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THE IDENTIFYING CODE NUMBER AND FUNCTIGRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a simple graph. A set D of vertices G is an identifying code of G, if for every two vertices x and y the sets $N_G[x] \cap D$ and $N_G[y] \cap D$ are non-empty and different. The minimum cardinality of an identifying code in graph G is the identifying code number of G and it is denoted by $\gamma^{ID}(G)$. Two vertices x and y are twin, when $N_G[x] = N_G[y]$. Graphs with at least two twin vertices are not identifiable graphs. In this paper, we deal with identifying code number of functigraph of G. Two upper bounds on identifying code number of functigraph are given. Also, we present some graph G with identifying code number $|V(G)| - 2$.

1. Introduction

All graphs throughout this paper considered simple, finite and undirected. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. If two vertices x and y are *adjacent*, then it denoted by $x \sim y$, otherwise, $x \nsim y$. The *closed neighborhood* of a vertex v in graph G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =| N_G(v) |$. We denote the *maximum* degree of G with $\Delta(G)$ and its *minimum degree* with $\delta(G)$. A vertex is called *universal* if it is adjacent to all of the vertices of graph.

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The *complement* of graph G is denoted by $\overline{G}$ and defineded as a graph with vertex set $V(G)$ which $e \in E(\overline{G})$ if and only if $e \notin E(G)$. For any $S \subseteq V(G)$, the *induced subgraph* on S, denoted by $G[S]$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we define the union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we define their join $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup K)$, where

$$K = \{u \sim v \mid u \in V_1,\ v \in V_2\}.$$

Let G be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$, $G^{'}$ be a copy of G with $V(G^{'}) = \{v_1^{'}, v_2^{'}, ..., v_n^{'}\}$ and $E(G^{'}) = \{v_i^{'} \sim v_j^{'} \mid v_i \sim v_j\}$, where $v_i^{'} \in V(G^{'})$ is corresponding to $v_i \in V(G)$. Then a functigraph G with function $\sigma : V(G) \to V(G^{'})$, ($\sigma$ is not necesserily bijective) is denoted by $C(G, \sigma)$, its vertices and edges are

$$V(C(G, \sigma)) = V(G) \cup V(G^{'})$$

and

$$\begin{aligned} E(C(G, \sigma)) &= E(G) \cup E(G^{'}) \cup \\ &\quad \{v_i \sim v_j^{'} \mid v_i \in V(G), v_j^{'} \in V(G^{'}), \sigma(v_i) = v_j^{'}\}, \end{aligned}$$

respectively. For $v_i^{'} \in V(G^{'})$,

$$R_{v_i^{'}} = \sigma^{-1}(\{v_i^{'}\}) = \{v_j \in V(G) \mid \sigma(v_j) = v_i^{'}\}$$

and for $\ell \in \{0, 1, 2, \cdots, n = |V(G)|\}$, we define

$$B_\ell = \{v_i^{'} \in V(G^{'}) \mid |R_{v_i^{'}}| = \ell\}.$$

For simplicity, the open neighborhood of x in $C(G, \sigma)$ is denoted by $N_C(x)$.

A set of vertices G such as D is a *dominating set* of graph G if for every vertex x of $V(G)$, is either in D or is adjacent to a vertex in D. It is clear that every isolated vertex is in every dominating set of G. Also a set D is called a *separating set* of G if for each pair u, v of vertices of G, $N_G[u] \cap D \neq N_G[v] \cap D$ (equivalently, $(N_G[u] \triangle N_G[v]) \cap D \neq \emptyset$). If a dominating set D in graph G is a separating set of G, then we say that D is an identifying code of graph G and if G has an identifying code, then we say that G is an *identifiable graph*. Given a graph G, the smallest size of an identifying code of G is called *identifying code number* of G and denoted by $\gamma^{ID}(G)$. A vertex x is a twin of another vertex y if $N_G[x] = N_G[y]$. A graph G is called twin free if no vertex has a twin. The first observation regarding the concept of identifying codes is that a graph is identifiable if and only if it is twin free [2].

Karpovsky et al [9] have shown that for every identifiable graph G of order n, $\gamma^{ID}(G) \geq \lceil \log_2(n+1) \rceil$. Also, they proved that

$$\gamma^{ID}(G) \geq \frac{2n}{\Delta(G)+2}.$$

For every identifiable graph G of order n with at least one edge, there exists a famous bound as $\gamma^{ID}(G) \leq n-1$ (see [3]). In 2012, Foucaud et al [4], had a conjecture that for every connected identifiable graph G, there exist a constant c such that $\gamma^{ID}(G) \geq n - \frac{n}{\Delta(G)} + c$. It is noteworthy that in 2006 Gravier et al [6] investigated the identifying code number of cycles. According to their theorems, this conjecture holds for graphs of maximum degree 2.

