Journal of Algebraic Systems
Vol. 10, No. 2, (2023), pp 203-224

$r-$CLEAN RINGS RELATIVE TO RIGHT IDEALS

H. HAKMI* AND B. ALHUSSEIN

Abstract. An associative ring R with identity is called $r-$clean if every element of R is the sum of a regular and an idempotent element. In this paper, we introduce the concept of $r-$clean rings relative to right ideal. We study various properties of these rings. We give some relations between $r-$clean ring and $r-$clean ring of 2×2 matrices over R relative to some right ideal P. We give some necessary and sufficient conditions for a ring R to be $r-$clean, in terms of $P-$regular, $P-$local and $P-$clean properties of a given ring. Also, we prove that every ring is $r-$clean relative to any maximal right ideal of it.

1. Introduction

In their fundamental work [2], Andrunakievich V. A and Ryabukhin Yu. M were the first who introduced the notion of rings relative to right ideals, they study the quasi-regularity and pimitivity relative to right ideals. Later in [1] the concept of rings relative to right ideals which was extended to regular ring relative to right ideals in as generalization of (Von Neumann) regular rings (also known as $P-$regular rings). In [5], H. Hakmi continued the study of $P-$regular and $P-$potent rings and in [6], he studied local ring relative to right ideal ($P-$local rings). In our paper we continue the study of rings relative to right ideals from a new point of view that, $r-$clean rings relative to right ideals. An element a of a ring R is said to be clean if $a = u + e$, where $e \in R$ is an idempotent and u is a unit in R. If every element of a ring R is

DOI: 10.22044/JAS.2021.10627.1525.
MSC(2010): Primary: 16E50, 16E70; Secondary: 16D70, 16D50, 16U99.
Keywords: ($P-$)idempotents; ($P-$)regular rings; $r-$clean ring relative to right ideal.
Received: 10 March 2021, Accepted: 31 December 2021.
*Corresponding author.

clean, then R is called a clean ring. Clean rings were introduced by W. K. Nicholson in his fundamental paper [7]. He proved that every clean ring is an exchange ring, and a ring with central idempotents is clean if and only if it is an exchange ring, where a ring R is said to be exchange if for each $a \in R$ there exists idempotent $e \in R$ such that $e \in aR$ and $(1-e) \in (1-a)R$. The clean rings were further extended to $r-$clean rings and the $r-$clean rings were introduced by Ashrafi and Nasibi [3]. They defined an element x of a ring R to be $r-$clean if $x = a+e$, where $a \in R$ is a regular element and $e \in R$ is an idempotent. A ring R is said to be $r-$clean if each of its element is $r-$clean. In our paper we study the concept of $r-$clean ring relative to some proper right ideal.

Throughout this paper, all rings are associative with identity. In Section 2, we study the fundamental properties of $P-$idempotents, where we proved that if $e \in R$ is an idempotent, then the set of all elements $f \in R$ such that $f - e \in P$ is a semi-group relative to multiplication on P. In Section 3, we study some properties of $P-$regular and $P-$clean elements. In Section 4, we study $r-$clean rings relative to right ideal P. Where we proved that every ring R is an $r-$clean relative to every maximal right ideal. In addition, we obtain that a ring R is $P-$local if and only if R is $r-$clean relative to P and the set of idempotents in R is $\{0, 1, p, 1+p\}$ for every $p \in P$.

Furthermore, we proved that, if in the ring R the set of all $P-$ idempotents is $\{0, 1, p, 1+p\}$ for every $p \in P$, then the ring R is $r-$clean relative to P if and only if R is $P-$clean. Also, if the set of all idempotents in R is $\{0, 1, p, 1+p\}$ for every $p \in P$, then the ring R is $r-$clean relative to P if and only if for every $x \in R$, either x or $1-x$ is the $P-$regular element. Also, in this section, we study the connection between the $r-$clean elements in a ring R and $r-$clean elements relative to P (relative to Q) in the ring of 2×2 matrices over R. We prove that an element a of a ring R is $r-$clean if and only if there exist $x, y \in R$ such that the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $r-$clean relative to some proper right ideal P of $M_2(R)$.

2. $P-$Idempotent Elements

Let R be a ring and $P \neq R$ be a right ideal of R. Recall that an element $e \in R$ is idempotent relative to right ideal P or $P-$idempotent for short, if $e^2 - e \in P$ and $eP \subseteq P$, [1]. Note that in previous definition it is easily verified that $0, 1 \in R$ are $P-$idempotents for every right ideal P of R. Also, if $P = 0$, then an element $e \in R$ is $P-$idempotent if and only if e is idempotent.

Lemma 2.1. *Let R be a ring and $P \neq R$ be a right ideal of R. For every $P-$ idempotent $e \in R$ the following hold;*

(1) *e^2 and $1-e$ are $P-$idempotents.*
(2) *For every positive integer k, e^k is $P-$idempotent.*
(3) *For every $p \in P$, $e+p \in R$ is $P-$ idempotent in R.*

Proof. (1) It is obvious.

(2) Since e is $P-$idempotent, $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. The proof can be done by induction on k. For $k = 1, 2$ the assertion holds by assumption and (1). Suppose that e^{k-1} is $P-$idempotent, then

$$(e^{k-1})^2 - e^{k-1} \in P \quad \text{and} \quad e^{k-1}P \subseteq P$$

So $(e^{k-1})^2 = e^{k-1} + p_1$ for some $p_1 \in P$. Thus

$$\begin{aligned} (e^k)^2 &= (e^{k-1})^2 e^2 \\ &= (e^{k-1} + p_1)(e + p_0) \\ &= e^k + e^{k-1}p_0 + p_1 e + p_1 p_0. \end{aligned}$$

Therefore $(e^k)^2 - e^k = p$, where $p = e^{k-1}p_0 + p_1 e + p_1 p_0 \in P$. This shows that

$$(e^k)^2 - e^k \in P \quad \text{and} \quad e^k P = ee^{k-1}P \subseteq eP \subseteq P.$$

(3) Since $e \in R$ is $P-$idempotent, $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Let $p \in P$ and suppose that $f = e + p$, then

$$\begin{aligned} f^2 &= (e+p)(e+p) \\ &= e^2 + ep + pe + p^2 \\ &= e + p_0 + ep + pe + p^2 \\ &= (e+p) + (-p + p_0 + ep + pe + p^2) \\ &= f + p_2. \end{aligned}$$

Where $p_2 = -p + p_0 + ep + pe + p^2 \in P$, thus $f^2 - f \in P$. On the other hand, for every $t \in P$, $ft = (e+p)t = et + pt \in eP + P \subseteq P$, so $fP \subseteq P$. Thus $f = e + p$ is $P-$idempotent. $\square$

