Journal of Algebraic Systems

Vol. 10, No. 2, (2023), pp 245-258

FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE MINIMAXNESS OF LOCAL COHOMOLOGY MODULES DEFINED BY A SYSTEM OF IDEALS

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Abstract. Let R be a commutative Noetherian ring and ϕ a system of ideals of R. We prove that, in certain cases, there are local–global principles for the finiteness and minimaxness of generalized local cohomology module $H^i_\phi(M,N)$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R and M, N are R-modules. The i-th generalized local cohomology functor $H^i_{\mathfrak{a}}(M,N)$ is defined by $H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n>0} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$ for all $i \in \mathbb{N}$.

Let ϕ be a non-empty set of ideals of R. We say that ϕ is a system of ideals of R if $\mathfrak{a}_1, \mathfrak{a}_2 \in \phi$, then there is an ideal $\mathfrak{b} \in \phi$ such that $\mathfrak{b} \subseteq \mathfrak{a}_1\mathfrak{a}_2$. In such a system, for every R-module N, one can define

DOI: 10.22044/JAS.2022.10587.1524.

MSC(2010): Primary: 13C05; Secondary: 13C11, 13E05.

Keywords: Generalized local cohomology modules; Faltings; Minimax modules.

Received: 27 February 2021, Accepted: 17 January 2022.

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$\Gamma_\phi(N)=\{x\in N|\mathfrak{a}x=0$ for some $\mathfrak{a}\in\phi\}$. For each $i\geq 0$, the i-th right derive functor of $\Gamma_\phi(-)$ is denoted by $H^i_\phi(-)$. Some basic properties of the local cohomology modules with respect to ϕ were shown in [3], [4].

Another generalization of local cohomology functor was given by Bijan-Zadeh [3]. For each $i\geq 0$, $H^i_\phi(-,-)$ is the functor defined by $H^i_\phi(M,N)=\varinjlim_{\mathfrak{a}\in\phi}\operatorname{Ext}^i_R(M/\mathfrak{a}M,N)$ for all R-modules M,N and $i\in\mathbb{N}_0$.

The functor $H^i_\phi(-,-)$ is R-linear which is contravariant in the first variable and covariant in the second variable. If $\phi=\{\mathfrak{a}^n|n\in\mathbb{N}\}$, then $H^i_\phi(-,-)$ is naturally equivalent to $H^i_\mathfrak{a}(-,-)$. An important theorem in local cohomology is Faltings' local-global principle for the finiteness Dimension of local cohomology modules [[8], satz 1], which states that for a positive integer r, then the $R_\mathfrak{p}$-module $H^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p})$ is finitely generated for all $i\leq r$ and for all $\mathfrak{p}\in\operatorname{Spec}(R)$ if and only if the R-module $H^i_\mathfrak{a}(M)$ is finitely generated for all $i\leq r$.

Faltings' local-global principle for the finiteness of local cohomology modules has been studied by several authors (for example see [7], [8]).

Aghapournahr et al. ([1], Theorem 2.8) studied the concept of the local-global principle for the minimaxness of ordinary local cohomology modules. The purpose of the present paper is genealization of Faltings' local-global principle of ordinary local cohomology to generalized local cohomology modules with respect to a systems of ideals. More precisely, we shall prove the following:

Theorem 1.1. *Let M,N be two finitely generated R-modules and $t\in\mathbb{N}$. If $S^{-1}\phi=\{S^{-1}\mathfrak{a}|a\in\phi\}$, then consider the following statements:*

(i) *$H^i_\phi(M,N)$ is finitely generated for all $i<t$;*

(ii) *There is an ideal $\mathfrak{c}\in\phi$ such that $\mathfrak{c}H^i_\phi(M,N)=0$ for all $i<t$;*

(iii) *There is an ideal* $\mathfrak{c} \in \phi$ *such that* $\mathfrak{c} \subseteq \operatorname{Rad}(0 :_R H^i_\phi(M, N))$ *for all* $i < t$.

