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GRADED I-PRIME SUBMODULES

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Abstract. Let $R = \bigoplus_{g \in G} R_g$ be a $G-$graded commutative ring with identity, I be a graded ideal and let M be a $G-$graded unitary R-module, where G is a semigroup with identity e. We introduce graded $I-$prime ideals (submodules) as a generalizations of the classical notions of prime ideals (submodules). We show that the new notions inherite the basic properties of the classical ones. In particular, we investigate the localization theory of these two concepts. We prove that for a faithfull flat module F, a graded submodule P of M is $I-$prime if and only if $F \otimes P$ is graded $I-$prime submodule of $F \otimes M$. As an application, for finitely generated graded module M over Noetherian graded ring R, the completion of graded $I-$prime submodules is $I-$prime submodule.

1. Introduction

Throughout this paper, R is a commutative graded ring with nonzero identity, I is a fixed graded ideal of R and M is a unitary graded R-module. The concept of weakly prime ideals was introduced by Anderson and Smith (2003) [6], which is a generalization of the concept of prime ideals. Weakly primary ideals were introduced and studied by Atani S. E. and Farzalipour F. in 2005, [7]. The class of prime submodules of modules was introduced and studied in 1992 as a generalization of the class of prime ideals of rings [14]. Then, many generalizations of prime submodules were studied such as primary,

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classical prime, weakly prime and classical primary submodules, see [11, 12, 15]. Graded prime ideals was introduced by Refai, Hailat and Obiedat in [17] whereas graded weakly prime ideals and graded weakly prime submodules were introduced and investigated by Atani S. E. [8] and [9]. The authors in [3, 5] introduced the notion I-prime ideals and I-prime submodules. This motivated us to study these concepts in the graded rings and graded modules. Let G be an arbitrary semigroup with identity e. A commutative ring R is called a *G-graded commutative ring* if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g\in G} R_g$ such that $1 \in R$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements of degree g. If $a \in R$, then a can be written uniquely as finite sum as $\sum_{g\in G} a_g$ where a_g is the component of a in R_g (of degree g). Also, we write $h(R) = \cup_{g\in G}R_g$. Moreover, if $R = \bigoplus_{g\in G} R_g$ is a graded ring, then R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$ [13]. Let P be an ideal of R. For $g \in G$, let $P_g := P \cap R_g$. Then P is a graded ideal of graded ring R if $P = \bigoplus_{g\in G} P_g$. In this case, P_g is a called the g-component of P. In this article, all ideals and submodules taken are graded and all elements taken are homogeneous.

An $R-$module M is said to be a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Note that $I \subseteq (N :_R M)$, $N = IM \subseteq (N :_R M)M \subseteq N$ so that $N = (N :_R M)M$, where $(N :_R M) = \{r \in R : rM \subseteq N\}$ and briefly we will write $(N : M)$. Hence an $R-$module M is a multiplication module if every submodule N of M has the form $N = (N : M)M$. For submodules $N = I_1M$ and $K = I_2M$ of a multiplication R-module M, the product NK is defined by $NK = I_1I_2M$. This product is independent of presentations of N and K, see [10]. A module M is called faithful if it has zero annihilator.

In section 2, we define graded I-prime ideals in G-graded commutative rings. The aim of this section is to explore some basic facts of these class of ideals. Various properties of graded I-prime ideals are considered. First, we give three equivalents to graded $I-$prime ideals (Theorem 2.3). Indeed, we prove that if r is a nonzero element in R with $(0 : Rr) \subseteq Rr$ and $IRr \subseteq Rr^2$, then Rr is a graded I-prime ideal if and only if Rr is graded prime ideal of R (Corollary 3.13).

Section 3 is devoted to introduce graded I-prime submodules. We prove that the class of graded I-prime submodules is closed under extensions, that is, for any two submodules $Q \subseteq P$ of M and any exact sequence $0 \to Q \to P \to \frac{P}{Q} \to 0$, the middle term is I-prime submodule if and only if so are the left and right terms (Theorem 3.9). Also we show that if M is a finitely generated faithful multiplication graded R-module and P is a proper graded submodule of M with $(IP : M) = I(P : M)$, then P is a graded I-prime submodule of M if and only if $(P : M)$ is a graded I-prime ideal of R (Theorem 3.17). Furthermore, for graded I-prime submodule P of M and a flat $R-$module F with $F \otimes P \neq F \otimes M$, $F \otimes P$ is a graded I-prime submodule of $F \otimes M$ (Theorem 3.24). If F is faithfuly flat, then the converse is also true. In particular, if M is finitely generated graded module over Noetherian graded ring R, then the completion of graded $I-$prime submodules is again graded $I-$prime (Corollary 3.25).

2. Graded I-prime ideals

Let R be a ring and I be a fixed ideal of R. We recall that a proper ideal P of R is *$I-$prime* if for $a, b \in R$ with $ab \in P?IP$ implies $a \in P$ or $b \in P$. Now we adjust the definition of $I-$prime ideal to graded rings appropriately.

Definition 2.1. Let I be a an ideal of R_e and P be a graded ideal of R. Then P is called *graded I-prime* ideal of R if $P \neq R$; and whenever $r, s \in h(R)$ with $rs \in P - IP$, we have $r \in P$ or $s \in P$.

Let Q be a graded ideal of a graded ring R. Then R/Q is a G-graded ring by $(R/Q)_g = (R_g + Q)/Q$ for all $g \in G$.

Example 2.2. Consider the graded ring $R = \mathbb{R}[x, y]$ where $\mathbb{R}$ is the set of real numbers and $G = \mathbb{N}_0$ is the set of nonnegative integers. Then R is a G-graded ring by

$$R_n = \oplus_{\substack{i+j=n,\\ i,j\geq 0}} \mathbb{R}x^i y^j$$

with $deg(x) = deg(y) = 1$. Thus $J = \langle x^2, xy, y^2\rangle$ is a graded ideal of R. Suppose that $S = R/J = \mathbb{R}[x, y]/\langle x^2, xy, y^2\rangle$. Then S is G-graded by $S_n = (R/J)_n = (R_n + J)/J$. Let $P = \langle \overline{x}, \overline{y}\rangle$ be a graded ideal of S. Then for any graded ideal I of S, P is a graded I-prime ideal of S.

It is easy to see that a graded proper ideal P of a graded ring R is a graded I-prime ideal of R if and only if $\frac{P}{IP}$ is graded $\{0\}-$prime in $\frac{R}{IP}$. Let I, J be two ideals of R_e with $I \subseteq J$ and P be a graded ideal of R. If P is graded I-prime ideal of R, then P is graded J-prime ideal. In the following theorem, we give some characterizations for graded I-prime ideals in graded rings.

