respect to the property $K \cap N = 0$.

Lemma 2.14. *Let M be a graded module over a graded ring R and L be a graded submodule of M such that L is graded prime R-module with $ann(L) = P$. If $L \cap PM = 0$, then any complement to L in M containing PM is a graded P-prime submodule of M i.e., the zero graded submodule of L can be lifted to M.*

Proof. Let K be a complement to L in M containing PM. Clearly, $K \neq M$. Let $IN \subseteq K$ for a graded ideal I of R and a graded submodule N of M such that $K \subset N$. Then $L \cap N \neq 0$ and $I(L \cap N) \subseteq L \cap K = 0$. Since L is a graded prime module, $I \subseteq P$ and hence $IM \subseteq PM \subseteq K$, so K is a graded prime submodule of M. □

Lemma 2.15. *Let M be a graded R-module. If the zero graded submodule is a graded radical submodule of M, then the zero graded submodule of graded prime module contained in M can be lifted to M.*

Proof. Let $\bigcap_{i \in I} K_i = 0$ where K_i is a graded prime submodule of M for all $i \in I$. Let L be a graded prime R-module in M with $P = ann(L)$.

Since $PL = 0 \subseteq K_i$ $(i \in I)$, we have $PM \subseteq K_i$ or $L \subseteq K_i$ $(i \in I)$. Thus $L \cap PM \subseteq K_i$ $(i \in I)$, and so $L \cap PM = 0$. By Lemma 2.14, there exists a graded prime submodule K of M such that $L \cap K = 0$. This completes the proof. □

Theorem 2.16. *Let M be a graded R-module. Suppose that M contains a graded essential submodule which is a direct sum of graded prime R-modules. Then the following statements are equivalent:*

(i) *The zero graded submodule is a graded radical submodule of M;*
(ii) *The zero graded submodule of every graded prime module contained in M can be lifted to M;*
(iii) *For every graded prime R-module L contained in M, $L \cap PM = 0$, where $P = ann(L)$;*
(iv) *For every graded prime R-module L contained in M, $L \nsubseteq PM$, where $P = ann(L)$.*

Proof. $(i) \Rightarrow (ii)$ Apply Lemma 2.15.

$(ii) \Rightarrow (iii)$ Let L be a graded prime R-module with $ann(L) = P$, and K be a graded prime submodule of M such that $L \cap K = 0$. Since $PL = 0 \subseteq K$, then $PM \subseteq K$.

$(iii) \Rightarrow (ii)$ Apply Lemma 2.14.

$(iii) \Rightarrow (iv)$ It is straightforward.

$(iv) \Rightarrow (iii)$ Let L be a graded prime R-module contained in M such that $L \cap PM \neq 0$ where $ann(L) = P$. Since L is graded prime and $L \cap PM$ is contained in L, $L \cap PM$ is again a graded prime R-module with the sum annihilator P. This contradicts (iv).

$(ii) \Rightarrow (i)$ It is similar to $(ii) \Rightarrow (i)$ of [15, Theorem 2.9] for non-graded case. □

Corollary 2.17. *Let M be a graded R-module. Suppose that either R has a.c.c. on graded ideals or R is commutative and M is graded Noetherian. Then the zero graded submodule is a graded radical submodule of M if and only if the zero graded submodule of every graded prime module contained in M can be lifted to M.*

3. Graded semiprime submodules

In this section, we study graded semiprime submodules over non-commutative graded rings.

Definition 3.1. Let M be a graded R-module. A graded submodule N of M is said to be graded semiprime, if $N \neq M$ and whenever $r_g R r_g m_h \subseteq N$ for some $r_g \in h(R)$ and $m_h \in h(M)$, then $r_g m_h \in N$. A graded module M is called graded semiprime if the zero graded submodule of M is graded semiprime. It can easily be seen that, for a proper graded left ideal I of R, I is graded semiprime submodule of ${}_RR$ if and only if $r_gRr_g \subseteq I$ implies $r_g \in I$ for every $r_g \in h(R)$ if and only if $J^2 \subseteq I$, then $J \subseteq I$ for every graded left ideal J of R. We call such a proper graded left ideal a graded semiprime left ideal of R.