Nowadays, identifying codes are an actively studied topic of its own like: the location of threats in facilities using sensors [12], error-detection schemes [9] and routing [10] in networks, terrorist network monitoring [13], as well as the structural analysis of RNA proteins [7]. For more details we refer reader to [5, 8, 11].

This concept was studied in a large number of various papers, investigating particular graphs or families of graphs. This paper deals with the study of functigraph of some graphs. Section 2, the identifying code number of of some special graphs are considered. Two upper bounds are presented. We prove that if G is an identifiable graph and $\delta(G) \geq 1$, then for every function $\sigma : V(G) \to V(G')$, graph $C(G, \sigma)$ is an identifiable graph and the upper bound $\gamma^{ID}(C(G,\sigma)) \leq n$ is achieved for σ as a permutation. Also, we show that for every identifiable graph G of order n, with $\delta(G) \geq 1$, $\gamma^{ID}(C(G,\sigma)) \leq 2\gamma^{ID}(G)$, where $\sigma : V(G) \to V(G')$ is a function and this bound is sharp. Section 3, we introduce some graphs with identifying code number $|V(G)| - 2$. Section 4, we discuss identifying code number of some graphs, which are not identifiable.

2. Identifying code number of some graphs which are identifiable

In this section, the identifiability of functigraph, is investigated.

Lemma 2.1. *Let G be a graph. Then $\gamma^{ID}(G) = 2$ if and only if $G \in \{\overline{K_2}, P_3\}$.*

Proof. By $\gamma^{ID}(G) \geq \lceil log_2(n+1) \rceil$, the proof is straightforward. □

Lemma 2.2. *If $\sigma : V(P_3) \to V(P_3')$ is a permutation, then*

$$\gamma^{ID}(C(P_3, \sigma)) = 3.$$

Proof. For every permutation $\sigma : V(P_3) \to V(P_3')$, $C(P_3, \sigma)$ is isomorphic to H_i ($i \in \{1, 2, 3, 4\}$) (see Figure 1). In H_1, $D_1 = \{v_2, v_1', v_3'\}$ is an identifying code of $C(P_3, \sigma)$. In H_2, $D_2 = \{v_2, v_3, v_1^{'}\}$ is an identifying code of $C(P_3, \sigma)$. In H_3 and H_4, $D_3 = \{v_2, v_3, v_2^{'}\}$ and $D_4 = \{v_2, v_2^{'}, v_3^{'}\}$ are identifying codes of $C(P_3, \sigma)$, respectively. So $\gamma^{ID}(C(P_3, \sigma)) \leq 3$. By Lemma 2.1, $\gamma^{ID}(C(P_3, \sigma)) = 3$. □

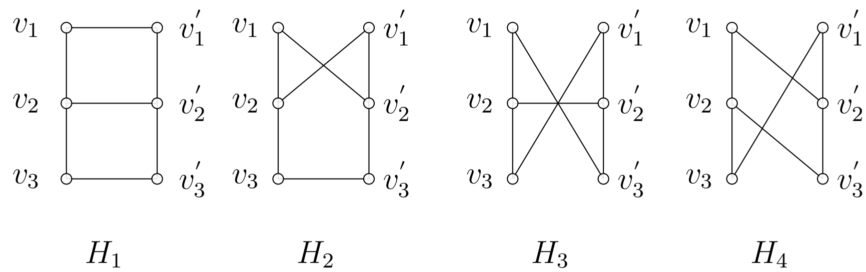

Figure 1

Lemma 2.3. *Let G be a graph and D be an identifying code of G.*

1) *If $N_G(x) = N_G(y)$, then $x \in D$ or $y \in D$.*
2) *If $N_G[x] \bigtriangleup N_G[y] = \{y_1, y_2\}$, then $y_1 \in D$ or $y_2 \in D$.*

Proof. Let $\{x, y\} \cap D = \emptyset$ or $\{y_1, y_2\} \cap D = \emptyset$. Then

$$N_G[x] \cap D = N_G[y] \cap D,$$

which is not true. □

It is clear that if $x \in V(G)$ and $\sigma(x) \in V(G^{'})$ are isolated vertices, then $C(G, \sigma)$ is not an identifiable graph.

Theorem 2.4. *Let G be an identifiable graph of order n. If $\delta(G) \geq 1$, then for every function $\sigma : V(G) \to V(G^{'})$, graph $C(G, \sigma)$ is an identifiable graph. If σ is a permutation, then $\gamma^{ID}(C(G, \sigma)) \leq n$. Furthermore, this bound is sharp.*

Proof. For each pair x, y of vertices of $C(G, \sigma)$, if $\{x, y\} \subseteq V(G)$, then since G is an identifilable graph, so $N_G[x] \neq N_G[y]$. Hence, $N_C[x] \neq N_C[y]$. Similarly, if $\{x, y\} \subseteq V(G^{'})$, then $N_C[x] \neq N_C[y]$. Now, let $x \in V(G)$ and $y \in V(G^{'})$. If $\sigma(x) \neq y$, then x is not adjacent to y in $C(G, \sigma)$. Hence, $N_C[x] \neq N_C[y]$. If $\sigma(x) = y$, then since G does not have any isolated vertex, so there exist $z \in N_C[y]$ such that $z \notin N_C[x]$. So $N_C[x] \neq N_C[y]$. Therefore, $C(G, \sigma)$ is an identifiable graph.