Theorem 2.3. *Let P be a proper graded ideal of R. Then the following conditions are equivalent:*

(1) *P is graded I-prime ideal in R;*
(2) *For $r \in h(R) - P$, $(P : r) = P \cup (IP : r)$;*
(3) *For $r \in h(R) - P$, $(P : r) = P$ or $(P : r) = (IP : r)$;*
(4) *For graded ideals J and K of R, if $JK \subseteq P$ and $JK \nsubseteq IP$, then $J \subseteq P$ or $K \subseteq P$.*

Proof. ($1 \Rightarrow 2$) Assume that P is a graded I-prime ideal in R and $r \in h(R) - P$. Suppose that $a \in h(P : r)$, that is $ra \in P$. Now if $ra \in IP$, then $a \in (IP : r)$. If $ra \notin IP$ and as P is graded I-prime ideal in R and $r \notin P$, then $a \in P$. Therefore $(P : r) \subseteq P \cup (IP : r)$. The other inclusion holds, because $IP \subseteq P$.

($2 \Rightarrow 3$) The union of two ideals is an ideal if one contains the other.

($3 \Rightarrow 4$) Suppose that J and K are two graded ideals of R such that $JK \subseteq P$. Let $J \nsubseteq P$ and $K \nsubseteq P$, we want to show that $JK \subseteq IP$. Assume $r \in h(J)$, first if $r \notin P$, then $rK \subseteq P$, that is $K \subseteq (P : r)$. Therefore by assumption $K \subseteq (IP : r)$. Hence $rK \subseteq IP$. Now if $r \in J \cap P$, take $s \in h(J) - P$. Then $r + s \in J - P$. By the first case, for each $t \in K$ we have $st \in IP$ and $(r + s)t \in IP$. Thus $rt + st \in IP$, that is $rt \in IP$. Therefore $JK \subseteq IP$.

($4 \Rightarrow 1$) Assume that for any two graded ideals J and K of R, if $JK \subseteq P$ and $JK \nsubseteq IP$, then $J \subseteq P$ or $K \subseteq P$. We claim that P is graded I-prime ideal in R. Let $rs \in P - IP$, where $r, s \in h(R)$. Take $J = Rr$ and $K = Rs$. Then $JK \subseteq P - IP$. By hypothesis, either $Rr \subseteq P$ or $Rs \subseteq P$, that is $r \in P$ or $s \in P$. □

Let G be a group with identity e and R be a G-graded ring and $S \subseteq h(R)$ be a multiplicative set. Then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \{\frac{a}{s} : a \in R_h, s \in S \cap R_{hg^{-1}}\}$ for all $g \in G$. If I is a graded ideal of R, then $S^{-1}I$ is a graded ideal of $S^{-1}R$. We generalize [3, Proposition 2.14] to the graded I-prime ideals.

Proposition 2.4. *Let G be a group and let $Q \subseteq P$ be two graded ideals of $G-$graded ring R. If P is a graded $I-$prime ideal of R, then the following statements hold:*

(1) *$\frac{P}{Q}$ is a graded I-prime ideal of $\frac{R}{Q}$;*
(2) *Suppose that S is a multiplicative closed subset of R_e with $P \cap S = \phi$. Then $S^{-1}P$ is a graded $S^{-1}I$-prime ideal of $S^{-1}R$.*

Proof. (1) For $r, s \in h(R)$, let

$$rs + Q = (r + Q)(s + Q) \in \tfrac{P}{Q} - I\tfrac{P}{Q} = \tfrac{P}{Q} - \tfrac{IP+Q}{Q}.$$

Then $rs \in P - (IP + Q)$, hence $rs \in P - IP$. As P is graded I-prime, either $r \in P$ or $s \in P$. Therefore either $r + Q \in \frac{P}{Q}$ or $s + Q \in \frac{P}{Q}$. Thus $\frac{P}{Q}$ is graded I-prime ideal in $\frac{R}{Q}$.

(2) For $rs \in h(R)$ and $u, v \in S$ let

$$\tfrac{r}{u}.\tfrac{s}{v} \in S^{-1}P - S^{-1}I.S^{-1}P \subseteq S^{-1}P - S^{-1}(IP).$$

Then there is $a \in S$ such that $rsa \in P$ and $rsa \notin IP$. As P is graded I-prime ideal in R , $r \in P$ or $sa \in P$, that is $\frac{r}{u} \in S^{-1}P$ or $\frac{s}{v} \in S^{-1}P$. Therefore $S^{-1}P$ is a graded $S^{-1}I$-prime ideal of $S^{-1}R$. $\square$

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be graded ring homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$. For any graded ideal J of R, We denote $\overline{f(J)}$ to the graded ideal generated by the elements of the set $f(J)$.

Proposition 2.5. *Let $f : R \longrightarrow S$ be a graded ring homomorphism. If P is a graded I-prime ideal of R with $Kerf \subseteq P$, then $\overline{f(P)}$ is a graded $\overline{f(I)}$-prime ideal of $f(R)$.*

Proof. By [16, Lemma 3.11], we have $\overline{f(P)}$ is graded ideal of $f(R)$. Let $rs \in \overline{f(P)} - \overline{f(I)f(P)} = \overline{f(P)} - \overline{f(IP)}$, where $r, s \in h(f(R))$. Then there are $x, y \in h(R)$ such that $r = f(x)$ and $s = f(y)$, that is $f(xy) = f(x)f(y) = rs \in \overline{f(P)} - \overline{f(IP)}$. Therefore there is $u \in P - IP$ such that $f(xy) = f(u)$. Thus $f(xy) - f(u) = f(xy - u) = 0$ and

$$xy - u \in Kerf \subseteq P.$$

Hence $xy \in P$. If $xy \in IP$, then $rs = f(xy) \in f(IP)$ which is a contradiction. Hence $xy \notin IP$. As P is graded I-prime ideal, $x \in P$ or $y \in P$, that is, $r = f(x) \in f(P)$ or $s = f(y) \in f(P)$. $\square$

A graded ideal P of a graded ring R is weakly prime if for all $a, b \in R$ with $ab \in P - \{0\}$, then $a \in P$ or $a \in P$. Thus a graded ideal is weakly prime if and only if it is $\{0\}-$prime. Now we characterize graded rings in which every graded ideal is $I-$prime for a fixed graded ideal I by the following.

Proposition 2.6. *If every graded ideal of a graded ring R is an $I-$prime and I is a graded ideal with $I^2 = 0$, then every graded ideal of R is weakly prime.*

Proof. Let P be a graded ideal of R and $a, b \in h(R)$ such that $0 \neq ab \in P$. If $ab \notin IP$, then we are done. If $ab \in IP$, then

$$ab \in IP - I.IP = IP - I^2P = IP - \{0\}$$

and as IP is $I-$prime ideal of R, $a \in IP \subseteq P$ or $b \in IP \subseteq P$, that is, $a \in P$ or $b \in P$. Hence P is Weakly Prime ideal of R. $\square$

3. Graded I-prime submodules

In this section we define the concept of I-prime submodules for graded modules. We investigate some basic properties of this notion. We begin this section by a definition.

Definition 3.1. Let P be a graded submodule of $M = \bigoplus_{g\in G} M_g$ and I a fixed ideal of R_e. Then

(1) P_g is an *I-g-prime submodule* of the R_e-module M_g, if $P_g \neq M_g$; and whenever $r \in R_e, m \in M_g$ with $rm \in P_g - I_eP_g$, then $r \in (P_g : M_g)$ or $m \in P_g$.

(2) P is *graded I-prime submodule* of M, if $P \neq M$; and whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P - IP$, then $r \in (P : M)$ or $m \in P$.

