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DIVISOR TOPOLOGIES AND THEIR ENUMERATION

F. ESMAEELI, K. MIRZAVAZIRI AND M. MIRZAVAZIRI*

Abstract. For a positive integer m, a subset of divisors of m is called a *divisor topology on* m if it contains 1 and m and it is closed under taking gcd and lcm. If $m = p_1 \dots p_n$ is a square free positive integer, then a divisor topology m corresponds to a topology on the set $[n] = \{1, 2, \dots, n\}$. Giving some facts about divisor topologies, we give a recursive formula for the number of divisor topologies on a positive integer.

1. Introduction

A *topology* τ on a set X is a collection of subsets of X, possessing $\emptyset$ and X, which is closed under arbitrary union and finite intersection. When X is finite, it should be closed under union and intersection. Among many problems concerning topologies on finite sets, the problem of enumerating the number of topologies is one of the oldest and hardest ones. The largest case, which is recorded for this problem, is the number of topologies on a finite set with 18 elements in 2007; see [17]. Various enumerating problems about specific topologies on finite sets are considered by mathematicians and computer scientists; see [2–12] and [14, 15].

Taking idea from the definition of a topology motivates us to consider a number theory version of this concept. By a *divisor topology*, we mean a finite set of positive divisors of a positive integer m, possessing 1 and m, which is closed under taking gcd and lcm. The original key of

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*Corresponding author.

this motivation is the fact that the gcd (resp. lcm) of two square free positive integers d and d' is the product of the elements of intersection (resp. union) of their prime divisors.

Definition of a divisor topology leads us to the notion of a *semi-divisor topology*. In the present paper, giving a two sided relation between the number of divisor and semi-divisor topologies, we establish a recursive formula to evaluate the number of divisor topologies on a positive integer m in terms of its divisors.

2. A Relation Between $|\Delta(m)|$ and $|\Delta'(m)|$

Let n be a positive integer. Recall that a *topology on the finite set* $[n] = \{1, 2, \dots, n\}$ is a family τ of subsets of $[n]$ such that $\emptyset, [n] \in \tau$ and τ is closed under taking union and intersection. The set of all topologies on $[n]$ is denoted by $T([n])$. To avoid ambiguity, we use the terminology *ordinary topology* for this ordinary concept of a topology on a set. We denote the number of ordinary topologies on a finite set with n elements by T_n. For a survey on finite topologies, the reader is referred to [13].

Furthermore, we recall that if p is a prime number, then $p^i \| n$ means $p^i | n$ but $p^{i+1} \not| n$. Moreover, we denote the number of positive divisors of n by $\mathbf{d}(n)$. We know that $\mathbf{d}(\prod_{i=1}^k p_i^{\alpha_i}) = \prod_{i=1}^k (\alpha_i + 1)$. The *Dirichlet inverse* $\mathbf{d}^{-1}$ of $\mathbf{d}$ under the *Dirichlet convolution* $*$ is then defined by

$$\mathbf{d}^{-1}(n) = \begin{cases} 1, & n = 1, \\ 0, & p^2 | n \text{ for some prime } p, \\ (-1)^r, & \text{otherwise, where } r \text{ is the number of distinct} \\ & \text{primes } p \text{ with } p \| n, \end{cases}$$

which has the property $\mathbf{d} * \mathbf{d}^{-1} = \mathbf{1}$, where $\mathbf{1}$ is the *unit function*. To study more about these notions, see [1].

Definition 2.1. Let m be a positive integer and let $D_m = \{d : d | m\}$. A subset δ of D_m is called a *divisor topology on* m if $1, m \in \delta$ and it is closed under taking gcd and lcm. We denote the set of all divisor topologies on m by $\Delta(m)$.

Note that each divisor topology is a sublattice of the bounded lattice D_m of all divisors of a natural number m under the division operation containing 1 and m.

The following simple result shows that $T_n = |\Delta(p_1 \dots p_n)|$.