Now, let σ be a permutation and $D = V(G)$. For each pair x, y of vertices of $C(G,\sigma)$, if $\{x,y\} \subseteq V(G)$, then $N_C[x] \cap D = N_G[x]$ and $N_C[y] \cap D = N_G[y]$. So $N_C[x] \cap D \neq N_C[y] \cap D$.

If $\{x,y\} \subseteq V(G^{'})$, then $N_C[x] \cap D = R_x$ and $N_C[y] \cap D = R_y$. Hence, $N_C[x] \cap D \neq N_C[y] \cap D$.

Finally, if $x \in V(G)$ and $y \in V(G^{'})$, then $N_C[x] \cap D = N_G[x]$ and $N_C[y] \cap D = R_y$. Since $\delta(G) \geq 1$ and σ is a permutation, so $N_C[x] \cap D \neq N_C[y] \cap D$.

However, $N_C[x] \cap D \neq N_C[y] \cap D$. Hence, $V(G)$ is an identifying code $C(G,\sigma)$. Therefore, $\gamma^{ID}(C(G,\sigma)) \leq |V(G)| = n$. By Lemma 2.2, this bound is Sharp. □

Corollary 2.5. *Let $G \cong K_{1,n-1}$, $n \geq 3$ and $\sigma : V(G) \to V(G^{'})$ be a permutation such that $\sigma(a) = a^{'}$, where a is the universal vertex of G and $a^{'} \in V(G^{'})$ is corresponding to a. Then $\gamma^{ID}(C(G,\sigma)) = n$.*

Proof. By Theorem 2.4, $C(G,\sigma)$ is an identifiable graph and $\gamma^{ID}(C(G,\sigma)) \leq n$.

Now, let $\gamma^{ID}(C(G,\sigma)) \leq n-1$ and D be an identifying code of $C(G,\sigma)$, where $\gamma^{ID}(C(G,\sigma)) = |D|$. Since for each $2 \leq i \leq n-1$, we have $N_C[v_1] = \{a, v_1, \sigma(v_1)\}$ and $N_C[v_i] = \{a, v_i, \sigma(v_i)\}$, so

$$|\{v_1, v_i, \sigma(v_1), \sigma(v_i)\} \cap D| \geq 1.$$

Hence, there is $A \subseteq V(X) \cup V(X^{'})$, such that $|A| \geq n-2$ and $A \subseteq D$, where $X = V(G) \setminus \{a\} = \{v_1, v_2, ..., v_{n-1}\}$. Since

$$N_C[v_1] \triangle N_C[\sigma(v_1)] = \{a, a^{'}\},$$

by Lemma 2.3, (2), $a \in D$ or $a^{'} \in D$. So $|D| \geq n-1$. Thus $|D| = n-1$. There is no loss of generality in assuming that $a \in D$ and $a^{'} \notin D$. Hence, there exists some $v_i \in V(G)$, such that $\sigma(v_i)$ is not dominated by D. It is a contradiction. □

Theorem 2.6. *Let G be an identifiable graph of order n, with $\delta(G) \geq 1$ and $\sigma : V(G) \to V(G^{'})$ be a function. Then $\gamma^{ID}(C(G,\sigma)) \leq 2\gamma^{ID}(G)$. Furthermore, this bound is sharp.*

Proof. By Theorem 2.4, $C(G,\sigma)$ is an identifiable graph. Let D_1 be an identifying code of G such that $\gamma^{ID}(G) = |D_1|$ and $D_1^{'} \subseteq V(G^{'})$ be corresponding to D_1. Let $X = \{v \in D_1 \mid N_G(v) \cap D_1 = \{v\}\}$ and $X^{'} \subseteq V(G^{'})$ be the corresponding to X. Also, let

$$Y^{'} = \{v^{'} \in X^{'} \mid R_{v^{'}} \cap D_1 = \{x\} \subseteq X\}.$$

If $Y^{'} = \emptyset$, then $D = D_1 \cup D_1^{'}$ is an identifying code of $C(G,\sigma)$ and so $\gamma^{ID}(C(G,\sigma)) \leq 2\gamma^{ID}(G)$.