The following example shows that the class of graded prime submodules is a subclass of graded $I-$prime submodules.

Example 3.2. Let $R = \mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$, $G = \mathbb{Z}_2$ and $M = \mathbb{Z}_{54}[i]$. Then M is a graded R-module with $M_0 = \mathbb{Z}_{54}$ and $M_1 = i\mathbb{Z}_{54}$. Take $I = 27R$ as an ideal of R and $P = \langle 27i\rangle$ be a submodule of $\mathbb{Z}_{54}[i]$ generated by the element $27i$. Therefore P is a graded I-prime submodule of M since

$$\begin{aligned} P - IP &= \langle 27 + i27\rangle - 27R\langle 27 + i27\rangle \\ &= \langle 27 + i27\rangle - \langle 27 + i27\rangle \\ &= \phi. \end{aligned}$$

In otherside P is not graded prime submodule, since $3(9i) = 27i \in P$ but neither $3 \in (P : M)$ nor $9i \in P$.

Next, the following example is a graded I-prime submodule that is not I-prime submodule.

Example 3.3. Let $R = \mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$, $G = \mathbb{Z}_2$ and $M = R$ be an R-module. Take $I = 2R$ as a graded ideal of R and $P = 2M$. Then P is a graded submodule of M and $IP = 4M$. Therefore P is a graded I-prime submodule of M, but P as a submodule of M is not I-prime submodule since $(1-i)(1+i) = 2 \in P-IP$ and neither $(1-i) \in (P : M)$ nor $(1 + i) \in P$.

It is well-known that $P_g - I_eP_g \subseteq P - IP$, for any graded submodule P of M. Hence we conclude that for a graded I-prime submodule P of

M, P_g is an I_e-g-prime submodule of M_g for all $g \in G$. The converse is not true. See the following counter example.

Example 3.4. Let $R = \mathbb{Z}[\sqrt{3}]$, $G = \mathbb{Z}_2$ and $M = R$ be R-module. Take $I = 3\mathbb{Z}[\sqrt{3}]$ and $P = 3M = 3\mathbb{Z} + 3\sqrt{3}\mathbb{Z}$. Then $IP = 9\mathbb{Z}[\sqrt{3}]$ and $P_0 = 3\mathbb{Z}$, $P_1 = 3\sqrt{3}\mathbb{Z}$ are g-prime submoduls of M_g and also I-g-prime submodules for $g = 0, 1$. But P is not graded I-prime submodule of M, since $(2\sqrt{3})(\sqrt{3}) = 6 \in P - IP$ with neither $2\sqrt{3} \in (P : M)$ nor $\sqrt{3} \in P$.

We give a set of charactrizations of graded $I - g-$prime submodules and graded $I-$prime submodules.

Theorem 3.5. *Let P be a graded submodule of an R-module M and I an ideal of R_e. Then the following conditions are equivalent:*

(1) *P_g is an I-g-prime submodule of M_g;*
(2) *For $m \in M_g - P_g$, $(P_g : m) = (P_g : M_g) \cup (IP_g : m)$;*
(3) *For $m \in M_g - P_g$, $(P_g : m) = (P_g : M_g)$ or $(P_g : m) = (IP_g : m)$;*
(4) *If whenever $AK \subseteq P_g - IP_g$ with A an ideal of R_e and K an R_e-submodule of M_g, then $A \subseteq (P_g : M_g)$ or $K \subseteq P_g$.*

Proof. $(1 \Rightarrow 2)$ Let $m \in M_g - P_g$ and $a \in (P_g : m) - (IP_g : m)$. Then $am \in P_g - IP_g$. As P_g is I-g-prime submodule of M_g and $m \notin P_g$. So $a \in (P_g : M_g)$.

$(2 \Rightarrow 3)$ It is obvious.

$(3 \Rightarrow 4)$ Let A be an ideal of R_e and K be a submodule of M_g such that $AK \subseteq P_g$. Assume that $A \nsubseteq (P_g : M_g)$ and $K \nsubseteq P_g$. We must prove that $AK \subseteq IP_g$. Let $a \in A$ and $m \in K$. First let $a \notin (P_g : M_g)$. Since $am \in P_g$, then $a \in (P_g : m)$, that is $(P_g : m) \neq (P_g : M_g)$. Hence by our assumption $(P_g : m) = (IP_g : m)$. Therefore $a \in (IP_g : m)$. Hence $am \in IP_g$. Now if $a \in A \cap (P_g : M_g)$. Let $u \in A - (P_g : M_g)$. Then $a + u \in A - (P_g : M_g)$. By the first case, for each $m \in K$ we have $um \in IP_g$ and $(a + u)m \in IP_g$, that is $am + um \in IP_g$. Thus $am \in IP_g$. It means that $AK \subseteq IP_g$.

$(4 \Rightarrow 1)$ Let $rm \in P_g - IP_g$. Take $A = Rr$ and $K = Rm$. Then $AK \subseteq P_g - IP_g$, that is $AK \subseteq P_g$ and $AK \nsubseteq IP_g$. Therefore by assuming $A \subseteq (P_g : M_g)$ or $K \subseteq P_g$. Hence $r \in A \subseteq (P_g : M_g)$ or $m \in K \subseteq P_g$. Thus $r \in (P_g : M_g)$ or $m \in P_g$. It means that P_g is I-g-prime submodule of M_g. □

Theorem 3.6. *Let P be a graded submodule of an $R-$module M. Then the following conditions are equivalent.*

(1) *P is a graded I-prime submodule of M.*
(2) *For $r \in h(R) - (P : M), (P : r) = P \cup (IP : r)$.*

(3) *For* $r \in h(R) - (P : M), (P : r) = P$ *or* $(P : r) = (IP : r)$.

Proof. ($1 \Rightarrow 2$) Suppose P is a graded I-prime submodule. Take $r \in h(R) - (P : M)$ and $m \in (P : r)$. So $rm \in P$. If $rm \notin IP$, then P graded I-prime gives $m \in P$. If $rm \in IP$, then $m \in (IP : r)$.

($2 \Rightarrow 3$) It is clear.

($3 \Rightarrow 1$) Let $rm \in P - IP$ for $r \in h(R)$ and $m \in h(M)$. If $r \notin (P : M)$, then by hypothesis $(P : r) = P$ or $(P : r) = (IP : r)$. Since $rm \notin IP$, $m \notin (IP : r)$. But $m \in (P : r)$ which means that $(P : r) \neq (IP : r)$. Hence $(P : r) = P$ and so $m \in P$. Therefore P is a graded I-prime submodule of M. □

Theorem 3.7. *Let* P *be a graded submodule of* M*. Then the following conditions are equivalent.*

(1) P *is a graded* I*-prime submodule of* M*.*
(2) *For* $m \in h(M) - P$, $(P : m) = (P : M) \cup (IP : m)$.
(3) *For* $m \in h(M) - P$, $(P : m) = (P : M)$ *or* $(P : m) = (IP : m)$*;*
(4) *If whenever* $AK \subseteq P$ *with* $AK \nsubseteq IP$*, where* A *a graded ideal of* R *and* K *a graded* $R-$*submodule of* M*, then* $A \subseteq (P : M)$ *or* $K \subseteq P$*.*

Proof. ($1 \Rightarrow 2$) Let $mh(M) - P$ and $a \in h(P : m) - (IP : m)$. Then $am \in P - IP$ and as P is graded I-prime submodule of M and $m \notin P$, we have $a \in (P : M)$.