Proposition 2.2. *Let $m = p_1 \dots p_n$ be a square free positive integer. Then there is a one-to-one correspondence between $\Delta(m)$ and $T([n])$.*

Proof. We can simply consider the mapping $\varphi : \Delta(m) \to T([n])$ defined by $\varphi(\delta) = \tau_\delta = \{A_d : d \in \delta\}$, where $A_d = \{i : p_i \text{ is a prime divisor of } d\}$. For each $\delta \in \Delta(m)$, it follows that $1, m \in \delta$. Therefore $A_1 = \varnothing$ and $A_m = [n]$ belong to τ_δ. Furthermore, since $A_d \cap A_{d'} = A_{\gcd(d,d')}$, $A_d \cup A_{d'} = A_{\mathrm{lcm}(d,d')}$, and τ_δ is closed under taking union and intersection, so $\varphi(\delta)$ is well defined. The inverse mapping $\varphi^{-1} : T([n]) \to \Delta(m)$ is defined by $\varphi^{-1}(\tau) = \delta_\tau = \{d_A : A \in \tau\}$, where $d_A = \prod_{i \in A} p_i$. Note that it follows from $\gcd(d_A, d_{A'}) = d_{A\cap A'}$ that $\mathrm{lcm}(d_A, d_{A'}) = d_{A\cup A'}$. Since m is square free, for different divisors d and d', we have $A_d \neq A_{d'}$. Therefore φ and φ^{-1} are one-to-one and onto.

□

Remark 2.3. Note that $\Delta(m)$ does not coincide with $T([n])$ for some n. For example, we know that there is no n with $|T([n])| = 12$, but for two prime numbers p and q, there are 12 divisor topologies on m, where $m = p^2q$. The 12 divisor topologies on m read as follows:

$$\begin{aligned}
&\delta_1 = \{1, p^2q\},\\
&\delta_2 = \{1, p, p^2q\}, \delta_3 = \{1, p^2, p^2q\}, \delta_4 = \{1, q, p^2q\}, \delta_5 = \{1, pq, p^2q\},\\
&\delta_6 = \{1, p, p^2, p^2q\}, \delta_7 = \{1, p, pq, p^2q\}, \delta_8 = \{1, p^2, q, p^2q\},\\
&\delta_9 = \{1, q, pq, p^2q\},\\
&\delta_{10} = \{1, p, q, pq, p^2q\}, \delta_{11} = \{1, p, p^2, q, p^2q\},\\
&\delta_{12} = \{1, p, p^2, q, pq, p^2q\}.
\end{aligned}$$

Definition 2.4. Let m be a positive integer. A subset ε of D_m is called a *semi-divisor topology on m* if it is closed under taking gcd and lcm. We denote the set of all semi-divisor topologies on m by $\Delta'(m)$.

Example 2.5. As an example, here is a list of all divisor topologies and semi-divisor topologies of $m = 2$:

$$\Delta(2) = \{\emptyset, \{1, 2\}\},$$

$$\Delta'(2) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

The following lemma gives the structure of a semi-divisor topology.

Lemma 2.6. *Let m be a positive integer and let ε be a nonempty semi-divisor topology on m. Then $\varepsilon \in \Delta'(m)$ if and only if $\frac{1}{d_0}\varepsilon \in \Delta(\frac{m_0}{d_0})$, where $d_0 = \min \varepsilon$ and $m_0 = \max \varepsilon$.*

Proof. Let $d_0 = \min \varepsilon$, let $m_0 = \max \varepsilon$, and let d be an arbitrary element of ε. Then $\gcd(d, d_0) \in \varepsilon$ implies that $d_0 \leqslant \gcd(d, d_0)$. However, $\gcd(d, d_0) \leqslant d_0$. Thus $d_0 = \gcd(d, d_0)$. This shows that $d_0 | d$. Therefore, $\frac{1}{d_0}\varepsilon$ is a subset of $D(\frac{m_0}{d_0})$, which is closed under taking gcd and lcm, $1 \in \frac{1}{d_0}\varepsilon$, and $\frac{m_0}{d_0} \in \frac{1}{d_0}\varepsilon$. As a result, $\frac{1}{d_0}\varepsilon$ is a divisor topology on $\frac{m_0}{d_0}$. The other side is obvious. $\square$