Let R be a ring and $P \neq R$ be a right ideal of R. Suppose that $Pid(R)$ be the set of all $P-$idempotent elements in R. It is clear that $Pid(R)$ is a non-empty subset of R, because $0, 1 \in Pid(R)$. For every $e, f \in Pid(R)$, we define the relation $(\sim)$ on $Pid(R)$ as following:

$$e \sim f \Leftrightarrow e - f \in P$$

It is easy to see that $(\sim)$ is an equivalent relation on $Pid(R)$. If $e \in Pid(R)$, then the equivalent class of e is:

$$\begin{aligned}[e] &= \{f : f \in Pid(R); \quad e \sim f\} \\ &= \{f : f \in Pid(R) : \quad e - f \in P\}\end{aligned}$$

Lemma 2.2. *Let R be a ring and $P \neq R$ be a right ideal in R. If $e, g \in R$ such that $e - g \in P$, then g is $P-$idempotent if and only if e is $P-$idempotent.*

Proof. Suppose that $e-g \in P$, then $e = g+p_1$ for some $p_1 \in P$. Assume that g is $P-$idempotent, then $g^2 - g \in P$, $gP \subseteq P$. So $g^2 = g + p_0$ for some $p_0 \in P$ and

$$\begin{aligned}e^2 &= (g + p_1)(g + p_1) \\ &= g^2 + gp_1 + p_1 g + p_1 p_1 \\ &= g + p_0 + gp_1 + p_1 g + p_1 p_1 \\ &= (g + p_1) + (-p_1 + p_0 + gp_1 + p_1 g + p_1 p_1),\end{aligned}$$

for

$$p' = -p_1 + p_0 + gp_1 + p_1 g + p_1 p_1 \in P$$

We have $e^2 - e = p' \in P$ and $eP \subseteq gP + p_1 P \subseteq P$. This shows that e is $P-$ idempotent. Similarly, we can prove the converse. ☐

Lemma 2.3. *Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$ idempotent $e \in R$ the following hold:*

1. *Every element $f \in [e]$ is $P-$idempotent.*
2. *For every $g \in [e]$, ge and eg are $P-$idempotents.*

Proof. Since $e \in R$ is $P-$idempotent, $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

(1) Let $f \in [e]$, then $f - e \in P$, so $f = e + p_1$ for some $p_1 \in P$. Thus,

$$\begin{aligned}f^2 - f &= (e + p_1)(e + p_1) - (e + p_1) \\ &= e^2 + ep_1 + p_1 e + p_1^2 - e - p_1 \\ &= (e^2 - e) + ep_1 + p_1 e + (p_1^2 - p_1) \in P.\end{aligned}$$

So $f^2 - f \in P$. For every $t \in P$;

$$ft = (e + p_1)t = et + p_1 t \in eP + P \subseteq P.$$

This shows that $f \in [e]$ is $P-$idempotent.

(2) Let $g \in [e]$, then $g - e \in P$, so $g = e + p_2$ for some $p_2 \in P$. On the other hand, since $g \in [e]$, by (1) g is $P-$idempotent, so $g^2 - g \in P$

and $gP \subseteq P$, therefore $g^2 = g + p_3$ for some $p_3 \in P$. Thus

$$\begin{aligned} ge &= (e + p_2)e \\ &= e^2 + p_2e \\ &= (e + p_0) + p_2e \\ &= e + p_0 + p_2e \\ &= e + p_4. \end{aligned}$$

Where, $p_4 = p_0 + p_2e \in P + PR \subseteq P$ and

$$\begin{aligned} (ge)^2 - ge &= (e + p_4)(e + p_4) - (e + p_4) \\ &= e^2 + ep_4 + p_4e + p_4^2 - e - p_4 \\ &= (e^2 - e) + ep_4 + p_4e + p_4^2 - p_4 \\ &\in eP + P \\ &\subseteq P. \end{aligned}$$

So $(ge)^2 - ge \in P$. Also, for every $t \in P$;

$$(ge)t = g(et) \in g(eP) \subseteq gP \subseteq P.$$

This shows that ge is $P-$idempotent. Similarly, we can prove that eg is a $P-$idempotent element. $\square$

Lemma 2.4. *Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:*

(1) *For every $f \in R$, $f \in [e]$ if and only if $1 - f \in [1 - e]$.*
(2) *For every element $f \in [e]$, $fe \in [e]$ and $ef \in [e]$.*
(3) *For every $f, g \in [e]$, $fg \in [e]$ and $gf \in [e]$.*
(4) *The set $[e]$ is closed under multiplication defined on R.*

Proof. Suppose that $e \in R$ is $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$.

(1) $(\Rightarrow)$. If $f \in [e]$, then $f - e \in P$, so

$$(1 - f) - (1 - e) = 1 - f - 1 + e = -(f - e) \in P.$$

So $1 - f \in [1 - e]$.

$(\Leftarrow)$. If $1-f \in [1-e]$, then $f-e = 1-1+f-e = (1-e)-(1-f) \in P$, so $f \in [e]$.

(2) Let $f \in [e]$, then $f - e \in P$, so $f = e+p_1$ for some $p_1 \in P$. Thus,

$$fe - e = (e + p_1)e - e = (e^2 - e) + p_1e \in P + PR \subseteq P;$$

$$ef - e = e(e + p_1) - e = (e^2 - e) + ep_1 \in P + eP \subseteq P.$$

This shows that $fe, ef \in [e]$.

(3) Let $f, g \in [e]$, then $f - e \in P$ and $g - e \in P$, so $f = e + p_2$ and $g = e + p_3$ for some $p_2, p_3 \in P$. Thus,

$$fg = (e + p_2)(e + p_3) = e^2 + ep_3 + p_2e + p_2p_3 = e^2 + p_4,$$

where, $p_4 = ep_3 + p_2e + p_2p_3 \in P$, so $fg - e = (e^2 - e) + p_4 \in P$. This shows that $fg \in [e]$. Similarly, we can prove that $gf \in [e]$. (4) Is clear by (2). □

From two Lemmas 2.3 and 2.4, we can obtain the following:

Corollary 2.5. *Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the set:*

$$[e] = \{f : f \in R; e - f \in P\}$$

is a semi-group in R.

3. $P-$Regular Elements and $P-$Clean Elements

An element a of a ring R is called (Von Neumann) regular, if $aba = a$ for some $b \in R$, [4]. A ring R is called regular if every element in R is regular, [4]. Recall that an element x of a ring R is clean if $x = a + e$, where $a \in R$ is unit and $e \in R$ is idempotent, [8]. A ring R is called a clean ring, if every element x in R is clean, [8].

Let R be a ring and $P \neq R$ be a right ideal of R. Recall that an element $a \in R$ is regular relative to right ideal P or $P-$regular for short, if there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$, [2]. A ring R is called a $P-$regular ring if every element a in R is $P-$regular, [2]. Also, an element $a \in R$ has a right $P-$inverse if $R = aR + P$. Note that an element $a \in R$ has a right $P-$inverse if and only if there exists $x \in R$ such that $1 - ax \in P$.