($2 \Rightarrow 3$) It is clear.

($3 \Rightarrow 4$) Let A be a graded ideal of R and K be a graded submodule of M such that $AK \subseteq P$. Assume that $A \nsubseteq (P : M)$ and $K \nsubseteq P$. Now we must prove that $AK \subseteq IP$. Let $a \in h(A)$ and $m \in h(K)$. First let $a \notin (P : M)$. Since $am \in P$, that is $a \in (P : m)$, we have $(P : m) \neq (P : M)$. Hence by our assumption $(P : m) = (IP : m)$. Therefore $a \in (IP : m)$. Thus $am \in IP$. Now if $a \in A \cap (P : M)$. Let $u \in h(A) - (P : M)$. Then $a + u \in A - (P : M)$. By the first case, for each $m \in h(K)$ we have $um \in IP$ and $(a + u)m \in IP$. Then $am + um \in IP$, that is $am \in IP$. Therefore $AK \subseteq IP$.

($4 \Rightarrow 1$) Let $rm \in P - IP$ for some $r \in h(R)$ and $m \in h(M)$. Take $A = Rr$ and $K = Rm$. Then $AK \subseteq P - IP$, that is $AK \subseteq P$ and $AK \nsubseteq IP$. Therefore by assuming $A \subseteq (P : M)$ or $K \subseteq P$. Thus $r \in A \subseteq (P : M)$ or $m \in K \subseteq P$. Hence $r \in (P : M)$ or $m \in P$. It is mean that P is graded I-prime submodule of M.

□

Theorem 3.8. *Let* P *be a graded submodule of* M*. Then the following conditions are equivalent.*

(1) P *is a graded* I*-prime submodule of* M*;*

(2) *For any graded submodule K of M with $K \nsubseteq P$,*
$$(P : K) = (P : M) \cup (IP : K).$$
(3) *For any graded submodule K of M with $K \nsubseteq P$,*
$$(P : K) = (P : M) \text{ or } (P : K) = (IP : K).$$

Proof. ($1 \Rightarrow 2$) Suppose that $K \subseteq M$ and $K \nsubseteq P$. Let $a \in h(P : K)$. Then $aK \subseteq P$. So $RaK \subseteq P$. If $RaK \nsubseteq IP$, and as P is graded I-prime submodule of M and $K \nsubseteq P$, $Ra \subseteq (P : M)$ and so $a \in (P : M)$. For the reverse inclusion, if $a \in h(P : M)$, then $aM \subseteq P$, and so $am \in P$ for all $m \in h(M)$. Hence $a \in (P : K)$. If $a \in (IP : K)$, then $aK \subseteq IP \subseteq P$, so $a \in (P : K)$. Thus $(P : K) = (P : M) \cup (IP : K)$.

($2 \Rightarrow 3$) It is clear.

($3 \Rightarrow 1$) Suppose that for $K \subseteq M$ and $K \nsubseteq P$, $(P : K) = (P : M)$ or $(P : K) = (IP : M)$, where K is submodule of M. Let $AK \subseteq P - IP$ where A is an ideal of R and K is a submodule of M. Since $K \nsubseteq P$, by Theorem 3.7 we must prove $A \subseteq (P : M)$. Since $AK \subseteq P$,
$$A \subseteq (P : K) = (P : M).$$
Hence we get (1). □

In the following we show that for any exact sequence
$$0 \to Q \to P \to \frac{P}{Q} \to 0$$
with $Q \subseteq P$ are graded submodules of M, P is graded I-prime submodule of M if and only if so are Q and $\frac{P}{Q}$.

Theorem 3.9. *Suppose that P and Q are graded submodules of M such that $Q \subseteq P$ with $P \neq M$. Then*

(1) *If P is a graded I-prime submodule of M, then $\frac{P}{Q}$ is a graded I-prime submodule of $\frac{M}{Q}$.*
(2) *If Q and $\frac{P}{Q}$ are graded I-prime submodule, then P is a graded I-prime submodule of M.*

Proof. (1) For all $r \in h(R), m \in h(M)$, let
$$r(m + Q) = rm + Q \in \tfrac{P}{Q} - I\tfrac{P}{Q}.$$
Then $rm + Q \in \frac{P}{Q} - \frac{IP+Q}{Q}$. So $rm \in P - IP$, because if $rm \in IP$, then $rm \in IP + Q$, so $rm + Q \in \frac{IP+Q}{Q}$ which is a contradiction. As P is graded I-prime submodule, $r \in (P : M)$ or $m \in P$. Thus $r \in (\frac{P}{Q} : \frac{M}{Q})$ or $m + Q \in \frac{P}{Q}$.

(2) Let $am \in P - IP$ where $a \in h(R)$ and $m \in h(M)$. Then $a(m+Q) = am + Q \in \frac{P}{Q}$. If $am \in Q$, since $am \notin IP$ and $Q \subseteq P$, then $IQ \subseteq IP$. So $am \nsubseteq IQ$. Hence $am \in Q - IQ$. As Q is a graded I-prime submodule, $a \in (Q : M) \subseteq (P : M)$ or $m \in Q \subseteq P$. If $am \notin Q$, then we have $am + Q \notin \frac{IP+Q}{Q}$ and $a(m+Q) \in \frac{P}{Q} - I\frac{P}{Q}$. Since $\frac{P}{Q}$ is a graded I-prime submodule, we have $m + Q \in \frac{P}{Q}$ or $a \in (\frac{P}{Q} : \frac{M}{Q})$. Hence $m \in P$ or $a \in (P : M)$. $\square$

Proposition 3.10. (1) *Let P be a graded submodule of M and $g \in G$. Then P_g is an I-g-prime submodule in M_g if and only if $\frac{P_g}{IP_g}$ is a $\{0\}$-g-prime submodule in $\frac{M_g}{IP_g}$.*

(2) *A graded submodule P of an R-module M is graded I-prime submodule of M if and only if $\frac{P}{IP}$ is graded $\{0\}$-prime submodule of $\frac{M}{IP}$.*

Proof. (1) Let P_g be an I-g-prime submodule in M_g. Let $r \in R_e$ and $m \in M_g$ such that

$$0 \neq r(m + IP_g) \in \frac{P_g}{IP_g}.$$

Then $rm \in P_g - IP_g$. So $r \in (P_g : M_g)$ or $m \in P_g$, because P_g is I-g-prime submodule in M_g. Therefore $r \in (\frac{P_g}{IP_g} : \frac{M_g}{IP_g})$ or $m + IP_g \in \frac{P_g}{IP_g}$. Hence $\frac{P}{IP}$ is $\{0\}$-g-prime submodule in $\frac{M_g}{IP_g}$. Conversely, suppose that $\frac{P_g}{IP_g}$ is $\{0\}$-g-prime submodule in $\frac{M_g}{IP_g}$. Let $r \in R_e$ and $m \in M_g$ such that $rm \in P_g - IP_g$. Then $0 \neq r(m + IP_g) \in \frac{P_g}{IP_g}$. So $m + IP_g \in \frac{P_g}{IP_g}$ or $r \in (\frac{P_g}{IP_g} : \frac{M_g}{IP_g})$. Thus $m \in P_g$ or $r \in (P_g : M_g)$. This means that P_g is I-g-prime submodule in M_g.