(iv) $H^i_{S^{-1}\phi}(S^{-1}M, S^{-1}N)$ *is finitely generated* $S^{-1}R$*-module for all* $i < t$, *where* $S = R - \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

Then the following implications are true.

(a) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

(b) (iv) $\Rightarrow$ (i), *if* $\operatorname{Max}(\phi)$ *is finite.*

Theorem 1.2. *Let* $(R, \mathfrak{m})$ *be a local ring and* M, N *be two finitely generated* R*-module and* $t \geq 1$ *be an integer. Consider the following statements:*

(i) $H^i_\phi(M, N)$ *is a minimax* R*-module for all* $i < t$;

(ii) *There exists* $\mathfrak{a} \in \phi$ *such that* $\mathfrak{a}H^i_\phi(M, N)$ *is a minimax* R*-module for all* $i < t$;

(iii) $H^i_{\phi_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ *is a finitely generated* $R_\mathfrak{p}$*-module for all* $i < t$ *and* $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$.

Then,

(a) (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii),

(b) (iii) $\Rightarrow$ (i), *If* $\operatorname{Max}(\phi)$ *is finite set.*

For any unexplained notion and terminology is eferred to [5] and [6].

2. Main results

In this section, we investigate the finiteness and minimaxness of local cohomology modules with respect to a system of ideals of R. In particular, we prove that there is a Falthings' local-global principle for the minimaxness of local cohomology modules with respect to a system of ideals.

Remark 2.1. Let ϕ be a non-empty set of ideals of R.

(i) We call ϕ a system of ideals of R whenever $\mathfrak{a}_1, \mathfrak{a}_2 \in \phi$, there is an ideal $\mathfrak{b} \in \phi$ such that $\mathfrak{b} \subseteq \mathfrak{a}_1\mathfrak{a}_2$. Define the relative $\leq$ on ϕ as follows; $\mathfrak{a} \leq \mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$. It is clear that this relation is a partial order on ϕ.

(ii) Using Zorn's Lemma ϕ has maximal element, we use $Max(\phi)$ to denote the set of all maximal elements of ϕ. Moreover, if $Max(\phi)$ is finite set, it has a unique maximal element.

(iii) The ϕ-torsion submodule $\Gamma_\phi(N)$ of N is defined as follows:

$$\Gamma_\phi(N) = \{x \in N | \mathfrak{a}x = 0 \, \text{for some} \, \mathfrak{a} \in \phi\}.$$

It is straightforward to see tat $\Gamma_\phi(N) = \cup_{\mathfrak{a}\in\phi}(0 :_M \mathfrak{a})$.

(iv) For each $i \geq 0$, the functors $H^i_\phi(M, -)$ and $\varinjlim_{\mathfrak{a}\in\phi} H^i_{\mathfrak{a}}(M, -)$ are naturally equivalent.

(v) $H^0_\phi(M, N) \cong \operatorname{Hom}_R(M, \Gamma_\phi(N))$. If $\Gamma_\phi(N) = N$, then

$$H^i_\phi(M, N) \cong \operatorname{Ext}^i_R(M, N),$$

see [3].

The following elementary lemmas are needed in the proof of our main theorems.

Lemma 2.2. *Let T be a ϕ-torsion finitely generated R-module. Then, there exists $\mathfrak{b} \in \phi$ such that $\mathfrak{b}T = 0$.*

Proof. The assertion follows immediately from definition. □

Lemma 2.3. *Let $L \longrightarrow M \longrightarrow N$ be an exact sequence of R-modules and R-homomorphisms, and suppose that there exist $\mathfrak{a}, \mathfrak{b} \in \phi$ such that $\mathfrak{a}L = 0$ and $\mathfrak{b}N = 0$. Then, $\mathfrak{c}M = 0$ for some $\mathfrak{c} \in \phi$.*