So suppose that $Y^{'} \neq \emptyset$ and $Y^{'} = \{v_1^{'}, \cdots, v_t^{'}\}$. Since $\delta(G) \geq 1$, for $1 \leq i \leq t$, $N_{G^{'}}(v_i^{'}) \neq \emptyset$, we set $Y_1^{'} = \{u_{i1}^{'} \in V(G^{'}) \mid u_{i1}^{'} \in N_{G^{'}}(v_i^{'})\}$. Then $D = D_1 \cup Y_1^{'} \cup D_1^{'} \setminus \sigma(Y^{'})$ is an identifying code of $C(G,\sigma)$. Thus $\gamma^{ID}(C(G,\sigma)) \leq |D| = \gamma^{ID}(G) + t + \gamma^{ID}(G) - t = 2\gamma^{ID}(G)$.
It is clear that $\gamma^{ID}(P_3) = 2$. Let $\sigma : V(P_3) \to V(P_3^{'})$ be a function, such that $\sigma(a) = \sigma(b) = \sigma(c) = b^{'}$, where $deg_{P_3}(b) = 2$. Then $\gamma^{ID}(C(P_3,\sigma)) = 4$. This show that this bound is sharp. □

Theorem 2.7. *Let G be a graph with $\delta(G) \geq 1$ such that G is not an identifiable graph and $\sigma : V(G) \to V(G^{'})$ be a function. Then $C(G,\sigma)$ is an identifiable graph if and only if two following conditions are hold.*

1) If $N_G[x] = N_G[y]$, then $\sigma(x) \neq \sigma(y)$.
2) If $N_{G^{'}}[x] = N_{G^{'}}[y]$, then $x \notin B_0$ or $y \notin B_0$.

Proof. Let conditions (1) and (2) are holding and x and y be two vertices of $C(G,\sigma)$. Let $\{x,y\} \subseteq V(G)$. If $N_G[x] = N_G[y]$, then $\sigma(x) \neq \sigma(y)$. So $\sigma(x) \in N_C[x]$ and $\sigma(x) \notin N_C[y]$. If $N_G[x] \neq N_G[y]$, then $N_C[x] \neq N_C[y]$. Suppose that $\{x,y\} \subseteq V(G^{'})$. If $N_{G^{'}}[x] \neq N_{G^{'}}[y]$, then $N_C[x] \neq N_C[y]$. If $N_{G^{'}}[x] = N_{G^{'}}[y]$ and $x \notin B_0$, then there exists $z \in V(G)$ such that $\sigma(z) = x$. So $z \in N_C[x]$ and $z \notin N_C[y]$. Now, assume that $x \in V(G)$, $y \in V(G^{'})$ and $N_C[x] = N_C[y]$. Then $\sigma(x) = y$ and y is an isolated vertex in $G^{'}$, which is contradiction with this fact that $\delta(G) \geq 1$.

Conversely, let $C(G,\sigma)$ be an identifiable graph. If $N_G[x] = N_G[y]$ and $\sigma(x) = \sigma(y)$. Then $N_C[x] = N_G[x] \cup \{\sigma(x)\}$ and

$$N_C[y] = N_G[y] \cup \{\sigma(y)\}.$$

Hence, $N_C[x] = N_C[y]$. Which is not true. If $N_{G^{'}}[x] = N_{G^{'}}[y]$ and $\{x,y\} \subseteq B_0$, then $N_C[x] = N_{G^{'}}[x]$ and $N_C[y] = N_{G^{'}}[y]$. Which is a contradiction. □

Let us mention two consequences of the Theorem 2.7.

Corollary 2.8. *Let G be a graph of order n with $\delta(G) \geq 1$. If G is not an identifiable graph, then for every permutation $\sigma : V(G) \to V(G^{'})$, $C(G,\sigma)$ is an identifiable graph.*

Proof. By Theorem 2.7, the proof is straightforward. □

Corollary 2.9. *Let $G \cong K_n$ and $n \geq 2$. Then $C(G,\sigma)$ is an identifiable graph if and only if $\sigma : V(G) \to V(G^{'})$ be a permutation.*

Proof. If σ is a permutation, then by Corollary 2.8, $C(G,\sigma)$ is an identifiable graph.

Conversely, let $C(G,\sigma)$ be an identifiable graph. On the contrary, let σ not be a permutation. Then $B_0 \neq \emptyset$. If $\{x,y\} \subseteq B_0$, then

$$N_C[x] = N_C[y] = V(K_n).$$

Which is contradiction. If $|B_0| = 1$, then $|B_2| = 1$. Let $y \in B_2$ and $\sigma(t) = \sigma(z) = y$. Then $N_C[t] = V(G) \cup \{y\} = N_C[z]$. So $C(G, \sigma)$ is not an identifiable graph. That is not true. □

3. Graphs $G = (V(G), E(G))$ with identifying code number $|V(G)| - 2$

Foucaud et al.[3], in 2011 classified all graphs with identifying code number $|V(G)| - 1$. In this section, we intruduce some graphs with identifying code number $|V(G)| - 2$.