(2) Let P be graded I-prime submodule of M. Let $r \in h(R)$ and $m \in h(M)$ such that $0 \neq r(m + IP) \in \frac{P}{IP}$. Then $rm \in P - IP$ and as P is graded $I-$prime submodule in M, $r \in (P : M)$ or $m \in P$, that is $r \in (\frac{P}{IP} : \frac{M}{IP})$ or $m + IP \in \frac{P}{IP}$. Thus $\frac{P}{IP}$ is graded $\{0\}$-prime submodule in $\frac{M}{IP}$. Conversely, suppose that $\frac{P}{IP}$ is graded $\{0\}$-prime submodule in $\frac{M}{IP}$. Let $r \in h(R)$ and $m \in h(M)$ such that $rm \in P - IP$. Then $0 \neq r(m + IP) \in \frac{P}{IP}$. Therefore $m + IP \in \frac{P}{IP}$ or $r \in (\frac{P}{IP} : \frac{M}{IP})$, that is $m \in P$ or $r \in (P : M)$. Hence P is graded I-prime submodule in M. $\square$

Proposition 3.11. *Suppose G is a group with identity e and R is a $G-$graded ring and M is a $G-$graded $R-$module. Then for a graded I-prime submodule P of M, we have*

(1) *If $S \subseteq h(R)$ is a multiplicative closed subset of R with $(P : M) \cap S = \phi$, then $S^{-1}P$ is graded I-prime submodule of $S^{-1}M$.*
(2) *Let S be a multiplicative closed subset of $h(R)$ with $S^{-1}P \neq S^{-1}M$ and $S^{-1}(IP) \subseteq S^{-1}IS^{-1}P$. Then $S^{-1}P$ is a graded $S^{-1}I$-prime submodule of $S^{-1}M$.*

Proof. (1) Let $r \in h(R)$ and $\frac{m}{t} \in h(S^{-1}M)$ with

$$r\tfrac{m}{t} \in S^{-1}P - IS^{-1}P.$$

Then $\frac{rm}{t} \in S^{-1}P - S^{-1}(IP)$. So there is $s \in S$ and $p \in h(P)$ such that $\frac{rm}{t} = \frac{p}{s}$. Then there is $u \in S$ such that

$$surm = utp \in P.$$

Moreover for any $v \in S, vrm \notin IP$. So $usrm \in P - IP$. Since P is graded I-prime submodule of M, then $rm \in P$ and so $rm \in P - IP$. Hence $r \in (P : M) \subseteq (S^{-1}P : S^{-1}M)$ or $m \in P$, that is $\frac{m}{t} \in S^{-1}P$.

(2) For all $\frac{r}{s} \in h(S^{-1}R)$, $\frac{m}{t} \in h(S^{-1}M)$, let

$$\tfrac{r}{s}.\tfrac{m}{t} \in S^{-1}P - S^{-1}IS^{-1}P \subseteq S^{-1}P - S^{-1}(IP).$$

So $urm \in P - IP$ for some $u \in S$, and as P is graded I-prime, either $ur \in (P : M)$ or $m \in P$. Therefore

$$\tfrac{ur}{us} = \tfrac{r}{s} \in S^{-1}(P : M) = (S^{-1}P : S^{-1}M)$$

or $\frac{m}{t} \in S^{-1}P$. This means that $S^{-1}P$ is graded $S^{-1}I$-prime submodule of $S^{-1}M$. □

Recall that a graded proper ideal P of a graded ring R is *graded prime* if for any $a, b \in h(R)$ with $ab \in P$, we get $a \in P$ or $b \in P$. A graded proper submodule P of a graded module M over graded ring R is *graded prime submodule* if for any $r \in h(R)$ and $m \in M$ with $rm \in P$, we get $m \in P$ or $rM \subseteq P$.

Theorem 3.12. *Let M be an R-module and $r \in h(R)$ such that $rM \neq M$ with $Ann_M(r) \subseteq rM$ and $I \subseteq (rm : M)$. Then rM is a graded I-prime submodule of M if and only if it is a graded prime submodule of M.*

Proof. It is clear that every graded prime is a graded I-prime. For the converse part, let $sm \in rM$ where $s \in h(R)$ and $m \in h(M)$. We have to prove $s \in (rM : M)$ or $m \in rM$. If $sm \notin I(rM)$, then as rM is graded I-prime, $s \in (rM : M)$ or $m \in rM$. Now, if $sm \in I(rM)$, then $(s + r)m \in rM$. If $(s + r)m \notin I(rM)$, then as rM is graded I-prime, $s + r \in (rM : M)$ or $m \in rM$, that is, $s \in (rM : M)$ or $m \in rM$. For the case $(s+r)m \in I(rM)$, $sm \in I(rM)$ implies $rm \in I(rM)$. Hence, there exists $y \in IM$ such that $rm = ry$ and so $m - y \in Ann_M(r)$. This implies that $m \in IM + Ann_M(r) \subseteq rM + Ann_M(r) \subseteq rM$ and hence we get the result. □

By a particular condition, graded $I-$prime ideals and graded prime ideals are coincide in graded principal ideal rings.

Corollary 3.13. *Let r be a nonzero graded element in a graded ring R with $(0 : Rr) \subseteq Rr$ and $IRr \subseteq Rr^2$. Then Rr is graded I-prime ideal if and only if Rr is graded prime ideal of R.*

Proof. Suppose that Rr is a graded I-prime ideal of R. If Rr is not graded prime, then there exist graded elements $a, b \in R$ such that $ab \in Rr$ but $a \notin Rr$ and $b \notin Rr$. Since Rr is a graded I-prime ideal, $ab \in IRr$. Now we have $a(b+r) \in Rr$. If $a(b+r) \notin IRr$, then $a \in Rr$ or $(b+r) \in Rr$, that is, $a \in Rr$ or $b \in Rr$, which is a contradiction. Therefore $a(b+r) \in IRr$ and we have $ar \in IRr$, that is $ar \in Rr^2$. So there is an element $d \in R$ such that $ar = dr^2$, that is $r(a - dr) = 0$ and as $(0 : r) \subseteq Rr$, we have $(a - dr) \in Rr$ and so $a \in Rr$ which is a contradiction. Hence Rr is a graded prime ideal of R.
The converse part is obvious. □

Example 3.14. Consider the graded ring $R = \dfrac{K[x]}{< x^2 - x^4 >}$ and the graded ideals $P = I =< \overline{x}^2 >$ where K is a field. Then the graded ideal $P =< \overline{x}^2 >$ is an $I-$prime since $P - IP = \emptyset$. But P is not graded prime cause the first condition $(0 :< x^2 >) =< 1-\overline{x}^2 > \nsubseteq < \overline{x}^2 >$ of the Corollary 3.13 does not satisfied. Similarly, if we take $P =< \overline{x}^3 >$ and $I =< \overline{x}^2 >$ in the graded ring $S = \dfrac{K[x]}{< x^3 - x^5 >}$. Then P is $I-$prime but not prime since the second condition of Corollary 3.13 does not satisfied cause $IP =< \overline{x}^3 > \nsubseteq P^2 =< \overline{x}^4 >$.