Theorem 2.7. *Let m be a positive integer. Then*
$$|\Delta'(m)| = 1 + \textstyle\sum_{d|m} \mathbf{d}(\frac{m}{d})|\Delta(d)|.$$

Proof. Using Lemma 2.6, we can write
$$\Delta'(m) = \{\emptyset\} \cup \left(\cup_{m'|m} \cup_{d'|m'} d'\Delta(\tfrac{m'}{d'})\right),$$
where the sets $d'\Delta(\frac{m'}{d'})$ are disjoint.

To see this, let $\varepsilon \in \Delta'(m)$ and let $\varepsilon \neq \{\emptyset\}$. If $d' = \min \varepsilon$ and $m' = \max \varepsilon$, then $d'|m'|m$ and $\varepsilon \in d'\Delta(\frac{m'}{d'})$. Conversely, let $\delta \in d'\Delta(\frac{m'}{d'})$. Then δ is clearly a semi-divisor topology on m.

Now we can write
$$|\Delta'(m)| = 1 + \sum_{m'|m} \sum_{d'|m'} |\Delta(\frac{m'}{d'})| = 1 + \sum_{d|m} \sum_{e|\frac{m}{d}} |\Delta(d)| = 1 + \sum_{d|m} \mathbf{d}(\frac{m}{d})|\Delta(d)|.$$
$\square$

Now we apply Theorem 2.7 for the following example.

Example 2.8. We have
$$\begin{aligned}
|\Delta'(12)| &= 1 + \mathbf{d}(12)|\Delta(1)| + \mathbf{d}(6)|\Delta(2)| + \mathbf{d}(4)|\Delta(3)| \\
&\quad + \mathbf{d}(3)|\Delta(4)| + \mathbf{d}(2)|\Delta(6)| + \mathbf{d}(1)|\Delta(12)| \\
&= 1 + 6 \times 1 + 4 \times 1 + 3 \times 1 + 2 \times 2 + 2 \times 4 + 1 \times 12 \\
&= 38.
\end{aligned}$$

Corollary 2.9. *Let $m = p_1 \dots p_n$ be a square free positive integer. Then*
$$|\Delta'(m)| = 1 + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} T_i.$$

Proof. Using Theorem 2.7, we have
$$\begin{aligned}
|\Delta'(m)| &= 1 + \sum_{d|m} \mathbf{d}(\frac{m}{d})|\Delta(d)| \\
&= 1 + \mathbf{d}(\frac{m}{1})|\Delta(1)| \\
&\quad + \sum_{i=1}^{n} \sum_{\{j_1,\dots,j_i\} \subseteq \{1,\dots,n\}} \mathbf{d}(\frac{p_1 \dots p_n}{p_{j_1} \dots p_{j_i}})|\Delta(p_{j_1} \dots p_{j_i})| \\
&= 1 + 2^n + \sum_{i=1}^{n} \binom{n}{i} 2^{n-i} T_i
\end{aligned}$$

$$= \quad 1 + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} T_i.$$

□

For example, $|\Delta'(6)| = 1 + 4T_0 + 4T_1 + T_2 = 1 + 4 + 4 + 4 = 13$.

Theorem 2.10. *Let m be a positive integer. Then*

$$\Delta(m) = \sum_{d|m} \mathbf{d}^{-1}(\tfrac{m}{d})(|\Delta'(d)| - 1).$$

Proof. Define two arithmetic functions f and g by

$$f(m) = |\Delta'(m)| - 1$$

and $g(m) = |\Delta(m)|$, respectively. Then Theorem 2.7 says that $f = g * \mathbf{d}$. Thus $g = f * \mathbf{d}^{-1}$. □