Example 3.2. Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in K \right\}$ where K is a field and $G = \mathbb{Z}_4$ (the group of integers modulo 4). Then R is G-graded by $R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in K \right\}$ and $R_2 = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \mid b \in K \right\}$ and $R_1 = R_3 = \{0\}$. Let $M = R$, then M is G-graded by $M_g = R_g$. Then M is a graded semiprime R-module.

Proof. Let $r_gRr_g = \{0\}$ where $r_g \in h(R) = R_0 \cup R_3$. We show that $r_g = \{0\}$. We have either $r_g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ or $r_g = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$ where $x, y \in K$. In the first case, since R has identity,

$$r_g^2 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so $x^2 = 0$, then $x = 0$ since K is a field. Hence $r_g = 0$. Similarly, in the second case, we get $r_g = 0$ and so M is a graded semiprime R-module. □

Lemma 3.3. *Let M be a graded R-module and N be a graded submodule of M. The following statements are equivalent:*

(i) *N is a graded semiprime submodule of M;*
(ii) *For every graded left ideal I of R and $m_g \in h(M)$, if $I^2m_g \subseteq N$, then $Im_g \subseteq N$;*
(iii) *For every $m_g \in h(M) - N$, $(N : m_g) = \{r \in R \mid rm_g \in N\}$ is a graded semiprime left ideal of R.*

Proof. $(i) \Rightarrow (ii)$ Let $a \in I$. Then we can write $a = \sum_{h \in G} a_h$ where $a_h \in I \cap h(R)$. Thus for any $h \in G$; $a_hm_g \in Im_g$. So

$$a_hRa_hm_g \subseteq I^2m_g \subseteq N,$$

and hence $a_hm_g \in N$ since N is graded semiprime. Therefore $am_g = \sum_{h \in G} a_hm_g \in N$, and so $Im_g \subseteq N$.

$(ii) \Rightarrow (iii)$ $(N : m_g) \neq R$ since $m_g \notin N$. Let $I^2 \subseteq (N : m_g)$ where I is a graded ideal of R. Hence $I^2m_g \subseteq N$, so by hypothesis, $Im_g \subseteq N$. Thus $I \subseteq (N : m_g)$, as needed.

$(iii) \Rightarrow (i)$ Let $r_gRr_gm_h \subseteq N$ where $r_g \in h(R)$ and $m_g \in h(M)$. So $r_gRr_g \subseteq (N : m_h)$, hence $r_g \in (N : m_h)$ since $(N : m_h)$ is a graded semiprime ideal of R. Therefore $r_g m_h \in N$, so N is a graded semiprime submodule of M. □

Let R be a graded ring and M be a graded R-module. For a graded submodule N of M and a graded ideal I of R, the set $\{m \in M \mid Im \subseteq N\}$ is denoted by $(N :_M I)$ or $(N : I)$. It can easy to see that $(N :_M I)$ is a graded submodule of M.

Lemma 3.4. *Let M be a graded R-module and I be a graded ideal of R. If N is a graded semiprime submodule of M, then either $IM \subseteq N$ or else $(N :_M I)$ is a graded semiprime submodule of M.*

Proof. Let $IM \nsubseteq N$. Then $(N :_M I) \neq M$. Now we show that $(N :_M I)$ is a graded semiprime submodule of M. Let $r_g R r_g m_{g'} \subseteq (N :_M I)$ where $r_g \in h(R)$ and $m_{g'} \in h(M)$, hence $I(r_g R r_g m_{g'}) \subseteq N$. Let $a \in I$, so $a = \sum_{h \in G} a_h$ where $a_h \in I \cap h(R)$. We have $(a_h r_g) R (a_h a_g) m_{g'} \subseteq N$ for every $h \in G$, so $a_h a_g m_{g'} \in N$ for any $h \in G$. Therefore, $a r_g m_{g'} \in N$, so $r_g m_{g'} \in (N :_M I)$. Hence $(N :_M I)$ is a graded semiprime submodule of M. □

Let M be a graded uniform R-module. We may construct a subset $P = \{r \in R : rN = 0$ *for some graded submodule N of M*$\}$ of R. It is clear that P is a graded ideal of R, and we call it the assassinator of M.