For an integer $k \geq 1$, let $A_k = (V_k, E_k)$ be the graph with vertex set $V_k = \{x_1, ..., x_{2k}\}$ and edge set $E_k = \{x_i \sim x_j \mid |i - j| \leq k - 1\}$. Also, let $\mathscr{A}$ be the closure of $\{A_i \mid i = 1, 2, \cdots\}$ with respect to operation $\bowtie$. In the next theorem, Foucaud et al. showed that for any twin free graph $G \notin \{K_{1,n-1}\} \cup (\mathscr{A}, \bowtie) \cup (\mathscr{A}, \bowtie) \bowtie K_1$, $\gamma^{ID}(G) \leq |V(G)| - 2$.

Theorem 3.1. [3] *Let G be an identifiable graph of order n. Then $\gamma^{ID}(G) = |V(G)| - 1$ if and only if $G \ncong \overline{K_2}$ and*

$$G \in \{K_{1,n-1}\} \cup (A, \bowtie) \cup (A, \bowtie) \bowtie K_1.$$

Theorem 3.2. *Let $G \cong K_{m,n}$, $m, n \geq 2$ and $G \ncong C_4$. Then*

$$\gamma^{ID}(G) = |V(G)| - 2.$$

Proof. Let the bipartition of $K_{m,n}$ be X and Y with $|X| = n$ and $|Y| = m$. Also, let D be an identifying code of $K_{m,n}$. By Lemma 2.3, (1), we have $|X \cap D| \geq n-1$ and $|Y \cap D| \geq m-1$. So $|D| \geq m+n-2$. By Theorem 3.1, $\gamma^{ID}(G) = m + n - 2$. □

Observation 3.3. If $\sigma : V(K_2) \to V(K_2')$ is a permutation, then $\gamma^{ID}(C(K_2, \sigma)) = 3$.

Proof. It is clear that $C(K_2, \sigma) \cong C_4$. Since $\gamma^{ID}(C_4) = 3$, so $\gamma^{ID}(C(K_2, \sigma)) = 3$. □

Theorem 3.4. *Let $G \cong K_n$, $n \geq 3$ and $\sigma : V(G) \to V(G')$ be a permutation. Then $\gamma^{ID}(C(G, \sigma)) = |V(C(G, \sigma))| - 2$.*

Proof. By Corollary 2.9, $C(G, \sigma)$ is an identifiable graph. Let

$$X = V(G) \setminus \{v_1\} \cup V(G') \setminus \{\sigma(v_1)\}.$$

Then for $2 \leq i \leq n$, we have $N_C[v_i] \cap X = V(G) \setminus \{v_1\} \cup \{\sigma(v_i)\}$, $N_C[v_1] \cap X = V(G) \setminus \{v_1\}$. If $v_i' \in V(G')$ and $v_i' \neq \sigma(v_1)$, then

$$N_C[v_i'] \cap X = V(G') \setminus \{\sigma(v_1)\} \cup \sigma^{-1}(v_i')$$

and $N_C[\sigma(v_1)] \cap X = V(G') \setminus \{\sigma(v_1)\}$. So for each pair $x,\ y$ in $C(G, \sigma)$, we have $N_C[x] \cap X \neq N_C[y] \cap X$. Hence, X is an identifying code of $C(G, \sigma)$ and so $\gamma^{ID}(C(G, \sigma)) \leq |X| = 2n - 2$.

Now, let D be an identifying code of graph $C(G, \sigma)$ and $\gamma^{ID}(C(G, \sigma)) = |D|$. Since $N_C[v_1] \triangle N_C[v_2] = \{\sigma(v_1), \sigma(v_2)\}$, so by Lemma 2.3, (2), we have $\sigma(v_1) \in D$ or $\sigma(v_2) \in D$. Let $\sigma(v_1) \notin D$. Then $\sigma(v_2) \in D$. Now, let $3 \leq i \leq n$. Since $N_C[v_1] \triangle N_C[v_i] = \{\sigma(v_1), \sigma(v_i)\}$, by Lemma 2.3, (2), $\sigma(v_i) \in D$. So there is $A \subseteq V(G)$, such that $A \subseteq D$ and $|A| \geq n-1$. Similarly, There is $A' \subseteq V(G')$, such that $A' \subseteq D$ and $|A'| \geq n-1$. Hence, $|D| \geq 2n-2$. Therefore, $\gamma^{ID}(C(G, \sigma)) = 2n-2$. □

Following Ashrafi et. al [1], a link of graphs G and H by vertices $y \in V(G)$ and $z \in V(H)$ is defined as the graph $(G \sim H)(y, z)$ obtained by joining y and z by an edge in the union of these graphs.