Proof. It is clear. □

In the following theorem, we generalized the main theorem of Bahmanpour and Naghipour ([2], Theorem 2.3) for local cohomology modules with respect to a system of ideals of R.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. Let M and N be two finitely generated R-modules. Let t be a nonnegative integer such that $H^i_\phi(M, N)$ be minimax for all $i < t$. Then, $\operatorname{Hom}_R(R/a, H^t_\phi(M, N))$ is finitely generated for all $\mathfrak{a} \in \phi$ and $\operatorname{Ass}_R(H^t_\phi(M, N)) \bigcap \operatorname{V}(\mathfrak{a})$ is finite.*

Proof. We use induction on t. By Remark 2.1, we have

$$H^0_\phi(M, N) \cong \operatorname{Hom}_R(M, \Gamma_\phi(N)),$$

and the assertion is true for $t = 0$. Let $t \geq 1$ and suppose that the claim has been proved for $t - 1$. From the exact sequence

$$0 \longrightarrow \Gamma_\phi(N) \longrightarrow N \longrightarrow N/\Gamma_\phi(N) \longrightarrow 0$$

we earn the exact sequence

$$\operatorname{Ext}^t_R(M, \Gamma_\phi(N)) \xrightarrow{f} H^t_\phi(M, N) \xrightarrow{g} H^t_\phi(M, N/\Gamma_\phi(N)).$$

So, Imf is finitely generated. From exact sequence

$$0 \longrightarrow Imf \longrightarrow H^t_\phi(M, N) \longrightarrow H^t_\phi(M, N/\Gamma_\phi(N)),$$

it is enough to show that $\operatorname{Hom}_R(R/a, H^t_\phi(M, N/\Gamma_\phi(N)))$ is finitely generated. Thus, we can assume that $\Gamma_\phi(N) = 0$. Let $\mathfrak{a} \in \phi$. Then, $\mathfrak{a}$ contains an N-regular element x. Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0.$$

This short exact sequence yields the exact sequence

$$H^{t-1}_\phi(M, N) \xrightarrow{h} H^{t-1}_\phi(M, N/xN) \xrightarrow{k} H^t_\phi(M, N) \xrightarrow{x} H^t_\phi(M, N).$$

We split the above exact sequence into the following two exact sequences

$$0 \longrightarrow Imh \longrightarrow H^{t-1}_{\phi}(M, N/xN) \longrightarrow Imk \longrightarrow 0$$

and $0 \longrightarrow Imk \longrightarrow H^t_{\phi}(M, N) \longrightarrow H^t_{\phi}(M, N)$ then we get the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(R/a, Imh) \longrightarrow \mathrm{Hom}_R(R/a, H^{t-1}_{\phi}(M, N/xN)) \\ &\longrightarrow \mathrm{Hom}_R(R/a, Imk) \longrightarrow \mathrm{Ext}^1_R(R/a, Imh) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(R/a, Imk) \longrightarrow \mathrm{Hom}_R(R/a, H^t_{\phi}(M, N)) \\ &\xrightarrow{x} \mathrm{Hom}_R(R/a, H^t_{\phi}(M, N)). \end{aligned} \tag{2.2}$$

Moreover, by the induction hypothesis, $\mathrm{Hom}_R(R/a, H^{t-1}_{\phi}(M, N/xN))$ is finitely generated. Hence, by the exact sequence (2.1) the R-module $\mathrm{Hom}_R(R/a, Imh)$ is finitely generated. It is clear that the R-module, $x\,\mathrm{Hom}_R(R/a, H^t_{\phi}(M, N)) = 0$ it follows that

$$\mathrm{Hom}_R(R/a, H^t_{\phi}(M, N)) \cong \mathrm{Hom}_R(R/a, Imk).$$

On the other hand, since Imh is minimax there exists two R-modules T_1, T_2 such that T_1 is finitely generated and T_2 is Artinian and

$$0 \longrightarrow T_1 \longrightarrow Imh \longrightarrow T_2 \longrightarrow 0$$