Example 3.15. Let $P = I = 2Z$ be ideals in the ring of integers Z which is a trivial graded principal ideal ring. Then the conditions of the Corollary 3.13 are satisfied and so P is $I-$prime ideal as well as it is a prime ideal of Z.

Lemma 3.16. *Let M be an R-module and $0 \neq x \in h(M)$ such that $Rx \neq M$ and $Ann(x) = 0$. Then $I-$primeness of Rx implies primeness of Rx for $I = (Rx : M)$.*

Proof. Suppose Rx is not graded prime submodule of M. Then there exist $r \in h(R)$ and $m \in h(M)$ such that $rm \in Rx$ with $r \notin (Rx : M)$ and $m \notin Rx$. If $rm \notin (Rx : M)x$, then Rx is not graded I-prime submodule of M. Assume $rm \in (Rx : M)x$. But $m + x \notin Rx$ and $r(m+x) \in Rx$. If $r(m+x) \notin (Rx : M)x$, then Rx is not graded I-prime submodule of M, since $r \notin (Rx : M)$. So we get $r(m+x) \in (Rx : M)x$. Then $rx \in (Rx : M)x$ and so $rx = sx$ for some $s \in (Rx : M)$. Thus $(r - s)x = 0$ and $r - s \in Ann(x) = 0$ which give us $r = s$ and so $r \in (Rx : M)$. Thus the result is obtained. $\square$

In the following we link graded I-prime submodules of M and I-prime ideals of R together as follows .

Theorem 3.17. *Let M be a finitely generated faithful multiplication graded R-module and P a proper graded submodule of M with $(IP : M) = I(P : M)$. Then P is graded I-prime submodule of M if and only if $(P : M)$ is a graded I-prime ideal of R.*

Proof. Assume that P is a graded I-prime submodule of M. We have to prove $(P : M)$ is graded I-prime ideal of R. Let $a, b \in h(R)$ such that $ab \in (P : M) - I(P : M)$. Then $abM \subseteq P$. If $abM \subseteq IP$, then $ab \in (IP : M) = I(P : M)$ which is a contradiction. Hence $abM \nsubseteq IP$. Then $abM \subseteq P$ and $abM \nsubseteq IP$. So $a \in (P : M)$ or $bM \subseteq P$. It means that $a \in (P : M)$ or $b \in (P : M)$, that is, $(P : M)$ is a graded I-prime ideal of R. For the converse, assume $(P : M)$ is a graded I-prime ideal of R. Let $rm \in P - IP$, where $r \in h(R)$ and $m \in h(M)$. Then $r(Rm : M) \subseteq (rRm : M) \subseteq (P : M)$ and $r(Rm : M) \nsubseteq I(P : M)$. Otherwise, we will get

$$rRm = r(Rm : M)M \subseteq I(P : M)M = IP$$

which is a contradiction. Hence

$$r(Rm : M) \subseteq (P : M)$$

and $r(Rm : M) \nsubseteq I(P : M)$. As $(P : M)$ is a graded I-prime ideal, $r \in (P : M)$ or $(Rm : M) \subseteq (P : M)$. So $r \in (P : M)$ or $Rm \subseteq P$. Therefore $r \in (P : M)$ or $m \in P$. This means that P is a graded I-prime submodule of M. $\square$

Remark 3.18. Let $I_1 \subseteq I_2$ be ideals of R_e. Then every graded I_1-prime submodule is a graded I_2-prime. For this suppose that P is graded I_1-prime. Take $r \in h(R)$ and $m \in h(M)$ with $rm \in P - I_2P$. Since $I_1 \subseteq I_2$, $I_1P \subseteq I_2P$ and thus $P - I_2P \subseteq P - I_1P$. Then $rm \in P - I_1P$. Since P is a graded I_1-prime submodule, we have $r \in (P : M)$ or $m \in P$. Hence P is graded I_2-prime.

Theorem 3.19. *Let P be a graded I-prime submodule of M with $(P : M) = J$. Then there is a one-to-one correspondence between* *graded I-prime submodules of $\frac{R}{J}$-module $\frac{M}{P}$ and the graded I-prime submodules of the graded R-module M containing P.*

Proof. Assume that Q is a proper graded submodule of M containing P. Suppose that Q is graded I-prime. We want to prove that $\frac{Q}{P}$ is graded I-prime submodule of $\frac{M}{P}$. Now $\frac{Q}{P}$ is a proper submodule of $\frac{M}{P}$ since Q is proper in M. Let

$$(a+P)(m+P) \in \tfrac{Q}{P} - I\tfrac{Q}{P},$$

where $a \in h(R)$ and $m \in h(M)$. Then $am + P \in \frac{Q}{P} - \frac{IQ+P}{P}$. Therefore $am + P \notin \frac{IQ+P}{P}$ and hence $am \notin IQ + P$, then $am \notin IQ$, because if $am \in IQ$, then $am \in IQ + P$. Thus $am \in Q - IQ$. Since Q is a graded I-prime submodule, we get that $a \in (Q : M)$ or $m \in Q$. So $m+P \in \frac{Q}{P}$ or $a+P \in (\frac{Q}{P} : \frac{M}{P})$. Conversely, suppose that $\frac{Q}{P}$ is a graded I-prime submodule of $\frac{M}{P}$ and by assumption P is graded I-prime. So by Theorem 3.9 (2) Q is graded I-prime. □

Since zero submodule is graded $I-$prime, by taking $P = 0$ in Theorem 3.19, we get a one to one correspondence between graded $I-$prime submodules of M as $R-$module and as $\dfrac{R}{Ann(M)}-$module. The following characterizes $I-$prime submodules in finitely generated faithful multiplication graded modules.