Lemma 3.1. *Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. If a is $P-$regular, then $a + p$ is $P-$regular for every $p \in P$.*

Proof. Suppose that $a \in R$ is $P-$regular, then there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$, so $a = aba + p_0$ for some $p_0 \in P$. Let $p \in P$, then

$$\begin{aligned}
(a+p)b(a+p) &= (ab + pb)(a + p) \\
&= aba + abp + pba + pbp \\
&= a - p_0 + abp + pba + pbp \\
&= (a + p) + (-p - p_0 + abp + pba + pbp) \\
&= (a + p) + p',
\end{aligned}$$

where, $p' = -p - p_0 + abp + pba + pbp \in P$. So

$$(a+p)b(a+p)-(a+p) \in P.$$

For every $t \in P$, $(a+p)bt = abt + pbt \in abP + pR \subseteq P$, thus

$$(a+p)P \subseteq P.$$

This shows that $a+p$ is $P-$regular. □

Let R be a ring and $M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the sets:

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\} \quad \textit{and} \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in $M_2(R)$ such that $P \neq M_2(R)$, $Q \neq M_2(R)$.

Proposition 3.2. *Let R be a ring. Then the following hold:*

(1) *If $e \in R$ is an idempotent, then for every $x, y \in R$ the element $\alpha = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$.*

(2) *An element $e \in R$ is idempotent in R if and only if the element $\alpha = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$, for some $x, y \in R$.*

(3) *If $e \in R$ is an idempotent, then for every $x, y \in R$ the element $\alpha = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in $M_2(R)$.*

(4) *An element $e \in R$ is idempotent in R if and only if the element $\alpha = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in $M_2(R)$, for some $x, y \in R$.*

Proof. (1) Suppose that $e \in R$ is idempotent. Let $x, y \in R$, then

$$\alpha^2 - \alpha = \begin{bmatrix} x^2 & xy + ye \\ 0 & e \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & 0 \end{bmatrix} \in P$$

For every $p = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, $\alpha p = \begin{bmatrix} xa & yb \\ 0 & 0 \end{bmatrix} \in P$, thus $\alpha P \subseteq P$. This shows that α is $P-$idempotent.

(2) If $e \in R$ is idempotent in R, then by (1) the element α is $P-$idempotent. Conversely, suppose that α is $P-$idempotent for some $x, y \in R$. Since $\alpha^2 - \alpha \in P$,

$$\begin{bmatrix} x^2 & xy + ye \\ 0 & e^2 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & e^2 - e \end{bmatrix} \in P$$

Thus $e^2 = e$. Similarly, we can prove (3) and (4). □

Proposition 3.3. *For any element $a \in R$ the following hold:*

(1) *If a is a regular element in R, then for every $x \in R$, the elements:*
$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$$
are $P-$regular in $M_2(R)$.

(2) *If for some $x \in R$, the element $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is $P-$regular in $M_2(R)$, then a is regular in R.*

(3) *If for some $x \in R$, the element $\alpha = \begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$ is $P-$regular in $M_2(R)$, then a is regular in R.*

(4) *If a is a regular element in R, then for every $x \in R$, the elements:*
$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$$
are $Q-$regular in $M_2(R)$.

(5) *If for some $x \in R$, the element $\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ is $Q-$regular in $M_2(R)$, then a is regular in R.*

(6) *If for some $x \in R$, the element $\alpha = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$*

is $Q-$regular in $M_2(R)$, then a is regular in R.

Proof. (1) Suppose that a is a regular element in R, then $a = aba$ for some $b \in R$. For every $x_1, y_1 \in R$,
$$\beta = \begin{bmatrix} x_1 & y_1 \\ 0 & b \end{bmatrix} \in M_2(R)$$
such that
$$\alpha\beta\alpha - \alpha = \begin{bmatrix} xx_1x & xy_1a \\ 0 & aba \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} xx_1x - x & xy_1a \\ 0 & aba - a \end{bmatrix} \in P$$
and for every $t = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \in P$ where $a', b' \in R$
$$\alpha\beta t = \begin{bmatrix} xx_1 & xy_1 \\ 0 & ab \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xx_1a' & xx_1b' \\ 0 & 0 \end{bmatrix} \in P$$
this shows that $\alpha\beta P \subseteq P$. Thus, α is an $P-$regular element in $M_2(R)$. Similarly, we can prove that for every $x', y' \in R$, the element:
$$\beta' = \begin{bmatrix} 0 & b \\ x_1 & y_1 \end{bmatrix} \in M_2(R)$$

such that $\alpha'\beta'\alpha' - \alpha' \in P$ and $\alpha'\beta'P \subseteq P$. i.e., α' is a $P-$regular element in $M_2(R)$.
(2) Suppose that α is $P-$regular in $M_2(R)$, then there exists

$$\beta = \begin{bmatrix} y & z \\ r & b \end{bmatrix} \in M_2(R)$$

where $y, z, r, b \in R$ such that $\alpha\beta\alpha - \alpha \in P$, so

$$\begin{bmatrix} xyx & xza \\ arx & aba \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} xyx - x & xza \\ arx & aba - a \end{bmatrix} \in P$$

This shows that $aba = a$. i.e., an element a is regular. (3) It is proved in the similar way as in the proof of (2). (4) It is proved in the similar way as in the proof of (1). (5) and (6) It is proved in the similar way as in the proof of (2) and (3). □

4. $r-$Clean Rings Relative to Right Ideal

Recall that an element x of a ring R is $r-$clean if $x = a + e$, where $a \in R$ is regular and $e \in R$ is idempotent, [3]. A ring R is $r-$clean if every element $x \in R$ is $r-$clean, [3].

Definition 4.1. Let R be a ring and $P \neq R$ be a right ideal of R. We say that an element x of a ring R is $r-$clean relative to right ideal P, if $x = a+e$, where $e \in R$ is $P-$idempotent and $a \in R$ is $P-$regular. Also, we say a ring R is $r-$clean relative to right ideal P, if every element x in R is $r-$clean relative to P.

Note that in previous definition, it is easy to see that for $P = 0$, a ring R is a $r-$clean relative to P if and only if R is $r-$clean. Furthermore, we have the following:

Lemma 4.2. *Let R be a ring and $P \neq R$ be a right ideal of R. Then the following hold:*

1. *Elements $1, -1, 0$ are $r-$cleans relative to P.*
2. *Every right invertible element of R is $r-$clean relative to P.*
3. *Every invertible element of R is $r-$clean relative to P.*

Proof. (1) It is obvious, because $1 = 1+0$, $0 = (-1)+1$, $-1 = -1+0$, where $1, 0$ are $P-$idempotents and $1, -1$ are $P-$regular elements.