is exact. This exact sequence induces the following exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(R/a, T_1) \longrightarrow \mathrm{Hom}_R(R/a, Imh) \longrightarrow \mathrm{Hom}_R(R/a, T_2) \\ &\longrightarrow \mathrm{Ext}^1_R(R/a, T_1) \longrightarrow \mathrm{Ext}^1_R(R/a, Imh) \longrightarrow \mathrm{Ext}^1_R(R/a, T_2) \end{aligned} \tag{2.3}$$

which implies that the R-module $\mathrm{Hom}_R(R/a, T_2)$ is of finite length and since

$$\operatorname{Supp} T_2 \subseteq V(\mathfrak{m}) \subseteq V(\mathfrak{a}).$$

By ([12], Proposition 4.1) $\operatorname{Ext}^i_R(R/a, T_2)$ is finitely generated. From the exact sequence (2.3) we get the R-module $\operatorname{Ext}^1_R(R/a, Imh)$ is finitely generated. It follows from the exact sequence (2.1) that the R-module $\operatorname{Hom}_R(R/a, Imk)$ is finitely generated. Now, we use the exact sequence (2.2) to obtain the result. □

Theorem 2.5. *Let M, N be two finitely generated R-modules and $t \in \mathbb{N}$. Then, the following statements are equivalent.*

(i) *$H^i_\phi(M, N)$ is finitely generated for all $i < t$;*

(ii) *There is an ideal $\mathfrak{c} \in \phi$ such that $\mathfrak{c}H^i_\phi(M, N) = 0$ for all $i < t$;*

(iii) *There is an ideal $\mathfrak{c} \in \phi$ such that $\mathfrak{c} \subseteq \operatorname{Rad}(0 :_R H^i_\phi(M, N))$for all $i < t$.*

Proof. The conclusion (ii) $\Longleftrightarrow$ (iii) is obviously true.

To prove (i) $\Rightarrow$ (ii), in view of the Theorem 2.2, there exists $\mathfrak{c}_i \in \phi$, such that $\mathfrak{c}_i H^i_\phi(M, N) = 0$. Since ϕ is a system of ideals, there is $\mathfrak{c} \in \phi$ such that $\mathfrak{c} \subseteq \mathfrak{c}_i$ for $i = 1, \ldots, t-1$. It follows that $\mathfrak{c}H^i_\phi(M, N) = 0$ for all $i < t$.

In order to show the implication (ii) $\Rightarrow$ (i) we use induction on t. When $t = 1$, there is nothing to prove. Now, suppose inductively $t > 1$ and that the assertion holds for $t - 1$. By this inductive assumption, $H^i_\phi(M, N)$ is finitely generated for all $i \leq t-2$ and it only remains to prove that $H^{t-1}_\phi(M, N)$ is finitely generated. Since

$$H^i_\phi(M, \Gamma_\phi(N)) \longrightarrow H^i_\phi(M, N) \longrightarrow H^i_\phi(M, N/\Gamma_\phi(N))$$

is exact for all $i > 0$, we assume $\Gamma_\phi(N) = 0$. Therefore, $\Gamma_\mathfrak{a}(N) = 0$ for all $\mathfrak{a} \in \phi$. Therefore, in view of hypothesis there exists $x \in \mathfrak{a}$ such that x is N-sequence and $xH_\phi^{t-1}(M,N)=0$. Using the exact sequence $0 \longrightarrow N \longrightarrow N \longrightarrow N/xN \longrightarrow 0$, we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow H_\phi^0(M,N) \longrightarrow H_\phi^0(M,N) \longrightarrow \cdots \longrightarrow H_\phi^{t-2}(M,N) \\ \longrightarrow H_\phi^{t-2}(M,N/xN) \longrightarrow H_\phi^{t-1}(M,N) \longrightarrow H_\phi^{t-1}(M,N). \end{aligned} \quad (2.4)$$

Now, use the exact sequence (2.4) together with Lemma 2.3 to see that there is $\mathfrak{c} \in \phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}$ and $\mathfrak{c}H_\phi^i(M,N/xN)=0$ for $i<t-1$. Therefore, by inductive hypothesis $H_\phi^{t-2}(M,N/xN)$ is finitely generated R-module. Again, using the long exact sequence (2.4), the result follows. □