Example 2.11. We have

$$\begin{aligned} |\Delta(12)| &= \mathbf{d}^{-1}(12)(|\Delta'(1)| - 1) + \mathbf{d}^{-1}(6)(|\Delta'(2)| - 1) \\ &+ \mathbf{d}^{-1}(4)(|\Delta'(3)| - 1) + \mathbf{d}^{-1}(3)(|\Delta'(4)| - 1) \\ &+ \mathbf{d}^{-1}(2)(|\Delta'(6)| - 1) + \mathbf{d}^{-1}(1)(|\Delta'(12)| - 1) \\ &= (-2) \times 1 + 4 \times 3 + 1 \times 3 + (-2) \times 7 \\ &+ (-2) \times 12 + 1 \times 37 \\ &= 12. \end{aligned}$$

3. Construction of $\Delta(p^\alpha k)$ via $\Delta'(k)$

Lemma 3.1. *Let k and α be two positive integers, let p be a prime, and let $\gcd(p, k) = 1$. Then δ is a divisor topology on $m = p^\alpha k$ if and only if there are $\varepsilon_0, \ldots, \varepsilon_\alpha$ such that*

i. *Each ε_i is a semi-divisor topology on k;*
ii. *$\delta = \cup_{i=0}^{\alpha} p^i \varepsilon_i$;*
iii. *$1 \in \varepsilon_0$;*
iv. *$k \in \varepsilon_\alpha$;*
v. *If $i < i'$ and $d \in \varepsilon_i, d' \in \varepsilon_{i'}$, then $\gcd(d, d') \in \varepsilon_i$ and $\mathrm{lcm}(d, d') \in \varepsilon_{i'}$.*

Moreover, δ is a semi-divisor topology on m if and only if $(i), (ii)$, and (v) hold.

Proof. The proof is easily obtained from $\varepsilon_i = \{\frac{d}{p^i} : p^i \| d \text{ and } d \in \delta\}$. □

Definition 3.2. Let k be a positive integer, let p be a prime, and let $\gcd(p, k) = 1$. Suppose that ε and ε' are two semi-divisor topologies on k. Then we say that ε *precedes or equals to* ε' and write $\varepsilon \preceq_p \varepsilon'$ if $\varepsilon \cup p\varepsilon' \in \Delta'(pk)$.

Lemma 3.3. *Let k be a positive integer, let p and q be two primes, and let* $\gcd(p,k)=\gcd(q,k)=1$*. Then*

i. $\varepsilon \preceq_p \varepsilon'$ *if and only if* $\varepsilon \preceq_q \varepsilon'$ *for each* $\varepsilon, \varepsilon' \in \Delta'(k)$*;*
ii. $\varepsilon \preceq_p \varepsilon$ *for each* $\varepsilon \in \Delta'(k)$*;*
iii. $\varepsilon \preceq_p \varepsilon'$ *and* $\varepsilon' \preceq_p \varepsilon$ *imply* $\varepsilon = \varepsilon'$ *for each* $\varepsilon, \varepsilon' \in \Delta'(k)$ *with* $\varepsilon \neq \{\emptyset\}$ *and* $\varepsilon' \neq \{\emptyset\}$*.*

Proof. i. This is clear by Lemma 3.1. Note that the parts (i), (ii), and (v) of Lemma 3.1 are independent of the choice of p.

ii. This is again obvious by Lemma 3.1.

iii. Let $d_0 = \min \varepsilon$ and let d' be an arbitrary element of ε'. Then $\gcd(d_0, d') \in \varepsilon$ implies that $d_0 \leqslant \gcd(d_0, d')$. However, $\gcd(d_0, d') \leqslant d_0$. Thus $d_0 = \gcd(d_0, d')$. Hence $d_0 | d'$. Now $\varepsilon' \preceq_p \varepsilon$ implies that $d' = \operatorname{lcm}(d_0, d') \in \varepsilon$. Thus $\varepsilon' \subseteq \varepsilon$. A similar argument shows $\varepsilon \subseteq \varepsilon'$. □

Regarding (i) of Lemma 3.3, we can use $\preceq$ instead of $\preceq_p$ without any ambiguity.