Theorem 3.20. *Let P be a proper graded submodule of a finitely generated faithful multiplication graded R-module M such that $I(P : M) = (IP : M)$. Then P is a graded I-prime submodule of M if and only if whenever A and B are graded submodules of M such that $A.B \subseteq P$ and $A.B \nsubseteq IP$, then either $A \subseteq P$ or $B \subseteq P$.*

Proof. Suppose that P is a graded I-prime submodule of M and A, B are two graded submodules of M. As M is multiplication R-module, $A = (A : M)M$ and $B = (B : M)M$ and so $A.B = (A : M)(B : M)M$. Suppose that $A.B \subseteq P$ and $A.B \nsubseteq IP$ with $A \nsubseteq P$ and $B \nsubseteq P$. Then $(A : M) \nsubseteq (P : M)$ and $(B : M) \nsubseteq (P : M)$. Since $(P : M)$ is graded I-prime by Theorem 3.17, either $(A : M)(B : M) \nsubseteq (P : M)$ or $(A : M)(B : M) \subseteq I(P : M)$. In the first case, we have

$$AB = (A : M)(B : M)M \nsubseteq (P : M)M = P,$$

which is a contradiction. Now in the other case we get

$$AB = (A : M)(B : M)M \subseteq I(P : M)M = IP$$

which is again contradiction. So either $A \subseteq P$ or $B \subseteq P$. For the converse, by Theorem 3.17, it is enough to prove that $(P : M)$ is a graded I-prime ideal of R. Let $rs \in (P : M) - I(P : M)$, where $r, s \in h(R)$ such that $r \notin (P : M)$ and $s \notin (P : M)$. Take $A = rM$ and $B = sM$. Then $AB = rsM \subseteq (P : M)M = P$ and $AB = rsM \nsubseteq IP$, because if $AB = rsM \subseteq IP$, then $rs \in (IP : M) = I(P : M)$ which is a contradiction. Hence $AB \subseteq P - IP$, so by hypothesis we get either $A = rM \subseteq P$ or $B = sM \subseteq P$. Therefore $r \in (P : M)$ or $s \in (P : M)$. We get the result. $\square$

Theorem 3.21. *Let M be a multiplication graded R-module and P a graded submodule of M. Then P is a graded I-prime submodule in M if and only if, for any graded ideal J of R and graded submodule N of M with $JN \subseteq P$ and $JN \nsubseteq IP$, implies that $J \subseteq (P : M)$ or $N \subseteq P$.*

Proof. Suppose that P is a graded I-prime submodule of M and there exists a graded ideal J of R and a graded submodule N of M such that $JN \subseteq P$ but $J \nsubseteq (P : M)$ and $N \nsubseteq P$. We claim that $JN \subseteq IP$. Let $r \in h(M)$ such that $r \in J - (P : M)$. Then $rN \subseteq P$ implies that $N \subseteq (P : Rr)$. Since P is a graded I-prime submodule and $N \nsubseteq P$, we get by Theorem 3.7 that $N \subseteq (IP : Rr)$. Thus $rN \subseteq IP$. Suppose $r \in J \cap (P : M)$ and let $s \in h(M)$ such that $s \in (N : M)$. If $s \in I$, then $rs \in I(P : M) \subseteq (IP : M)$ and so $rsM \subseteq IP$. If $s \in (N : M) - I$, then $sM \subseteq N$ and so $sJM \subseteq JN \subseteq P$. Thus $JM \subseteq (P : Rs)$ and $JM \nsubseteq P$ since by our assumption $J \nsubseteq (P : M)$. Again by Theorem 3.7 we get that $JM \subseteq (IP : Rs)$ and so $sJM \subseteq IP$. As s is arbitrary in $(N : M)$, $rN = r(N : M)M \subseteq IP$. Hence $JN \subseteq IP$. For the converse, the condition that M is a multiplication R-module is not required. Let $rm \in P - IP$ for $r \in h(R)$ and $m \in h(M)$. Then $(Rr)(Rm) \subseteq P - IP$ and so either $Rr \subseteq (P : M)$ or $Rm \subseteq P$. Therefore, $r \in (P : M)$ or $m \in P$. This means that P is a graded I-prime submodule of M. $\square$

Theorem 3.22. *Let R be a G-graded ring and M_1, M_2 be two graded R-modules with a graded R-epimorphism $f \colon M_1 \to M_2$. Then a graded submodule Q of M_2 such that $f(M_1) \nsubseteq Q$ is graded I-prime submodule of M_2 if and only if $f^{-1}(Q)$ is a graded I-prime submodule of M_1.*

Proof. To show that $f^{-1}(Q)$ is a proper submodule of M_1, let $f^{-1}(Q) = M_1$ and $m \in h(M_1)$. Thus $m \in f^{-1}(Q)$, and $f(m) \in Q$, that is, $f(M_1) \subseteq Q$ which contradics our assumption. Now, suppose Q is a graded I-prime submodule of M_2 and let $r \in h(R)$, $m \in h(M_1)$ such that

$$rm \in f^{-1}(Q) - If^{-1}(Q) = f^{-1}(Q) - f^{-1}(IQ) = f^{-1}(Q - IQ).$$

Assume that $m \notin f^{-1}(Q)$. This implies that $f(m) \notin Q$. Since $rm \in f^{-1}(Q - IQ)$, then $f(rm) \in Q - IQ$, that is $rf(m) \in Q - IQ$. As Q is a graded I-prime submodule of M_2 and $f(m) \notin Q$, $r \in (Q : M_2)$. Hence $rM_2 \subseteq Q$ and $f(M_1) \subseteq M_2$. Therefore $rf(M_1) \subseteq rM_2 \subseteq Q$. This means $rM_1 \subseteq f^{-1}(Q)$, so $r \in (f^{-1}(Q) : M_1)$. Conversely, suppose that $f^{-1}(Q)$ is a graded I-prime submodule of M_1 and let $r \in h(R)$ and $m \in h(M_2)$ such that $rm \in Q - IQ$ and $m \notin Q$. Since f is graded epimorphism, there exists $n \in h(M_1)$ such that $f(n) = m$. Thus $f(n) \notin Q$ and so $n \notin f^{-1}(Q)$. Hence $rf(n) = f(rn) \in Q - IQ$, so $rn \in f^{-1}(Q) - If^{-1}(Q)$. Since $f^{-1}(Q)$ is graded I-prime submodule of M_1 and $n \notin f^{-1}(Q)$, $r \in (f^{-1}(Q) : M_1)$. Thus $rM_1 \subseteq f^{-1}(Q)$ and $rM_2 \subseteq Q$. Therefore Q is graded I-prime submodule of M_2. □

Let $f : M_1 \to M_2$ be an epimorphism of $R-$modules. According to Theorem 3.22, there is a one to one correspondence between graded $I-$prime submodules P of M_1 with $kerf \subseteq P$ and proper graded $I-$prime submodules $f(P)$ of M_2. In the following, we study graded $I-$prime submodules in decomposition graded modules (bigraded modules).

Theorem 3.23. *Let R_i be a graded ring and M_i a graded R_i-module for $i = 1, 2$ with $R = R_1 \times R_2$ and $M = M_1 \times M_2$ a graded R-module. Let I_1 and I_2 be two ideals of $(R_1)_e$ and $(R_2)_e$ respectively with $I = I_1 \times I_2$ and P_1, P_2 be proper graded submodules of M_1 and M_2 respectively.*

(1) *If $I_2M_2 = M_2$, then P_1 is a graded I_1-prime submodule of M_1 if and only if $P_1 \times M_2$ is a graded I-prime submodule of M.*
(2) *If $I_1M_1 = M_1$, then P_2 is a graded I_2-prime submodule of M_2 if and only if $M_1 \times P_2$ is a graded I-prime submodule of M.*
(3) *If $I_iP_i = P_i$ for $i = 1, 2$, then $P_1 \times P_2$ is graded I-prime submodule of M.*