Definition 2.6. Let M,N be two finitely generated R-modules. Using Theorem 2.5 and Remark 2.1, we define the finiteness dimension $f_\phi(M,N)$ relative to ϕ by

$$\begin{aligned} f_\phi(M,N) &= \inf\{i \in \mathbb{N} | H_\phi^i(M,N) \text{ is not finitely generated}\} \\ &= \inf\{i \in \mathbb{N} |\, \mathfrak{a}H_\phi^i(M,N) \neq 0 \text{ for all } \mathfrak{a} \in \phi\} \\ &= \inf\{i \in \mathbb{N} |\, \mathfrak{a} \nsubseteq \operatorname{Rad}(0 :_R H_\phi^i(M,N))\}. \end{aligned}$$

At this stage the following remark is needed.

Remark 2.7. (i) Let $f: R \longrightarrow R'$ be a ring homomorphism. For any ideal $\mathfrak{a}$ of R we denote its extension to R' by $\mathfrak{a}^e$. If ϕ is a system of ideals of R, then the set $\phi^e = \{\mathfrak{a}^e : \mathfrak{a} \in \phi\}$ is a system of ideals of R'. Moreover, suppose that S is a multiplicatively closed subset of R and ϕ is a system of ideals of R. Let $S^{-1}\phi = \{S^{-1}\mathfrak{a} | \mathfrak{a} \in \phi\}$. Then, the connected right sequences of covariant functors, from category R-modules to category $S^{-1}R$-modules and $\{S^{-1}R^i\Gamma_\phi(-)\}_{i\geq 0}$ and

$$\{R^i\Gamma_{S^{-1}\phi}(S^{-1}(-))\}_{i\geq 0}$$

are isomorphic. In particular, for any R-module N,

$$S^{-1}R^i\Gamma_{\phi}(N) \cong R^i\Gamma_{S^{-1}\phi}(S^{-1}(N))$$

for all $i \geq 0$. For Example,

$$\tilde{W}(I,J) = \{\mathfrak{a} | \, \mathfrak{a} \text{ is an ideal of } R \text{ and } I^n \subseteq \mathfrak{a} + J, \text{ for some } n \gg 0\},$$

is system of ideals and

$$S^{-1}\tilde{W}(I,J) = \{S^{-1}\mathfrak{a} | \, \mathfrak{a} \in \tilde{W}(I,J)\} \neq \tilde{W}(S^{-1}I, S^{-1}J).$$

(ii) Let $R \longrightarrow R'$ be a flat extension of rings, M and T be R-modules. If M is finitely generated, then

$$\operatorname{Ext}^i_R(M,T) \otimes_R R' \cong \operatorname{Ext}^i_{R'}(M \otimes R', T \otimes R')$$

is finitely generated for all $i \geq 0$ (see [11]).

(iii) Suppose that $R \longrightarrow R'$ is faithfully flat. Then, $T \otimes R'$ is finitely generated as an $R'-$module if and only if T is finitely generated as an R-module.

(iv) Let $(R, \mathfrak{m})$ be a local ring and $\hat{R}$ its completion with respect to $\mathfrak{m}$, and T an R-module. If T has support only at $\mathfrak{m}$, Then $T \otimes R'$ has support only at $\mathfrak{m}\hat{R}$.

(v) Let $(R, \mathfrak{m}, k)$ be a complete local Noetherian ring and let T be an R-module. Then T is Artinian if and only if $\operatorname{Supp} T = \{\mathfrak{m}\}$ and $\operatorname{Hom}(K,T)$ is finitely generated (see [10]).