Definition 3.4. Let k be a positive integer, let p be a prime, and let $\gcd(p,k)=1$. Using $\Delta'(k)$ as an index set, we can define a matrix $A_{p,k} = [a_{\delta\delta'}]_{r\times r}$, where $r = |\Delta'(k)|$ and

$$a_{\delta\delta'} = \begin{cases} 1, & \delta \preceq \delta', \\ 0 & \textit{othewise}. \end{cases}$$

For $1 \leqslant j \leqslant r$, the *$j$-deciding map* is the mapping

$$F_j : \Delta'(k)^{j+1} \to \{0,1\}$$

defined by $F_j(\delta_0, \delta_1, \ldots, \delta_j) = \prod_{0\leqslant s<t\leqslant j} a_{\delta_s\delta_t}$. The $(j+1)$-tuple $(\delta_0, \delta_1, \ldots, \delta_j)$ is called a *path in $\Delta'(k)$ of length j* if

$$F_j(\delta_0, \delta_1, \ldots, \delta_j) = 1.$$

A path $(\delta_0, \delta_1, \ldots, \delta_j)$ is called a *topological path* if $1 \in \delta_0$ and $k \in \delta_j$.

The following result in fact summarizes our considerations in Lemma 3.1 and the fact that (v) of that lemma is equivalent to $\varepsilon_i \preceq \varepsilon_{i'}$ for any $i < i'$, so we omit its proof.

Theorem 3.5. *Let k and α be two positive integers, let p be a prime, and let* $\gcd(p,k)=1$*. Then*

i. $\Delta'(p^\alpha k) = \{\cup_{i=0}^{\alpha} p^i \varepsilon_i : (\varepsilon_0, \ldots, \varepsilon_\alpha)$ *is a path in* $\Delta'(k)\}$*;*
ii. $\Delta(p^\alpha k) = \{\cup_{i=0}^{\alpha} p^i \varepsilon_i : (\varepsilon_0, \ldots, \varepsilon_\alpha)$ *is a topological path in* $\Delta'(k)\}$;
iii. $|\Delta'(p^\alpha k)|$ *is the number of paths of length α in* $\Delta'(k)$*;*
iv. $|\Delta(p^\alpha k)|$ *is the number of topological paths of length α in* $\Delta'(k)$*.*

Corollary 3.6. *Let α be a positive integer and let p and q be two primes. Then*

$$|\Delta'(p^\alpha)| = 2^{\alpha+1}, |\Delta(p^\alpha)| = 2^{\alpha-1};$$
$$|\Delta'(p^\alpha q)| = (\alpha^2+9\alpha+16)2^{\alpha-2}, |\Delta(p^\alpha q)| = (\alpha^2+13\alpha+18)2^{\alpha-4}.$$

Proof. The first part is obvious. To evaluate $|\Delta(p^\alpha q)|$, we use Theorem 3.5 and the facts that

$$\begin{aligned}\Delta'(q) &= \{\varepsilon_0=\emptyset, \varepsilon_1=\{1\}, \varepsilon_2=\{q\}, \varepsilon_3=\{1,q\}\},\\ \Delta'(pq) &= \{\varepsilon_i \cup p\varepsilon_j : (i,j)\notin\{(2,1),(2,3),(4,2)\}\}.\end{aligned}$$

This helps us to write

$$A_{p,q} = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&0&1&0\\1&0&1&1\end{pmatrix}.$$

A topological path should begins with ε_1 or ε_3 and ends with ε_2 or ε_3. Moreover, if for a path $P=(\varepsilon_0,\varepsilon_1,\dots,\varepsilon_\alpha)$ of length α the notation $P=(i_0i'_0, i_1i'_1,\dots,i_\alpha i'_\alpha)$ means that ε_j can be ε_{i_j} or $\varepsilon_{i'_j}$, then we can say that a topological path of length α should be one of the following forms:

$$\begin{aligned}&(1,01,01,\dots,01,23),\\&(1,01,01,\dots,01,2,02,02,\dots,02,2),\\&(1,01,01,\dots,01,3,03,03,\dots,03,23),\\&(1,01,01,\dots,01,3,03,03,\dots,03,2,02,02,\dots,02,2),\\&(3,03,03,\dots,03,23),\\&(3,03,03,\dots,03,2,02,02,\dots,02,2).\end{aligned}$$

For example, the first form means that $\varepsilon_0=1$, $\varepsilon_1,\dots,\varepsilon_{\alpha-1}$ can be ε_0 or ε_1, and ε_α can be ε_2 or ε_3.