Lemma 3.5. *Let M be a graded uniform R-module and P be an assassinator of M. If $PK = 0$, for some non-zero graded submodule K of M, then P is a graded prime ideal of R.*

Proof. Let $IJ \subseteq P$ for some graded ideals I and J of R and let $I \nsubseteq P$, so there exists $a \in I$ such that $a \notin P$. Let $b \in J$. Thus $ab \in IJ$, hence $ab \in P$. Therefore $a(bN) = (ab)N = 0$ for some non-zero graded submodule N of M. Then $bN = 0$ since $a \notin P$, so $b \in P$, as required. □

Lemma 3.6. *Let M be a graded uniform R-module. Then M is a graded prime R-module if and only if M is a graded semiprime R-module.*

Proof. It is clear that every graded prime R-module is a graded semiprime R-module. Let M be a graded semiprime R-module. Let $r_g R m_h = 0$ for some $r_g \in h(R)$ and $m_h \in h(M)$. Suppose that $r_g M \neq 0$ and $0 \neq m_h$. Then $R r_g M \cap R m_h \neq 0$ since M is a graded uniform R-module. Hence there exist $s_{g'}, t_{g''} \in h(R)$ and $n_{h'} \in h(M)$ such that $0 \neq s_{g'} r_g n_{h'} = t_{g''} m_h$. Thus $R s_{g'} r_g n_{h'} = R t_{g''} m_h \subseteq R m_h$, and so $s_{g'} r_g R s_{g'} r_g n_{h'} \subseteq s_{g'} r_g R m_h = 0$. As M is graded semiprime,

$s_{g'} r_g n_{h'} = 0$, a contradiction. Hence M is a graded prime R-module. □

Theorem 3.7. *Let L be a graded semiprime submodule of a graded R-module M. If M/L has finite uniform dimension then the following statements are equivalent:*

- (i) *L is a graded radical submodule of M;*
- (ii) *L can be lifted to M whenever it is contained as a graded prime submodule in a graded submodule of M;*
- (iii) *For every graded uniform submodule U/L of M/L, there exists a graded prime submodule K of M such that $U \cap K = L$;*
- (iv) *For every graded uniform submodule U/L of M/L with assassinator P, $U \cap PM \subseteq L$;*
- (iv) *For every graded uniform submodule U/L of M/L with assassinator P, $U \nsubseteq L + PM$.*

Proof. Without loss of generality, we may assume that $L = 0$. Thus M is a graded semiprime R-module having finite uniform dimension. Since any graded uniform submodule of M is graded prime by Lemma 3.6, then the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ hold by Theorem 2.16.
$(iv) \Rightarrow (i)$ Since M has graded finite uniform dimension, so by Lemma 3.6, there exist a (finite) direct sum of graded prime modules which is essential in M. Let L be a graded prime R-module contained in M with annihilator P. Suppose that $L \cap PM \neq 0$. As M has graded finite uniform dimension, $L \cap PM$ contains a graded uniform module U. Clearly, P is the assassinator of U. By (iv), $U \cap PM = 0$, which is impossible, because $U \subseteq PM$. Hence every graded prime R-module L contained in M with $ann(L) = P$, we must have $L \cap PM = 0$. Then by Theorem 2.16, the proof holds. □

4. Graded modules over commutative graded rings

Throughout this section, R will denote a commutative G-graded ring with identity. In this section, we study graded semiprime submodules of graded modules over commutative graded rings.

Lemma 4.1. *Let R be a commutative graded ring with identity and M be a graded R-module. Then N is a graded semiprime submodule of M if and only if $r_g^2 m_h \in N$, where $r_g \in h(R)$ and $m_h \in h(M)$, then $r_g m_h \in N$.*

Proof. The proof is completely straightforward. □

We know that every graded prime submodule is a graded semiprime submodule, but the converse is not true in general. Consider the following example.

Example 4.2. Let R be a commutative domain which is not a field. Consider $M = R \oplus R$ and $G = \mathbb{Z}_2$ the group of integers modulo 2. Then R is G-graded by $R_0 = R$ and $R_1 = \{0\}$ and M is a G-graded R-module by $M_0 = R \oplus \{0\}$ and $M_1 = \{0\} \oplus R$. Let P be a non-zero graded prime ideal of R. Then $N = P \oplus \langle 0 \rangle$ is a graded semiprime submodule of M which is not a graded prime submodule of M.