Theorem 2.8. *Let M, N be two finitely generated R-modules and $t \in \mathbb{N}$. If $S^{-1}\phi = \{S^{-1}\mathfrak{a} | \, a \in \phi\}$, consider the following statements:*

(i) *$H^i_{\phi}(M,N)$ is finitely generated for all $i < t$,*

(ii) *$H^i_{S^{-1}\phi}(S^{-1}M, S^{-1}N)$ is finitely generated $S^{-1}R$- module for all $i < t$, where $S = R - \mathfrak{p}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then, the following implications are true.*

(a) (i) $\Rightarrow$ (ii).

(b) (ii) $\Rightarrow$ (i), *if* $\operatorname{Max}(\phi)$ *is finite.*

Proof. (i) $\Rightarrow$ (ii) Using Remark 2.7, shows that

$$H^i_{S^{-1}\phi}(S^{-1}M, S^{-1}N) \cong S^{-1}(H^i_\phi(M,N))$$

for all $i \in \mathbb{N}$ and $\mathfrak{p} \in \operatorname{Spec}(R)$, this implication is clear. In order to show that (ii) implies (i), we proceed by induction on t. If $t = 1$ there is nothing to show. Suppose that $t > 1$ and the case $t-1$ is settled. By inductive hypothesis the R-module $H^i_\phi(M,N)$ is finitely generated for all $i < t-1$, and so it is enough to show that the R-module $H^{t-1}_\phi(M,N)$ is finitely generated. Using 2.4, $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_\phi(M,N))$ is finitely generated. In other hand $\operatorname{Ass} H^{t-1}_\phi(M,N) \subseteq \cup_{\mathfrak{a}\in\phi} v(\mathfrak{a})$, the $\operatorname{Ass} H^{t-1}_\phi(M,N)$ is finite set, by assumption. Let

$$\operatorname{Ass} H^{t-1}_\phi(M,N) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s\}.$$

Since $(H^{t-1}_\phi(M,N))_{\mathfrak{p}_i}$ is finitely generated $R_{\mathfrak{p}_i}$-module for all $\mathfrak{p}_i \in \operatorname{Spec}(R)$, it follows from Theorem 2.5 and Remark 2.7 that there exists $\mathfrak{b}_i \in \phi$ such that $\mathfrak{b}_{i\mathfrak{p}_i} H^{t-1}_\phi(M,N)_{\mathfrak{p}_i} = 0$. Hence, there is $\mathfrak{c} \in \phi$ such that $\mathfrak{c}_\mathfrak{p} \subseteq (\mathfrak{b}_i)_\mathfrak{p}$ for all $i = 1, \ldots, s$. It follows that $\mathfrak{c}_\mathfrak{p} H^{t-1}_\phi(M,N)_{\mathfrak{p}_i} = 0$ for all $i = 1, \ldots, s$. Therefore $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s\} \nsubseteq \operatorname{Supp}(\mathfrak{c} H^{t-1}_\phi(M,N))$. On the other hand,

$$\operatorname{Ass} \mathfrak{c} H^{t-1}_\phi(M,N) \subseteq \operatorname{Ass} H^{t-1}_\phi(M,N),$$

then $\operatorname{Ass} \mathfrak{c} H^{t-1}_\phi(M,N) = \emptyset$ and $\mathfrak{c} H^{t-1}_\phi(M,N) = 0$. Now, the result follows from Theorem 2.5. □

Lemma 2.9. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and T an R-module. Then, $T/(0:_\mathfrak{a} T)$ is isomorphic to a submodule $(\mathfrak{a}T)^n$ for some $n \in \mathbb{N}$.*

Proof. Suppose $\mathfrak{a} = (x_1, \ldots, x_n)$ and define $f : T \longrightarrow (\mathfrak{a}T)^n$ by $f(m) = (x_1 m, \ldots, x_n m)$. Since $\ker(f) = (0:_\mathfrak{a} T)$, as required. □

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a local ring and M, N be two finitely generated R-modules and $t \geq 1$ an integer. Consider the following statements:*

(i) *$H^i_\phi(M, N)$ is a minimax R-module for all $i < t$;*

(ii) *There exists $\mathfrak{a} \in \phi$ such that $\mathfrak{a}H^i_\phi(M, N)$ is a minimax R-module for all $i < t$;*

(iii) *$(H^i_\phi(M, N))_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $i < t$ and $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$.*

Then,

(a) (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii),

(b) (iii) $\Rightarrow$ (i), *if* $\operatorname{Max}(\phi)$ *is a finite set.*

Proof. The implication (i) $\Rightarrow$ (ii) is obviously true.