The number of the above forms is $2^\alpha, (\alpha-1)2^{\alpha-2}, (\alpha-1)2^{\alpha-1}$, $\binom{\alpha-1}{2}2^{\alpha-3}, 2^\alpha, (\alpha-1)2^{\alpha-2}$, respectively. Summing these, we have

$$|\Delta(p^\alpha q)| = (\alpha^2+13\alpha+18)2^{\alpha-4}.$$

Now, using Theorem 2.7, we evaluate $|\Delta'(p^\alpha q)|$. We have

$$\begin{aligned}|\Delta'(p^\alpha q)| &= 1+\sum_{d\mid p^\alpha q}\mathbf{d}(\frac{p^\alpha q}{d})|\Delta(d)|\\ &= 1+\sum_{i=0}^{\alpha}\mathbf{d}(\frac{p^\alpha q}{p^i})|\Delta(p^i)| + \sum_{i=0}^{\alpha}\mathbf{d}(\frac{p^\alpha q}{p^i q})|\Delta(p^i q)|\end{aligned}$$

$$\begin{aligned} &= 1+\sum_{i=0}^{\alpha}2(\alpha-i+1)2^{i-1} \\ &+ \sum_{i=0}^{\alpha}(\alpha-i+1)(i^2+13i+18)2^{i-4}. \end{aligned}$$

A bothersome simplification gives the result. □

Although the number of topologies on a finite set is still an open problem, the enumeration of divisor topologies and semi-divisor topologies creates a wide area of research that can be extended in some different topics. As a collection of some enumerative problems concerning divisor topologies, one can consider divisor topologies with some constraints, the relation between divisor topologies and ordinary topologies on a finite set, and some partition problems dealing with divisor topologies.

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Fahimeh Esmaeeli
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad, Iran.
Email: fahimeh.smaily@gmail.com

Kamyar Mirzavaziri
Department of Computer Science, School of Mathematics, Statistics and Computer Science, University of Tehran, P.O. Box 141556619, Tehran, Iran.
Email: kmirzavaziri@gmail.com

Madjid Mirzavaziri
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad, Iran.
Email:mirzavaziri@um.ac.ir

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توپولوژی‌های مقسوم علیهی و شمارش آن‌ها

فهیمه اسماعیلی[۱]، کامیار میرزاوزیری[۲]، مجید میرزاوزیری[۳]

[۱,۳]دانشکده علوم ریاضی، دانشگاه فردوسی مشهد، مشهد، ایران

[۲]دانشکده آمار و علوم کامپیوتر، دانشگاه تهران، تهران، ایران

برای عدد صحیح و مثبت m، یک مجموعه از مقسوم علیه‌های عدد m یک توپولوژی مقسوم علیهی نامیده می‌شود، هرگاه این زیرمجموعه شامل ۱ و m باشد و به علاوه نسبت به (ب.م.م) و (ک.م.م) بسته باشد. در حالتی که $m = p_1 \dots p_n$ یک عدد صحیح مثبت خالی از مربع باشد، هر توپولوژی مقسوم علیهی برای m متناظر است با یک توپولوژی روی مجموعه‌ی $[n] = \{1, 2, \dots, n\}$. در مقاله حاضر، ضمن ارائه‌ی نتایجی برای توپولوژی‌های مقسوم علیهی، یک رابطه بازگشتی برای شمارش تعداد توپولوژی‌های مقسوم علیهی برای هر عدد صحیح مثبت ارائه می‌شود.

کلمات کلیدی: توپولوژی، توپولوژی مقسوم علیهی، شبه توپولوژی مقسوم علیهی.