Theorem 3.5. *Let $\mathcal{B}$ be a family of graphs of order n, with identifying code number $n-1$. Also, let $G \in \mathcal{B}$, $u \in V(G)$ and $v \in V(K_1)$, such that $(G \sim K_1)(u, v) \notin \mathcal{B}$. Then $\gamma^{ID}((G \sim K_1)(u, v)) = n - 1$.*

Proof. Since $(G \sim K_1)(u, v) \notin \mathcal{B}$, so

$$\gamma^{ID}((G \sim K_1)(u, v)) \leq |(G \sim K_1)(u, v)| - 2 = n + 1 - 2 = n - 1.$$

Let D be an identifying code of $(G \sim K_1)(u, v)$ and

$$\gamma^{ID}((G \sim K_1)(u, v)) = |D|.$$

Then $|D| \leq n-1$. If $v \notin D$, then D is an identifying code of G. Hence, $\gamma^{ID}(G) \leq |D|$. Thus $n - 1 \leq |D|$ and so $|D| = n - 1$. Now, let $v \in D$. Then there exists some $x \in V(G)$, such that $x \in N_G(u) \cap D$. Since G is an identifiable graph, so there exists $z \in V(G)$, such that $z \sim x$ and $z \not\sim u$ or $z \sim u$ and $z \not\sim x$. It is easy to see that $D \setminus \{v\} \cup \{z\} = D_1$ is an identifying code of G. So $|D_1| \geq n - 1$. Hence, $|D| \geq n - 1$ and so $|D| = n - 1$. Therefore,

$$\gamma^{ID}((G \sim K_1)(u, v)) = |V(G \sim K_1)(u, v))| - 2.$$

□

Theorem 3.6. *Let $G \cong (K_{1,r} \sim K_{1,s})(a, b)$, where a and b be the universal vertices of $K_{1,r}$ and $K_{1,s}$, respectively. Then $\gamma^{ID}(G) = |V(G)|-2$.*

Proof. Let $V(K_{1,r}) = \{a, v_1, v_2, \cdots, v_r\}$ and

$$V(K_{1,s}) = \{b, u_1, u_2, \cdots, u_s\},$$

such that a and b be the universal vertices of $K_{1,r}$ and $K_{1,s}$, respectively. Then $D_1 = V(K_{1,r}) \setminus \{a\} \cup V(K_{1,s}) \setminus \{b\}$ is an identifying code of G. So $\gamma^{ID}(G) \leq |D_1| = s + r$.

Now, let D be an identifying code of G, where $\gamma^{ID}(G) = |D|$. For each $1 \leq i \leq r$, we have $N_G[v_1] \triangle N_G[v_i] = \{v_1, v_i\}$, by Lemma 2.3, (2), $v_1 \in D$ or $v_i \in D$. Hence, there is $A \subseteq V(K_{1,r}) \setminus \{a\}$, such that $|A \cap D| \geq r - 1$. Similarly, there is $F \subseteq V(K_{1,s}) \setminus \{b\}$, such that $|F \cap D| \geq s - 1$. So $|D| \geq r + s - 2$. Since D is a dominating set of G, so $|A| = r$ or $|A| = r - 1$ and $a \in D$. Similarly, $|F| = s$ or $|F| = s - 1$ and $b \in D$. However, $|D| \geq s + r$. Therefore, $\gamma^{ID}(G) = s + r$. □

4. Identifying code number of $C(G, \sigma)$, where G is not an identifiable graph

In this section, we consider the identifying code number of $C(G, \sigma)$, where $\sigma : V(G) \to V(G')$ is a function and G is not an identifiable graph.

Theorem 4.1. *Let H be an empty graph of order s and $G \cong H \bowtie K_r$, where $(s, r) \notin \{(0, 2), (1, 1)\}$. Also, let $\sigma : V(G) \to V(G')$ be a permutation, such that $\sigma(V(H)) = V(H')$. Then*

$$\gamma^{ID}(C((H \bowtie K_r), \sigma)) = \begin{cases} 2r - 2, & s = 0 \\ 2r, & s = 1 \\ s + 1, & r = 1 \\ 2r + s - 3, & o.w. \end{cases}$$

Proof. By Corollary 2.8, $C(G, \sigma)$ is an identifiable graph. If $s \in \{0, 1\}$, then by Theorem 3.4, the proof is straightforward. If $r = 1$, then by Theorem 2.5, $\gamma^{ID}(C(G, \sigma)) = s + 1$.

Let $r, s \geq 2$, $V(H) = \{v_1, v_2, \cdots, v_s\}$ and $V(K_r) = \{u_1, u_2, \cdots, u_r\}$. Then $D_1 = V(K_r) \setminus \{u_1\} \cup V(K_r') \setminus \{\sigma(u_1)\} \cup \{v_1, v_2, \cdots, v_{s-1}\}$ is an identifying code of $C(G, \sigma)$. So $\gamma^{ID}(C(G, \sigma)) \leq 2r + s - 3$.