(2) If $a \in R$ has a right inverse, then $R = aR \subseteq aR + P \subseteq R$, so $R = aR + P$ and so $1 = ab + p_0$ for some $b \in R$, $p_0 \in P$. Thus, $a - aba = p_0a \in PR \subseteq P$. For every $t \in P$, $t = abt + p_0t$, so $abt = t - p_0t \in P$, i.e., $abP \subseteq P$. This shows that a is $P-$regular and $a = a + 0$. Thus a is $r-$clean relative to P.

(3) It is obvious by (2). □

Lemma 4.3. *Let R be a ring and $P \neq R$ be a right ideal of R. Then the following hold:*

1. *Every right $P-$invertible element of R is $r-$clean relative to P.*
2. *Every $P-$idempotent element in R is $r-$clean relative to P.*
3. *Every $P-$clean element in R is $r-$clean relative to P.*
4. *Every $P-$regular element in R is $r-$clean relative to P.*

Proof. (1) Let $a \in R$ has a right $P-$inverse, then $R = aR + P$, so $1 = ax + p_0$ for some $x \in R$ and $p_0 \in P$. Thus $a = axa + p_0a$ and so $axa - a = -p_0a \in PR \subseteq P$. For every $t \in P$,

$$axt = (1 - p_0t) = t - p_0t \in P,$$

so $axP \subseteq P$. This shows that a is $P-$regular. Thus we van write $a = a + 0$, hence a is $r-$clean relative to P.

(2) Let $e \in R$ be a $P-$idempotent element, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Thus,

$$eee = ee^2 = e(e + p_0) = e^2 + ep_0 = e + p_0 + ep_0$$

so $e^3 - e = p_0 + ep_0 \in P$ and $e^2P \subseteq eP \subseteq P$. This shows that e is $P-$regular and so e is $r-$clean relative to P.

(3) Let $x \in R$ be a $P-$clean element, then $x = a + e$, where $e \in R$ is $P-$ idempotent and $a \in R$ has a right $P-$inverse, so $R = aR + P$ and so $1 = ax + p_0$ for some $x \in R$ and $p_0 \in P$, therefore

$$axa - a = -p_0a \in PR \subseteq P.$$

For every $t \in P$, $axt = (1 - p_0)t = t - p_0t \in P$, i.e., $axP \subseteq P$, this shows that a is $P-$regular, thus x is $r-$clean relative to P.

(4) It is clear. □

Also, we have the following:

Lemma 4.4. *Let R be a ring, $P \neq R$ be a right ideal. Suppose that $x, y \in R$ are such that $x - y \in P$. Then y is $r-$clean relative to P if and only if x is $r-$clean relative to P.*

Proof. Let $x, y \in R$ such that $x - y \in P$, then $x = y + p_0$ for some $p_0 \in P$. Suppose that y is $r-$clean relative to P, then $y = a + e$, where $a \in R$ is $P-$regular and $e \in R$ is $P-$idempotent. So

$$x = (a + p_0) + e = a + (e + p_0),$$

where $a \in R$ is $P-$regular and $e + p_0 \in R$ is $P-$idempotent by Lemma 2.1. Thus x is $r-$clean relative to P. We can prove the converse in the similar way. □

Proposition 4.5. *Let R be a ring, $P \neq R$ be a right ideal of R and $x \in R$, then x is $r-$clean relative to P if and only if $1-x$ is $r-$clean relative to P.*

Proof. Let $x \in R$ be an $r-$clean element relative to P. Then write $x = a + e$, where $a \in R$ is $P-$regular and $e \in R$ is $P-$idempotent. Thus, $1 - x = (-a) + (1 - e)$. Since $a \in R$ is $P-$regular, $aba - a \in P$ for some $b \in R$ and $abP \subseteq P$, so

$$(-a)(-b)(-a) - (-a) = -(aba - a) \in P$$

and $(-a)(-b)P = abP \subseteq P$. This shows that $-a \in R$ is $P-$regular. Also, since $e \in R$ is $P-$idempotent, $1-e \in R$ is $P-$idempotent. Thus, $1 - x$ is $r-$clean relative to P. Conversely, if $1 - x$ is $r-$clean relative to P, write $1 - x = a + e$, where $a \in R$ is $P-$regular and $e \in R$ is $P-$idempotent. Thus, $x = -a + (1 - e)$, like previous part, $-a \in R$ is $P-$regular and $1 - e \in R$ is $P-$idempotent. Therefore, x is $r-$clean relative to P. □

Theorem 4.6. *Every ring R is $r-$clean relative to any maximal right ideal of R.*

Proof. Let R be a ring and M be a maximal right ideal of R. Let $a \in R$, then:

Case 1. If $a \in M$, then $a - axa \in M$ for every $x \in R$. Also, for every $m \in M$, $axm \in M$, i.e., $axM \subseteq M$, this shows that a is $M-$regular. Since $a = a + 0$ and 0 is $M-$idempotent, a is $r-$clean relative to M.

Case 2. If $a \notin M$, then $R = aR + M$, so $1 = ax + p_0$ for some $x \in R$ and $p_0 \in M$. So $a - axa = p_0a \in M$. Also, for every $t \in M$ we have $axt = (1 - p_0)t = t - p_0t \in P$, this shows that a is $M-$regular. Since $a = a + 0$ and 0 is $M-$idempotent, a is $M-$regular. Therefore R is $r-$clean relative to M. □

Let R be a ring and $P \neq R$ be a right ideal of R. A ring R is called $P-$local if for every element $x \in R$, either x or $1 - x$ has a right $P-$inverse, [6].

Proposition 4.7. *Let R be a ring and $P \neq R$ be a right ideal of R. If R is a $P-$local ring, then R is $r-$clean relative to right ideal P.*

Proof. Suppose that R is a $P-$local ring. Let $x \in R$, then either x or $1 - x$ has a right $P-$inverse.

Case 1. If x has a right $P-$inverse, then $R = xR + P$, so $1 = xy + p_0$ for some $y \in R$, $p_0 \in P$. Thus $x - xyx = p_0x \in PR \subseteq P$. On the other hand, for every $t \in P$ we have $t = xyt + p_0t$ and so $xyt = t - p_0t \in P$, therefore $xyP \subseteq P$. This shows that x is an $P-$regular element. Since

$x = x + 0$ and 0 is $P-$idempotent, x is $r-$clean relative to P.
Case 2. If $1 - x$ has a right $P-$inverse, then $R = (1 - x)R + P$, so $1 = (1 - x)z + p_1$ for some $z \in R$, $p_1 \in P$. Thus,

$$1 - x = (1 - x)z(1 - x) + p_1(1 - x)$$

$$x - 1 = (x - 1)(-z)(x - 1) + p_1(x - 1)$$

$$(x - 1) - (x - 1)(-z)(x - 1) = p_1(x - 1) \in PR \subseteq P$$

Also, for every $t \in P$ we have

$$t = (1 - x)zt + p_1 t = (x - 1)(-z)t + p_1 t$$

and so $(x-1)(-z)t = t - p_1 t \in P$. This shows that $(x-1)(-z)P \subseteq P$. Thus $x - 1$ is a $P-$regular element. Since $x = (x - 1) + 1$, implies that x is $r-$clean relative to P. Thus a ring R is $r-$clean relative to P. □

Theorem 4.8. *Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:*

(1) *R is $P-$local.*
(2) *R is $r-$clean relative to P and the set of $P-$idempotents in R is $\{0, 1, p, 1 + p\}$ for every $p \in P$.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is $P-$local, then by Proposition 4.7 R is $r-$clean relative to P. Let $e \in R$ be a $P-$idempotent. If $e = 0$ or $e = 1$, our proof is completed. Suppose that $e \neq 0$, $e \neq 1$, then by assumption either e or $1 - e$ has a right $P-$inverse.