Proof. (1) Suppose that P_1 is a graded I_1-prime submodule of M_1 and $I_2M_2 = M_2$. Let $(r_1, r_2) \in h(R)$ and $(m_1, m_2) \in h(M)$ such that

$$\begin{aligned}(r_1, r_2)(m_1, m_2) &= (r_1m_1, r_2m_2) \in (P_1 \times M_2) - I(P_1 \times M_2)\\ &= (P_1 \times M_2) - (I_1 \times I_2)(P_1 \times M_2)\\ &= (P_1 \times M_2) - (I_1P_1 \times I_2M_2)\\ &= (P_1 \times M_2) - (I_1P_1 \times M_2)\\ &= (P_1 - I_1P_1) \times M_2.\end{aligned}$$

Thus $r_1m_1 \in P_1 - I_1P_1$ and P_1 is a graded I_1-prime submodule of M_1, so $r_1 \in (P_1 :_{R_1} M_1)$ or $m_1 \in P_1$. Hence

$$(r_1, r_2) \in (P_1 :_{R_1} M_1) \times R_2 = (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2)$$

or $(m_1, m_2) \in P_1 \times M_2$. Therefore $P_1 \times M_2$ is a graded I-prime submodule of M. Conversely, assume that $P_1 \times M_2$ is a graded I-prime submodule of M. Take $r_1 \in h(R_1)$ and $m_1 \in h(M_1)$ with $r_1 m_1 \in P_1 - I_1 P_1$. Then

$$\begin{aligned} (r_1 m_1, 0) \in (P_1 - I_1 P_1) \times M_2 &= (P_1 \times M_2) - (I_1 P_1 \times M_2) \\ &= (P_1 \times M_2) - (I_1 P_1 \times I_2 M_2) \\ &= (P_1 \times M_2) - I(P_1 \times M_2) \end{aligned}$$

and as $P_1 \times M_2$ is a graded I-prime,

$$(r_1, 0) \in (P_1 \times M_2 :_{R_1 \times R_2} M_1 \times M_2) = (P_1 :_{R_1} M_1) \times R_2$$

or $(m_1, 0) \in P_1 \times M_2$. Therefore $r_1 \in (P_1 :_{R_1} M_1)$ or $m_1 \in P_1$, and P_1 becomes a graded I_1-prime submodule of M_1.

(2) The proof is similar to part (1).

(3) $I_1 P_1 = P_1$ and $I_2 P_2 = P_2$ implies

$$P_1 \times P_2 = I_1 P_1 \times I_2 P_2 = I(P_1 \times P_2).$$

Hence $P_1 \times P_2$ is a graded I-prime submodule of M. □

Now we state the graded $I-$prime submodules version of [5, Theorem 2.16].

Theorem 3.24. *Assume P is a graded I-prime submodule of M and F a flat $R-$ module such that $F \otimes P \neq F \otimes M$. Then $F \otimes P$ is graded I-prime submodule of $F \otimes M$.*

Proof. Let P be a graded I-prime submodule of graded R-module M and $r \in h(R) - (P : M)$. Theorem 3.6 implies $(P : r) = P$ or $(P : r) = (IP : r)$ and [5, Lemma 2.15] implies

$$(F \otimes P : r) = F \otimes (P : r) = F \otimes P$$

or $(F \otimes P : r) = F \otimes (P : r) = F \otimes (IP : r) = (I(F \otimes P) : r)$. Therefore $F \otimes P$ is graded I-prime submodule of $F \otimes M$. □

The converse of Theorem 3.24 is true under assumption that F is graded faithfully flat $R-$module. In fact $F \otimes P \neq F \otimes M$ if and only if $P \neq M$, and for $r \in h(R)$, $r \notin (P : m)$ implies $r \notin (F \otimes P : F \otimes m)$. Also $F \otimes (P : r) = F \otimes P$ and $F \otimes (P : r) = F \otimes (IP : r)$ implies $(P : r) = P$ and $(P : r) = (IP : r)$ respectively by faithfully flatness of F. Hence the result follows by Theorem 3.6.

It is well-known that if M is finitely generated graded module M over Noetherian graded ring R the completion $\hat{M}$ of a graded $R-$module M is isomorphic to $\hat{R} \otimes M$. Hence we deduce the following corollary.

Corollary 3.25. *Let P be a graded I-prime submodule of a finitely generated R-module M and R a Notherian graded ring. Then its completion $\hat{P}$ is graded I-prime submodule of the complete $\hat{R}$-module $\hat{M}$.*

Finally, by taking polynomial ring as an especial example of graded rings we have the following result in which the ideals and the submodules not necessarily need be graded.

Proposition 3.26. *Let P be I-prime submodule of an R-module M. Then $P[x]$ is an $I[x]$-prime submodule of $R[x]$-module $M[x]$, where x is an indeterminate.*

Proof. Assume that $r(x)m(x) \in P[x] - I[x]P[x] = P[x] - (IP)[x]$, for $r(x) \in R[x]$, $m(x) \in M[x]$. Let $r(x) = \sum_{i=0}^{n} r_i x^i$ and

$$m(x) = \sum_{j=1}^{m} m_j x^j$$

for $r_i \in R$, $m_j \in M$. Since

$$r(x)m(x) = \sum_{k=0}^{n+m} (\sum_{i+j=k} r_i m_j) x^k \in (P - IP)[x]$$

for $i = 0, 1, ..., n$ and $j = 0, 1, ..., m$, $\sum_{i+j=k} r_i m_j \in P - IP$ for all $k = 0, 1, ..., n+m$ and as P is I-prime submodule of M, we have recursively that $r_i \in (P : M)$ or $m_j \in P$ and this means

$$r_i x^i \in (P[x] : M[x])$$

or $m_j x^j \in P[x]$. Therefore $r(x) \in (P[x] : M[x])$ or $m(x) \in P[x]$ and so $P[x]$ is $I[x]$-prime submodule of $M[x]$. $\square$

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Journal of Algebraic Systems

GRADED I-PRIME SUBMODULES

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زیرمدول‌های I-اول مدرج

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فرض کنید $R=\bigoplus_{g\in G} R_g$ یک حلقه‌ی جابه‌جایی G-مدرج یکدار، I یک ایده‌آل مدرج و M یک R-مدول مدرج یکدار باشد، جایی‌که G یک نیم‌گروه با عضو همانی e است. ما ایده‌آل‌های (زیرمدول‌های) I-اول مدرج را معرفی می‌کنیم که تعمیمی از ایده‌آل‌های (زیرمدول‌های) اول هستند. این مفاهیم جدید، دارای خواصی مشابه مفاهیم کلاسیک هستند. به ویژه، به مطالعه‌ی قضیه موضعی‌سازی این دو مفهوم می‌پردازیم. نشان می‌دهیم که برای یک مدول تخت وفادار F، زیرمدول مدرج P از M، I-اول است اگر و تنها اگر $F\otimes P$، زیرمدول I-اول مدرج از $F\otimes P$ باشد. به عنوان یک کاربرد، اگر M یک مدول مدرج متناهیاً تولید شده روی حلقه‌ی مدرج نوتری R باشد، آن‌گاه مکمل زیرمدول‌های I-اول مدرج، زیرمدول I-اول هستند.

کلمات کلیدی: ایده‌آل‌های I-اول، زیرمدول I-اول، ایده‌آل اول مدرج، زیرمدول اول مدرج.