Proof. Let

$$r_g^2 m_h \in N = P \oplus \langle 0 \rangle,$$

where $r_g \in h(R)$ and $m_h \in h(M) = M_0 \cup M_1$. So $m_h = (x, 0)$ or $m_h = (0, y)$ for some $x, y \in R$. If $r_g^2(x, 0) \in P \oplus \langle 0 \rangle$, then $r_g^2 x \in P$, so $r_g \in P$ or $x \in P$, therefore in any case, $r_g(x, 0) \in P \oplus \langle 0 \rangle$. If $r_g^2(0, y) \in P \oplus \langle 0 \rangle$, then $r_g^2 y = 0$, then $r_g = 0$ or $y = 0$, and so in any case, $r_g(0, y) \in P \oplus \langle 0 \rangle$. Thus $N = P \oplus \langle 0 \rangle$ is a graded semiprime submodule of M. But it is not a graded prime submodule of M. Since $P \neq \{0\}$, so there exists $0 \neq x \in P$. Hence $x(1, 0) \in P \oplus \langle 0 \rangle$, but $(1, 0) \notin P \oplus \langle 0 \rangle$ and $x \notin (P \oplus \langle 0 \rangle : M)$. □

Definition 4.3. Let G be a group with identity e. A proper R_e-submodule N of a G-graded R-module M is said to be semiprime R_e-submodule of M if whenever $r_e^2 m \in N$ for some $r_e \in R_e$ and $m \in M$, then $r_e m \in N$.

Theorem 4.4. *Let $G = \{e, g\}$ and M be a G-graded R-module. Then the following hold:*

- (i) *If N is a semiprime R_e-submodule of M, then $N \oplus M_g$ is a semiprime R_e-submodule of M.*
- (ii) *If N is a semiprime R_e-submodule of M_g, then $M_e \oplus N$ is a semiprime R_e-submodule of M.*

Proof. (i) Since M_g is a R_e-module, so $N \oplus M_g$ is an R_e-submodule of M. Let $r_e \in R_e$ and $m \in M$ be such that $r_e^2 m \in N \oplus M_g$. As $m \in M = M_e \oplus M_g$, so $m = m_e + m_g$ for some $m_e \in M_e$ and $m_g \in M_g$ and since $r_e^2 m = r_e^2(m_e + m_g) = r_e^2 m_e + r_e^2 m_g \in N \oplus M_g$, then $r_e^2 m_e \in N$. Therefore $r_e m_g \in N$ since N is a semiprime R_e-submodule of M_e. Then $r_e(m_e + m_g) \in N \oplus M_g$ and so $r_e m \in N \oplus M_g$, as needed.
(ii) The proof is similar to (i). □

A proper graded submodule Q of a graded R-module M is said to be graded primary, if whenever $r_g m_h \in Q$ for some $r_g \in h(R)$ and

$m_h \in h(M)$, then $m_h \in Q$ or there exists a positive integer k such that $r_g^k \in (Q : M)$. We know that if Q is a graded primary submodule of a graded R-module M, then $P = Grad(Q : M)$, the graded prime radical of the graded ideal $(Q : M)$, is a graded prime ideal of R (see [14, Lemma 1.8]). In this case, we say that Q is a P-graded primary submodule of M.

Given a graded submodule N of a graded R-module M has a decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ where for all i, Q_i is a graded submodule of M, is called irredundant if $N \neq Q_1 \cap \cdots \cap Q_{i-1} \cap Q_{i+1} \cap \cdots \cap Q_n$ for all $1 \leq i \leq n$. A graded submodule N of M is said to have a graded primary decomposition if N is the intersection of a finite collection of graded primary submodules of M. By [14, Corollary 2.16], every proper graded submodule of a graded Noetherian module over a commutative graded ring has a graded primary decomposition.

A graded submodule N of a graded R-module M is said to have a normal graded primary decomposition, if there exists a positive integer n, distinct graded prime ideals P_i $(1 \leq i \leq n)$ of R and P_i-graded primary submodules Q_i $(1 \leq i \leq n)$ of M such that $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is an irredundant decomposition. It is clear any graded primary decomposition of a graded submodule can be reduced to a normal one.