If e has a right $P-$inverse, then $R = eR + P$, so $1 = ex + p_1$ for some $x \in R$ and $p_1 \in P$, thus $e = e^2x + ep_1$. Since e is $P-$idempotent, then $e^2 = e + p_0$ for some $p_0 \in P$. So

$$\begin{aligned} e &= e^2x + ep_1 \\ &= (e + p_0)x + ep_1 \\ &= ex + p_0x + ep_1 \\ &= 1 - p_1 + p_0x + ep_1 \\ &= 1 + p \end{aligned}$$

where, $p = -p_1 + p_0x + ep_1 \in P$. Thus $e = 1 + p$.

If $1 - e$ has a right $P-$inverse, then $R = (1 - e)R + P$, so $1 = (1 - e)y + p_2$ for some $y \in R$, $p_2 \in P$, thus $e = (e - e^2) + ep_2 \in P$. Our proof is completed.

(2) $\Rightarrow$ (1) Let $x \in R$, by assumption $x = a + e$, where $a \in R$ is a $P-$regular element and $e \in R$ is $P-$idempotent. Since a is $P-$regular, there exists $b \in R$ such that $a - aba \in P$ and $abP \subseteq P$, so $a = aba + p_0$ for some $p_0 \in P$. By assumption either $e = 0$, $e = 1$, $e = p$ or $e = 1+p$, where $p \in P$.

If $e = 0$, then $x = a$. Since $ab \in R$ is $P-$idempotent, either $ab = 0$, $ab = 1$, $ab = p'$ or $ab = p' + 1$, where $p' \in P$.

I) - If $ab = 0$, then $x = a = p_0 \in P$, so

$$R = aR + (1-a)R \subseteq (1-a)R + P \subseteq R$$

Therefore $R = (1-a)R + P$, this shows that $1 - x = 1 - a$ has a right $P-$inverse.

II) - If $ab = p'$, then

$$a = aba + p_0 = p'a + p_0 \in PR + P \subseteq P$$

and so

$$R = aR + (1-a)R \subseteq (1-a)R + P \subseteq R.$$

Therefore $R = (1-a)R + P$, this shows that $1 - x = 1 - a$ has a right $P-$inverse.

III) - If $ab = 1$, then $R = aR + (1 - ab) = aR$ and so $R = aR + P$ this shows that $x = a$ has a right $P-$inverse.

IV) - If $ab = 1 + p'$, then

$$R = aR + (1 - ab) = aR + (-p')R = aR + P$$

and so $R = aR + P$, this shows that $x = a$ has a right $P-$inverse.

If $e = p$, then

$$x = a + e = (aba + p_0) + p = aba + (p_0 + p) = aba + p'',$$

where, $p'' = p_0 + p \in P$.

I) - If $ab = 0$, then $x = aba + p'' = p'' \in P$, so

$$R = xR + (1-x)R \subseteq (1-x)R + P \subseteq R.$$

Therefore $R = (1-x)R+P$, this shows that $1-x$ has a right $P-$inverse.

II) - If $ab = p'$, then

$$a = aba + p_0 = p'a + p_0 \in PR + P \subseteq P,$$

so $x = a + e = a + p \in P$, thus

$$R = xR + (1-x)R \subseteq (1-x)R + P \subseteq R.$$

Therefore $R = (1-x)R+P$, this shows that $1-x$ has a right $P-$inverse.

III) - If $ab = 1$, then

$$x = a + e = (aba + p_0) + p = a + (p_0 + p) = a + p_1,$$

where, $p_1 = p_0 + p \in P$. Since $R = aR + (1 - ab)R = aR$, $R = aR$ and so

$$R = aR = (a + p_1 + (-p_1))R \subseteq (a + p_1)R + (-p_1)R \subseteq xR + P \subseteq R.$$

Thus $R = xR + P$, this shows that x has a right $P-$inverse.

IV) - If $ab = 1 + p'$, then

$$\begin{aligned} x &= a + e \\ &= (aba + p_0) + p \\ &= (1 + p')a + (p_0 + p) \\ &= a + p'a + p_0 + p \\ &= a + p_2, \end{aligned}$$

where, $p_2 = p'a + p_0 + p \in P$. Thus,

$$R = aR + (1 - ab)R = aR + (-p')R \subseteq aR + P \subseteq R.$$

So $R = aR + P$, therefore,

$$\begin{aligned} R &= aR + P \\ &= (a + p_2 - p_2)R + P \\ &\subseteq (a + p_2)R + (-p_2)R + P \\ &\subseteq xR + P \\ &\subseteq R \end{aligned}$$

so $R = xR + P$, this shows that x has a right $P-$inverse.

If $e = 1$, then $x = a + e = a + 1$.

I) - If $ab = 0$, then $a = aba + p_0 = p_0$, so $x = 1 + p_0$. Thus

$$R = (1 + p_0 - p_0)R \subseteq (1 + p_0)R + (-p_0)R \subseteq xR + P \subseteq R$$

So $R = xR + P$, this shows that x has a right $P-$inverse.

II) - If $ab = p'$, then $a = aba + p_0 = p'a + p_0 \in P$, so $x = a + e = a + 1$, thus

$$R = (1 + a - a)R \subseteq (1 + a)R + (-a)R \subseteq xR + P \subseteq R$$

therefore $R = xR + P$, this shows that x has a right $P-$inverse.

III) - If $ab = 1$, then $x = a + e = a + 1$, so $a = x - 1$ and so $-a = 1 - x$. Thus $R = aR + (1 - ab)R = aR = (-a)R = (1 - x)R$, therefore $R = (1-x)R+P$, this shows that $1-x$ has a right $P-$inverse.

IV) - If $ab = 1 + p'$, then

$$a = aba + p_0 = (1 + p')a + p_0 = a + p'a + p_0 = a + p_3,$$

where, $p_3 = p'a + p_0 \in P$. Since $x = a + e = a + 1$, $-a = 1 - x$. Thus,

$$R = aR + (1 - ab)R = aR + (-p')R \subseteq aR + P \subseteq R.$$

So

$$R = aR + P = (-a)R + P = (1 - x)R + P$$

this shows that $1 - x$ has a right $P-$inverse.