Now, let D be an identifying code of $C(G, \sigma)$ and

$$\gamma^{ID}(C(G, \sigma)) = |D|.$$

For every $i, j \in \{1, \cdots, r\}$, we have $N_C[u_i] \triangle N_C[u_j] = \{\sigma(u_i), \sigma(u_j)\}$. By Lemma 2.3, (2), $\sigma(u_i) \in D$ or $\sigma(u_j) \in D$. So there is $A' \subseteq V(K_r')$, such that $|A'| \geq r - 1$ and $A' \subseteq D$. Similarly, there is $A \subseteq V(K_r)$, such that $|A| \geq r - 1$ and $A \subseteq D$. Hence, $|D| \geq 2r - 2$.

Now, let $|D| \leq 2r + s - 4$ and $F \subseteq (V(H) \cup V(H')) \cap D$. Then $|F| \leq s - 2$. Let $|F \cap V(H)| = \ell \leq s - 2$ and $\{x, y\} \subseteq V(H) \setminus F$. Since $N_C[x] \triangle N_C[y] = \{\sigma(x), \sigma(y)\}$, by Lemma 2.3, (2), $\sigma(x) \in D$ or $\sigma(y) \in D$. Thus there is $X \subseteq V(H')$, such that $|X| \geq (s - \ell) - 1$ and $X \subseteq D$. Hence, $|F| \geq \ell + s - \ell - 1 = s - 1$, which is not true. So $|D| \geq 2r + s - 3$. Therefore, $\gamma^{ID}(C(G, \sigma)) = 2r + s - 3$. □

Theorem 4.2. *Let G be a graph of order n and a be an universal vertex of G. Also, let $G \setminus \{a\} = \overline{K_s} \bigcup_{i=1}^{r} K_{n_i}$, $r \geq 2$, $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ and*

$$\sigma : V(G) \to V(G^{'})$$

be a permutation, such that $\sigma(V(K_{n_i})) = V(K^{'}_{n_i})$, for each $1 \leq i \leq r$ and $\sigma(a) = a^{'}$. Then

$$\gamma^{ID}(C(G,\sigma)) = \begin{cases} 2n - 2r - 1, & s = 0, n_1 = 2 \\ 2n - 2r - 2, & s = 0, n_1 \geq 3 \\ 2n - 2r - s - 1, & s \geq 1 \end{cases}$$

Proof. By Corollary 2.8, $C(G,\sigma)$ is an identifiable graph. Let

$$V(K_{n_i}) = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$$

and $V(G) = V(\bigcup_{i=1}^{r} K_{n_i}) \cup \{v_j \mid 1 \leq j \leq s\} \cup \{a\}$.

Let $s = 0$, $n_1 = 2$ and

$$X_1 = V(G) \setminus \{v_{i1} \mid 1 \leq i \leq r\} \cup V(G^{'}) \setminus \{\sigma(v_{i1}), a^{'} \mid 1 \leq i \leq r\}.$$

Then X_1 is an identifying code of $C(G,\sigma)$. Thus

$$\gamma^{ID}(C(G,\sigma)) \leq |X_1| = 2n - 2r - 1. \tag{4.1}$$

Assume that $s = 0$, $n_1 \geq 3$ and

$$X_2 = V(G) \setminus \{a, v_{i1} \mid 1 \leq i \leq r\} \cup V(G^{'}) \setminus \{a^{'}, \sigma(v_{i1}) \mid 1 \leq i \leq r\}.$$

Then X_2 is an identifying code of $C(G,\sigma)$ and so

$$\gamma^{ID}(C(G,\sigma)) \leq |X_2| = 2n - 2r - 2. \tag{4.2}$$

Also, let $s \geq 1$ and

$$\begin{aligned} X_3 &= V(G) \setminus \{v_{i1}, v_s \mid 1 \leq i \leq r\} \\ &\cup V(G^{'}) \setminus (\{\sigma(v_{i1}), v^{'}_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}. \end{aligned}$$

Then X_3 is an identifying code of $C(G,\sigma)$. Thus

$$\gamma^{ID}(C(G,\sigma)) \leq |X_3| = 2n - 2r - s - 1. \tag{4.3}$$

Now, let D be an identifying code of $C(G,\sigma)$ with

$$\gamma^{ID}(C(G,\sigma)) = |D|.$$

Since $N_C[v_{i1}] \bigtriangleup N_C[v_{ij}] = \{\sigma(v_{i1}), \sigma(v_{ij})\}$, so by Lemma 2.3, (2), $\sigma(v_{i1}) \in D$ or $\sigma(v_{ij}) \in D$. Thus there is $A^{'} \subseteq \bigcup_{i=1}^{r} V^{'}(K_{n_i})$, such that $|A^{'} \cap D| \geq \sum_{i=1}^{r}(n_i - 1)$. Also, we have

$$N_C[\sigma(v_{i1})] \bigtriangleup N_C[\sigma(v_{ij})] = \{v_{i1}, v_{ij}\},$$

so by Lemma 2.3, (2), we have $v_{i1} \in D$ and $v_{ij} \in D$. So there is $A \subseteq V(\bigcup_{i=1}^{r} K_{n_i})$, such that $|A \cap D| \geq \sum_{i=1}^{r}(n_i - 1)$. Thus