Lemma 4.5. *Let N be a graded semiprime submodule of a graded R-module M. Let $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a normal graded primary decomposition of N in M, where $Grad(Q_i : M) = P_i$ for all $i = 1, \ldots, n$. Set $N_i = Q_1 \cap \cdots \cap Q_{i-1} \cap Q_{i+1} \cap \cdots \cap Q_n$ $(1 \leq i \leq n)$. Then the following hold:*

(i) *$P_iN_i \subseteq N$ for every $i = 1, \ldots, n$;*
(ii) *$(N_i :_M P_i) = N$ if P_i is a maximal member of the set $\{P_1, \ldots, P_n\}$;*
(iii) *N is a graded P_i-prime submodule of N_i for all $i = 1, \ldots, n$;*
(iv) *If P_i is a minimal member of the set $\{P_1, \ldots, P_n\}$, then Q_i is a P_i-prime submodule of M.*

Proof. (i) Let $r = \sum_{g\in G} r_g \in P_i$ and $m = \sum_{h\in G} m_h \in N_i \setminus Q_i$. There exists $n \in \mathbb{N}$ such that $r_g^n M \subseteq Q_i$ for all $g \in G$. Hence for all $g \in G$ and $h \in G$, $r_g^n m_h \in Q_i$, then $r_g^n m_h \in Q_i \cap N_i = N$, so for all $g \in G$ and $h \in G$, $r_g m_h \in N$ because N is graded semiprime. Therefore $rm \in N$, as required.

(ii) Let P_i be maximal among $P_1, P_2, \ldots, P_n$. By part (i),

$$N_i \subseteq (N :_M P_i).$$

Now we show $(N :_M P_i) \subseteq N_i$. Let $m = \sum_{h\in G} m_h \in M$ with $P_i m \subseteq N$, so $P_i m_h \subseteq N$ for any $h \in G$. By [5, Theorem 2], and since P_i is maximal among $P_1, P_2, \ldots, P_n$, we get

$$P_i \nsubseteq P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n.$$

Hence there exists $r_g \in P_i \cap h(R)$ such that

$$r_g \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_n.$$

Then for any $h \in G$, $r_g m_h \in N = N_i \cap Q_i$. Therefore since Q_i is a P_i-graded primary $(1 \leq i \leq n)$, we have for any $h \in G$, $m_h \in Q_i$ and so for any $h \in G$, $m_h \in N_i$. Thus $m = \sum_{h \in G} m_h \in N_i$.

(iii) By normality of the graded primary decomposition, $N \neq N_i$, and by part (i), $P_i \subseteq (N :_M N_i)$. Let $r_g \in h(R) - P_i$ and $m_h \in h(N_i)$ such that $r_g m_h \in N$. Then $r_g m_h \in Q_i$ and so $m_h \in Q_i$. This gives that $m_h \in N_i \cap Q_i = N$.

(iv) Let P_i be a minimal element of the set $\{P_1, \ldots, P_n\}$. It is enough to show that $(Q_i :_R M) = P_i$. Let $r = \sum_{g \in G} r_g$. Then there exists $t \in \mathbb{N}$ such that $r_g^t M \subseteq Q_i$. Choose

$$s_{g'} \in (P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n) \bigcap (h(R) - P_i).$$

Thus $s_{g'}^k M \subseteq N_i$ for some $k \in \mathbb{N}$. Therefore

$$s_{g'}^k r_g^t M \subseteq Q_i \cap N_i = N$$

and since N is graded semiprime, $s_{g'} r_g M \subseteq N$. Now, we have $s_{g'} r_g M \subseteq Q_i \cap N_i = N$ and $s_{g'} \in h(R) - P_i$. It follows that $r_g M \subseteq Q_i$ for all $g \in G$, so $rM \subseteq Q_i$. □

Theorem 4.6. *Let N be a graded semiprime submodule of a graded Noetherian R-module M, with the notation of the previous lemma, the following statements are equivalent:*

- (i) *N is a graded radical submodule of M;*
- (ii) *N can be lifted to M whenever it is contained as a P_i-graded prime submodule $(1 \leq i \leq n)$ in a graded submodule of M;*
- (iii) *For every $i = 1, \ldots, n$, there exists a P_i-graded prime submodule K_i of M such that $N = N_i \cap K_i$.*

Proof. $(i) \Rightarrow (ii)$ follows from Lemma 2.15 and $(ii) \Rightarrow (iii)$ holds by Lemma 4.5(iii).