If $e = 1 + p$, then $x = a + e + 1$.

I) - If $ab = 0$, then $a = aba + p_0 = p_0$, so

$$x = a + e = p_0 + p + 1 = 1 + p_4,$$

where, $p_4 = p_0 + p \in P$. Thus,

$$R = (1 + p_4 - p_4)R \subseteq (1 + p_4)R + (-p_4)R \subseteq xR + P \subseteq R.$$

So $R = xR + P$, this shows that x has a right $P-$inverse.

II) - If $ab = p'$, then $a = aba+p_0 = p'a+p_0 \in P$. Suppose that $a = t$, where $t \in P$. Then $x = a+e = t+1+p = 1+p_5$ where $p_5 = t+p \in P$. Thus,

$$R = (1 + p_5 - p_5)R \subseteq (1 + p_5)R + (-p_5)R \subseteq xR + P \subseteq R$$

Therefore $R = xR + P$, this shows that x has a right $P-$inverse.

III) - If $ab = 1$, then $x = a + e = a + 1 + p$, so $1 - x = -a - p$. Since

$$R = aR + (1 - ab)R = aR,$$

$$R = aR = (-a)R = (-a-p+p)R \subseteq (-a-p)R+pR \subseteq (1-x)R+P \subseteq R.$$

Thus, $R = (1 - x)R + P$, this shows that $1 - x$ has a right $P-$inverse.

IV) - If $ab = 1 + p'$, then

$$R = aR + (1 - ab)R = aR + (-p')R \subseteq aR + P \subseteq R$$

So $R = aR + P$. Since $x = a + e = a + (1 + p)$, $1 - x = -a - p$. Thus

$$\begin{aligned} R &= aR + P \\ &= (-a)R + P \\ &= (-a - p + p)R + P \\ &\subseteq (-a - p)R + pR + P \\ &\subseteq (1 - x)R + P \\ &\subseteq R \end{aligned}$$

Therefore $R = (1 - x)R + P$, this shows that $1 - x$ has a right $P-$invertible. So the proof is completed. □

Lemma 4.9. *Let R be a ring, $P \neq R$ be a right ideal of R. If R is a $P-$clean ring, then R is $r-$clean relative to P.*

Proof. Suppose that R is a $P-$clean ring. Let $x \in R$, then $x = a + e$, where $a \in R$ has a right $P-$inverse and $e \in R$ is $P-$idempotent. So $R = aR + P$ and therefore $1 = ab + p_0$ for some $b \in R$, $p_0 \in P$, so $a - aba = p_0a \in PR \subseteq P$. Also, for every $t \in P$, $abt = t - p_0t \in P$, i.e., $abP \subseteq P$, thus a is $P-$regular. Therefore R is $r-$clean relative to P. □

Theorem 4.10. *Let R be a ring, $P \neq R$ be a right ideal of R. If R is a $r-$clean ring relative to P and the $P-$idempotents of R are the only $0, 1, p$ and $1+p$ for every $p \in P$, then R is a $P-$clean ring.*

Proof. Let $x \in R$, then $x = a + e$, where $a \in R$ is a $P-$regular element and $e \in R$ is $P-$idempotent, so $e^2 = e + p_0$, $eP \subseteq P$ for some $p_0 \in P$ and $a = aba + p_1$, $abP \subseteq P$ for some $p_1 \in P$. Now we consider several cases.

If $a = 0$, then $x = e = (2e-1) + (1-e)$, where $1 - e \in R$ is $P-$ idempotent and $2e - 1 \in R$ has a right $P-$inverse, hence

$$(2e-1)(2e-1) = 4e^2 - 4e + 1 = 4(e + p_0) - 4e + 1 = 1 + 4p_0$$

i.e., $1 - (2e-1)(2e-1) \in P$. Thus, $x = e$ is a $P-$clean element.

Suppose that $a \neq 0$. Since $ab \in R$ is $P-$idempotent, by assumption either $ab = 0$, $ab = 1$, $ab = p$ or $ab = 1 + p$ where $p \in P$.

If $ab = 0$, then $a = aba + p_1 = p_1 \in P$ and so

$$x = a + e = p_1 + e = (2e-1) + (1-e) + p_1 = (2e-1) + ((1-e) + p_1)$$

Since $e \in R$ is $P-$idempotent, $1 - e \in R$ is $P-$idempotent and by Lemma 2.1, $(1-e) + p_1 \in R$ is $P-$idempotent. Thus x is $P-$clean, hence $2e - 1$ has a right $P-$inverse.

If $ab = 1$, then $R = aR + (1-ab)R = aR$, so $R = aR + P$, i.e., a has a right $P-$inverse, thus x is $P-$clean.

If $ab = p$, then $a = aba + p_1 = pa + p_1 \in P$. Suppose that $a = p_2$, where $p_2 \in P$. Then

$$x = a + e = p_2 + e = (2e-1) + (1-e) + p_2 = (2e-1) + ((1-e) + p_2)$$

Since $e \in R$ is $P-$idempotent, $1 - e \in R$ is $P-$idempotent and by Lemma 2.1, $(1-e) + p_2 \in R$ is $P-$idempotent. Thus x is $P-$clean, hence $2e - 1$ has a right $P-$inverse.

If $ab = 1 + p$, then

$$R = aR + (1-ab)R = aR + (-p)R \subseteq aR + P \subseteq R$$

so $R = aR+P$, This shows that a has a right $P-$inverse. Thus $x = a+e$ is $P-$clean. Therefore a ring R is $P-$clean. □

From Theorem 4.8 and Theorem 4.10, we can obtain the following:

Corollary 4.11. *Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:*

(1) *R is $P-$local.*
(2) *R is $r-$clean relative to P and the set of $P-$idempotents in R is $\{0, 1, p, 1+p\}$ for every $p \in P$.*

(3) *R is $P-$clean and the set of $P-$idempotents in R is $\{0, 1, p, 1+p\}$ for every $p \in P$.*

Proof. (1) $\Leftrightarrow$ (2) It is clear by Theorem 4.8.
(2) $\Rightarrow$ (3) It can be obtained by Theorem 4.10.
(3) $\Rightarrow$ (2) It follows from Lemma 4.9. □

Proposition 4.12. *Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements are equivalent:*

(1) *R is $r-$clean relative to right ideal P.*
(2) *For every $x \in R$, $x = a - e$, where $a \in R$ is $P-$regular and $e \in R$ is $P-$idempotent.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is a $r-$clean ring relative to P. Let $x \in R$, then $-x \in R$ and $-x = a + e$, where $a \in R$ is an $P-$regular element and $e \in R$ is $P-$idempotent, so $x = (-a) - e$. Since a is $P-$regular, there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$. Thus,

$$(-a)(-b)(-a) - (-a) = -aba + a = -(aba - a) \in P$$

and $(-a)(-b)R = abP \subseteq P$, this shows that $-a \in R$ is a $P-$regular element.