$$|D| \geq 2(\sum_{i=1}^{r}(n_i - 1)) = 2\sum_{i=1}^{r} n_i - 2r.$$

Case 1: Let $s = 0$, $n_1 = 2$ and $\{v_{11}, \sigma(v_{11})\} \cap D = \emptyset$. If

$$|D| = 2\sum_{i=1}^{r} n_i - 2r,$$

then $D \cap \{a, a^{'}\} = \emptyset$ and so $N_C[v_{12}] \cap D = N_C[\sigma(v_{12})] \cap D$, which is not true. So $D \cap \{a, a^{'}\} \neq \emptyset$. Hence, $|D| \geq 2\sum_{i=1}^{r} n_i - 2r + 1$. By (1), $\gamma^{ID}(C(G, \sigma)) = 2n - 2r - 1$.
Case 2: Let $s = 0$ and $n_1 \geq 3$. We have $|D| \geq 2\sum_{i=1}^{r} n_i - 2r$. By (2), $\gamma^{ID}(C(G, \sigma)) = 2n - 2r - 2$.
Case 3: Let $s \geq 1$. For $1 \leq i \leq s$, we have $N_C[v_1] = \{a, v_1, \sigma(v_1)\}$ and $N_C[v_i] = \{a, v_i, \sigma(v_i)\}$. So $|\{v_1, v_i, \sigma(v_1), \sigma(v_i)\} \cap D| \geq 1$. Thus there is $F \subseteq \{v_i, \sigma(v_i) \mid 1 \leq i \leq s\}$, such that $|F \cap D| \geq s - 1$. Hence $|D| \geq 2\sum_{i=1}^{r} n_i - 2r + s - 1 = 2n - 2r - s - 3$. Now, if $|D| = 2n - 2r - s - 3$, then $\{a, a^{'}\} \cap D = \emptyset$. It is clear that $N_C[v_i] \cap D = N_C[\sigma(v_i)] \cap D$, which is a contradiction. Hence, $D \cap \{a, a^{'}\} \neq \emptyset$. Let $|D \cap \{a, a^{'}\}| = 1$. Then $|D| \geq 2n - 2r - s - 2$. If $|D| = 2n - 2r - s - 2$ and $a \in D$, then $a^{'} \notin D$ (or if $a^{'} \in D$, then $a \notin D$). Thus there is an x in $\{v_i^{'} \mid 1 \leq i \leq s\}$ such that x is not dominated by D. It is impossible. Hence, $\{a, a^{'}\} \subseteq D$ and so $|D| \geq 2n - 2r - s - 1$. By (3), we have $\gamma^{ID}(C(G, \sigma)) = 2n - 2r - s - 1$. □

Corollary 4.3. *Let $G \cong K_3^r$ be a graph, $r \geq 2$ and $\sigma : V(G) \to V(G^{'})$ be a permutation. Then $\gamma^{ID}(C(G, \sigma)) = 2r + 1$.*

Proof. By Theorem 4.2, the proof is straigtforward. □

Conjecture 4.4. [4] There exists a constant c such that for any non-trivial connected twin-free graph G of maximum degree $\Delta(G)$,

$$\gamma^{ID}(G) \leq n - \frac{n}{\Delta(G)} + c.$$

Note: The conjecture 4.4, holds for graphs which are presented in Theorems 4.1 and 4.2 with $c = 0$.

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THE IDENTIFYING CODE NUMBER AND FUNCTIGRAPHS

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عدد کد شناساگر و گراف‌های تابعی

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فرض کنید $G = (V(G), E(G))$ یک گراف ساده باشد. یک مجموعه‌ی D از رئوس G یک کد شناساگر G هست، اگر برای هر دو رأس x و y، مجموعه‌های $N_G[x] \cap D$ و $N_G[y] \cap D$ ناتهی و متمایز باشند. تعداد اعضای یک کد شناساگر گراف G با کمترین عضو، عدد کد شناساگر G نامیده شده و با نماد $\gamma^{ID}(G)$ نشان داده می‌شود.
دو رأس x و y دوقلو هستند، هرگاه $N_G[x] = N_G[y]$. گراف‌هایی که حداقل دو رأس دوقلو دارند، کدپذیر نیستند. در این مقاله، عدد کد شناساگر گراف تابعی G را مورد بررسی قرار می‌دهیم. دو کران بالا برای عدد کد شناساگر گراف تابعی بیان شده است. همچنین، برخی از گراف‌هایی که دارای عدد کد شناساگر $|V(G)| - ۲$ می‌باشند را ارائه می‌کنیم.

کلمات کلیدی: کد شناساگر، گراف کدپذیر، گراف تابعی.