$(iii) \Rightarrow (i)$ We use induction on n. Let $n = 1$. Then by part (iv) of Lemma 4.5, N is a graded prime submodule and so a graded radical submodule of M. Now, let $n > 1$ and assume that our claim is true for $n-1$. Without loss of generality, assume that P_n is a maximal member of the set $\{P_1, \ldots, P_n\}$. If $n = 2$, then by part (iii) of assumption and the part (iv) of Lemma 4.5, we have N is an intersection of two graded prime submodules of M. Hence we can take $n > 2$. By part (ii) of Lemma 4.5, $(N :_M P_n) = N_n = Q_1 \cap \cdots \cap Q_{n-1}$. It is easy to see that

N_n is a graded semiprime submodule of M. We show that N_n satisfies the condition of (iii), that is, for every $j = 1, 2, \ldots, n-1$, there exists a P_i-prime submodule K_j of M such that

$$N_n = Q_1 \cap \cdots \cap Q_{j-1} \cap Q_{j+1} \cap \cdots \cap Q_{n-1} \cap K_j.$$

It is enough to take $j = n-1$. By assumption (iii), there exists a P_{n-1}-graded prime submodule K of M such that $N = N_{n-1} \cap K$. Let $m = \sum_{g \in G} m_g \in N_n$, so $m_g \in N_n$ for any $g \in G$. Then $P_n m_g \subseteq N$. By maximality of P_n, $P_n \nsubseteq P_1 \cap \cdots \cap P_{n-1}$, and so $m_g \in Q_1 \cap \cdots \cap Q_{n-2} \cap K$. Hence $m \in Q_1 \cap \cdots \cap Q_{n-2} \cap K$. This shows that

$$N_n \subseteq Q_1 \cap \cdots \cap Q_{n-2} \cap K.$$

Conversely, let $m = \sum_{g \in G} m_g \in Q_1 \cap \cdots \cap Q_{n-2} \cap K$. Hence for all $g \in G$, $m_g \in Q_1 \cap \cdots \cap Q_{n-2} \cap K$. Since M is graded Noetherian, $P_n^k M \subseteq Q_n$ for some positive integer k. Thus $P_n^k m_g \subseteq N_{n-1} \cap K = N$. As N is graded semiprime, we have $P_n m_g \subseteq N$. Therefore, $N_n = Q_1 \cap \cdots \cap Q_{n-2} \cap K$. By induction hypothesis, N_n is a graded radical submodule of M. On the other hand, by (iii), there exists a P_n-graded prime submodule K' of M such that $N = N_n \cap K'$. Hence N is a graded radical submodule of M. □

5. Conclusions

The concepts of graded prime R-modules over non-commutative graded rings, graded semiprime and graded radical submodules of graded R-modules over non-commutative graded rings have been studied and some results where established. In fact, some of the results concerning of prime and semiprime submodules are not hold for graded prime and graded semiprime submodules. The notion of graded semiprime submodules was proposed and basic properties of them based on their formations were introduced. We also explored some equivalent conditions for a graded module to have zero graded radical submodule. In future works, we will focus our research on other generalizations of graded prime submodules over non-commutative graded rings.

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GRADED SEMIPRIME SUBMODULES OVER NON-COMMUTATIVE GRADED RINGS

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زیرمدول‌های نیم‌اول مدرج روی حلقه‌های مدرج ناجابجایی

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فرض کنید G یک گروه با عضو همانی e، R یک حلقه مدرج شرکت‌پذیر و M یک $R-$مدول $G-$مدرج باشد. در این مقاله، مفهوم زیرمدول‌های نیم‌اول مدرج روی حلقه‌های مدرج ناجابجایی را معرفی می‌کنیم. ابتدا، مدول‌های اول مدرج روی حلقه‌های مدرج ناجابجایی را بیان و برخی خاصیت‌ها از این نوع مدول‌های مدرج را مورد بررسی قرار می‌دهیم. در ادامه، زیرمدول‌های نیم‌اول مدرج و زیرمدول‌های رادیکال مدرج از مدول‌های مدرج را مورد مطالعه قرار می‌دهیم. به‌عنوان مثال، برخی شرایط معادل برای اینکه یک مدول مدرج دارای زیرمدول رادیکال مدرج صفر باشد را به‌دست می‌آوریم.

کلمات کلیدی: مدول اول مدرج، زیرمدول رادیکال مدرج، زیرمدول نیم‌اول مدرج.