(2) $\Rightarrow$ (1) Let $x \in R$, then $-x \in R$ and $-x = a - e$, where $a \in R$ is a $P-$regular element and $e \in R$ is $P-$idempotent, so $x = (-a) + e$. Since a is $P-$regular, there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$. Thus,

$$(-a)(-b)(-a) - (-a) = -aba + a = -(aba - a) \in P$$

and $(-a)(-b)R = abP \subseteq P$, this shows that $-a \in R$ is a $P-$regular element. Thus, R is $r-$clean relative to P. □

Theorem 4.13. *Let R be a ring and $P \neq R$ be a right ideal of R. If the set of $P-$ idempotents in R is $\{0, 1, p, 1+p\}$ for every $p \in P$. Then the following conditions are equivalent:*

(1) *R is $r-$clean relative to P.*
(2) *For every $x \in R$, either x or $1 - x$ is a $P-$regular element.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is $r-$clean relative to P. Let $x \in R$, then $x = a+e$, where $a \in R$ is $P-$regular and $e \in R$ is $P-$idempotent. So by assumption:

If $e = 0$, then $x = a$ is $P-$regular.

If $e = p$, then $x = a+p$. Since $p \in P$ and a is $P-$regular, by Lemma 3.1, $x = a + p$ is $P-$regular.

If $e = 1$, then $x = a + 1$, so $1 - x = -a$. Since a is $P-$regular then $aba - a \in P$ and $abP \subseteq P$ for some $b \in R$, so

$$(-a)(-b)(-a) - (-a) = -aba + a = -(aba - a) \in P,$$

$$(-a)(-b)P = abP \subseteq P.$$

This shows that $-a \in R$ is a $P-$regular element, therefore $1 - x = -a$ is $P-$regular.

If $e = 1+p$, then $x = a+e = a+1+p$, so $1-x = -(a+p)$. Since a is $P-$regular, by Lemma 3.1, $a+p$ is $P-$regular and so $1-x = -(a+p)$ is a $P-$regular element.

(2) $\Rightarrow$ (1). Let $x \in R$, by assumption, either x or $1-x$ is $P-$regular. If x is $P-$regular, then $x = x + 0$ is $r-$clean relative to P. Suppose that $1 - x$ is $P-$regular, $x - 1$ is $P-$regular and so $x = (x - 1) + 1$ is $r-$clean relative to P. Thus, R is a $r-$clean ring relative to P. $\square$

Let R be a ring and $S = M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the sets:

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\} \quad and \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in S such that $P \neq S$, $Q \neq S$. Now we provide the connection between the $r-$clean elements in R and $r-$clean elements relative to P (relative to Q) in S, in the following:

Theorem 4.14. *For any element $a \in R$ the following hold:*

(1) *If a is a $r-$clean element in R, then for every $x, y \in R$, the element $\alpha = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P in S.*

(2) *If for some $x, y \in R$ the element $\alpha = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P in S, then the element a is $r-$clean in R.*

(3) *Let a be a $r-$clean element in R, then for every $x, y \in R$, the element $\alpha = \begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $r-$clean relative to Q in S.*

(4) *If for some $x, y \in R$ the element $\alpha = \begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $r-$clean relative to Q in S, then the element a is $r-$clean in R.*

Proof. (1) Suppose that a is $r-$clean in R, then $a = u+e$, where $u \in R$ is a regular element and $e \in R$ is idempotent. So, for every $x, y \in R$

$$\alpha = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & u + e \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} 0 & y \\ 0 & e \end{bmatrix}.$$

Since $u \in R$ is regular, by Proposition 3.3, the element $\beta = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix}$ is $P-$regular in S. Also, since $e \in R$ is idempotent, by Proposition 3.2, the element $\gamma = \begin{bmatrix} 0 & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in S. Thus, the element $\alpha = \beta + \gamma$ is $r-$clean relative to P in S.

(2) Suppose that for some $x, y \in R$, the element $\alpha = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P in S. Then $\alpha = \beta + \gamma$, where $\beta \in S$ is $P-$regular and $\gamma \in S$ is $P-$idempotent.

Suppose that $\gamma = \begin{bmatrix} x_2 & y_2 \\ z_2 & e \end{bmatrix} \in S$, where $x_2, y_2, z_2, e \in R$. Since γ is $P-$ idempotent, then $\gamma P \subseteq P$ which implies that $z_2 = 0$. So $\gamma = \begin{bmatrix} x_2 & y_2 \\ 0 & e \end{bmatrix}$ and by Proposition 3.2, $e \in R$ is idempotent in R.

Suppose that $\beta = \begin{bmatrix} x_1 & y_1 \\ z_1 & u \end{bmatrix} \in S$, where $x_1, y_1, z_1, u \in R$. Since

$\alpha = \beta + \gamma$ implies that $z_1 = 0$ and $a = u + e$. Since $\beta = \begin{bmatrix} x_1 & y_1 \\ 0 & u \end{bmatrix}$ is

$P-$ regular in S, by Proposition 3.3, $u \in R$ is regular. This shows that an element a is $r-$clean.

(3) It is proved in the similar way as in (1).

(4) It is proved in the similar way as in (2). □

From Theorem 4.14, we can obtain the following:

Corollary 4.15. *Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. Then the following statements hold:*

(1) *The element a is $r-$clean in R if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P in S.*
(2) *The element a is $r-$clean in R if and only if there exists $x \in R$ such that the element $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P in S.*
(3) *The element a is $r-$clean in R if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $r-$clean relative to Q in S.*
(4) *The element a is $r-$clean in R if and only if there exists $x \in R$ such that the element $\begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ is $r-$clean relative to Q in S.*

Lemma 4.16. *Let R be a ring, $P \neq R$ be a right ideal of R and $a \in R$. Then the following statements hold:*

(1) *The element a is $r-$clean in R if and only if the element* $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ *is $r-$clean relative to P in S.*

(2) *The element a is $r-$clean in R if and only if the element* $\alpha = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ *is $r-$clean relative to Q in S.*

Proof. (1)($\Rightarrow$) Suppose that a is $r-$clean, then $a = x + e$, where $x \in R$ is regular and $e \in R$ is idempotent, so $x = xyx$ for some $y \in R$. Thus,

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \ \gamma = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \in S$$

$$\beta\gamma\beta - \beta = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & xyx - x \end{bmatrix} \in P$$

and for any $\lambda = \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \in P$, where $u, v \in R$ and $\beta\gamma\lambda = \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \in P$. This shows that $\beta\gamma P \subseteq P$. Thus, $\beta \in S$ is a $P-$regular element is S. Also, since $e \in R$ is idempotent, $\delta = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \in S$ is $P-$idempotent and

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x + e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} = \beta + \delta$$

where $\beta \in S$ is $P-$regular and $\delta \in S$ is $P-$idempotent, thus α is $r-$clean relative to P in S.

($\Leftarrow$) Suppose that $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is $r-$clean relative to P is S, then $\alpha = \beta + \delta$, where $\beta \in S$ is $P-$regular and $\delta \in S$ is $P-$idempotent. Suppose that $\delta = \begin{bmatrix} u & v \\ w & e \end{bmatrix}$, where $u, v, w, e \in R$, since $\delta P \subseteq P$, $w = 0$, so $\delta = \begin{bmatrix} u & v \\ 0 & e \end{bmatrix}$. Also, since $\delta^2 - \delta \in P$, $e^2 = e$, so $e \in R$ is idempotent. Suppose that $\beta = \begin{bmatrix} x & y \\ z & b \end{bmatrix}$, where $x, y, z, b \in R$, since $\alpha = \beta + \delta$, $z = 0$, so $\beta = \begin{bmatrix} x & y \\ 0 & b \end{bmatrix}$ and $a = b + e$. Since $\beta = \begin{bmatrix} x & y \\ 0 & b \end{bmatrix}$ is $P-$regular in S, so by Theorem 4.14 the element $b \in R$ is regular. Thus, a is a regular element in R. Similarly, we can prove (2). □

Let R be a ring and $S = M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the set

$$S_0 = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R \right\}$$

is a subring in S with identity element. Also, the sets

$$P_0 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R \right\} \quad \textit{and} \quad Q_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in R \right\}$$

are right ideals in S_0 and $P_0 \neq S_0$, $Q_0 \neq S_0$. Then we have the following:

Theorem 4.17. *For any ring R the following hold:*

(1) *A ring R is a $r-$clean ring if and only if the ring S_0 is a $r-$clean ring relative to right ideal P_0.*
(2) *A ring R is a $r-$clean ring if and only if the ring S_0 is a $r-$clean ring relative to right ideal Q_0.*
(3) *The ring S_0 is a $r-$clean ring relative to right ideal P_0 if and only if the ring S_0 is a $r-$clean ring relative to right ideal Q_0.*

Proof. (1)($\Rightarrow$) Suppose that a ring R is $r-$clean. Let $\alpha = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix} \in S_0$, where $x, u \in R$. Since R is $r-$clean, then $u = a + e$ where $a \in R$ is a $P-$regular element and $e \in R$ is idempotent. Then

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & a+e \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$$

Let $\beta = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ and $\gamma = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$, then $\alpha = \beta + \gamma$.
Since $a \in R$ is regular, by Proposition 3.3, $\beta \in S_0$ is a P_0-regular element in S_0.

On the other hand, since $e \in R$ is idempotent, by Proposition 3.2, $\gamma \in S_0$ is $P-$ idempotent. Thus α is a $r-$clean element relative to P_0 in S_0. Therefore a ring S_0 is $r-$clean relative to P_0.

($\Leftarrow$) Let $x \in R$, then $\alpha = \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \in S_0$. Since S_0 is a $r-$clean ring relative to P_0, $\alpha = \beta + \gamma$ where $\beta = \begin{bmatrix} y & 0 \\ 0 & a \end{bmatrix} \in S_0$ is P_0-regular, for some $y, a \in R$ and $\gamma = \begin{bmatrix} z & 0 \\ 0 & e \end{bmatrix} \in S_0$ is P_0-idempotent, for some $z, e \in R$.

Since β is P_0-regular in S_0, by Proposition 3.3 a is regular in R.

Since γ is P_0-idempotent in S_0, by Proposition 3.2 a is idempotent in R. Thus $x = a + e$ is a $r-$clean element, therefore a ring R is $r-$clean. Similarly, we can prove (2).

(3) Follows immediately from (1) and (2). $\square$

Acknowledgments

The author would like to thank the referee for careful reading the manuscript. The valuable suggestions have simplified and clarified the paper.

References

1. A. V. Andrunakievich and V. A. Andrunakievich, Rings that are regular relative to right ideal, *Mat. Zametki.*, **49**(3) (1991), 3–11.
2. V. A. Andrunakievich and Yu. M. Ryabukhin, Quasiregularity and Pimitivity Relative to Right Ideals of a Ring, *Mat. Sb.*, **134**(4) (1987), 451–471.
3. N. Ashrafi and E. Nasibi, r-clean rings, *Math. Reports.*, **15(65)** (2), (2013), 125–132.
4. K. R. Goodearl, *Von Neumann regular rings*, Pitman. London, 1979.
5. H. Hamza, On Some Characterizations of Regular and Potent Rings Relative to Right Ideals, *Novi Sad J. Math.*, **48**(1) (2018), 1–7.
6. H. Hamza, P-Regular and P-Local Rings, *J. Algebra Relat. Topics*, **9**(1) (2021), 1–19.
7. W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, **229** (1977), 269–278.
8. H. Zhang and V. Camillo, On clean rings, *Comm. Algebra*, **44** (2016), 2475–1481.

Hamza Hakmi
Department of Mathematics, Damascus University, Damascus, Syria.
Email:hhakmi-64@hotmail.com

Bashar Alussein
Department of Mathematics, Damascus University, Damascus, Syria.
Email:bashar2014.h@gmail.com

Journal of Algebraic Systems

r-CLEAN RINGS RELATIVE TO RIGHT IDEALS

H. HAKMI AND B. ALHUSSEIN

حلقه‌های r-تمیز نسبت به ایده‌آل‌های راست

حمزه حکمی[۱] و بشار الحسین[۲]

[۱,۲]گروه ریاضی، دانشگاه دمشق، دمشق، سوریه

حلقه‌ی شرکت‌پذیر و یکدار R را r-تمیز می‌نامیم هرگاه هر عنصر R، جمع یک عنصر منظم و یک عنصر خودتوان باشد. در این مقاله، مفهوم حلقه‌های r-تمیز نسبت به ایده‌آل راست را معرفی کرده و خواص مختلف این حلقه‌ها را بررسی خواهیم کرد. در این راستا، روابطی بین حلقه‌ی r-تمیز و حلقه‌ی r-تمیز ماتریس‌های ۲ $\times$ ۲ روی حلقه‌ی R نسبت به یک ایده‌آل راست P را ارائه می‌دهیم. برخی شرایط لازم و کافی برای اینکه حلقه‌ی R حلقه‌ای r-تمیز باشد، بر اساس خواص P-منظم، P-موضعی و P-تمیز حلقه ارائه خواهیم کرد. همچنین، نشان می‌دهیم یک حلقه نسبت به هر ایده‌آل ماکسیمال خودش، r-تمیز است.

کلمات کلیدی: (P)-خودتوان؛ حلقه‌ی (P)-منظم؛ حلقه‌ی r-تمیز نسبت به یک ایده‌آل.