In order to show (ii) $\Rightarrow$ (i), we proceed by induction on t. If $t = 1$, there is nothing to show, because $H^0_\phi(M, N) \cong \operatorname{Hom}(M, \Gamma_\phi(N))$ is a minimax R-module. Suppose that $t > 1$ and that the desired result has been proved for $t - 1$. By the inductive hypothesis, the R-module $H^i_\phi(M, N)$ is minimax for all $i < t-1$, and it is enough to show that the R-module $H^{t-1}_\phi(M, N)$ is minimax. By assumption there exists $\mathfrak{a} \in \phi$ such that $\mathfrak{a}H^{t-1}_\phi(M, N)$ is a minimax R-module. In one hand, using Theorem 2.4 $(0 :_\mathfrak{a} H^{t-1}_\phi(M, N))$ is finitely generated for all $\mathfrak{a} \in \phi$. On the other hand, since $\mathfrak{a}H^{t-1}_\phi(M, N)$ is minimax R-module, Lemma 2.9 implies that $H^{t-1}_\phi(M, N)/(0 :_\mathfrak{a} H^{t-1}_\phi(M, N))$ is minimax R-module. We consider the exact sequence

$$\begin{aligned} 0 \longrightarrow (0 :_\mathfrak{a} H^{t-1}_\phi(M, N)) \longrightarrow H^{t-1}_\phi(M, N) \\ \longrightarrow H^{t-1}_\phi(M, N)/(0 :_\mathfrak{a} H^{t-1}_\phi(M, N)) \longrightarrow 0. \end{aligned} \tag{2.5}$$

Since $(0 :_{\mathfrak{a}} H^{t-1}_{\phi}(M,N))$ and $H^{t-1}_{\phi}(M,N)/(0 :_{\mathfrak{a}} H^{t-1}_{\phi}(M,N))$ are minimax, it follows that $H^{t-1}_{\phi}(M,N)$ is minimax, as required.

(i) $\Rightarrow$ (iii) As $H^{i}_{\phi}(M,N)$ is a minimax R-module, there is an exact sequence of R-modules $0 \longrightarrow T \longrightarrow H^{i}_{\phi}(M,N) \longrightarrow T' \longrightarrow 0$, such that T is a finitely generated and T' is an Artinian R-module. Since $T'_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$, then $T_{\mathfrak{p}} \cong (H^{i}_{\phi}(M,N))_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$, the result follows.

(iii) $\Rightarrow$ (i) We use induction on t. If $t = 1$, then the assertion holds by assumption. So assume that $t > 1$ and the result has been proved for $t-1$. By the inductive hypothesis $H^{i}_{\phi}(M,N)$ is minimax R-module for all $i < t-1$. Using Theorem 2.4 $(0 :_{\mathfrak{a}} H^{t-1}_{\phi}(M,N))$ is finitely generated for all $\mathfrak{a} \in \phi$. On the other hand, use the

$$\operatorname{Ass} H^{t-1}_{\phi}(M,N) \subseteq \cup_{\mathfrak{a}\in\phi} V(\mathfrak{a})$$

in conjunction with the assumption $\operatorname{Ass} H^{t-1}_{\phi}(M,N)$ is finite set. Let

$$\operatorname{Ass} H^{t-1}_{\phi}(M,N) - \{\mathfrak{m}\} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s\}.$$

By assumption $(H^{t-1}_{\phi}(M,N))_{\mathfrak{p}_i}$ is finitely generated for all $i = 1, \ldots, s$. Using Theorem 2.5 and Remark 2.7, there exist $\mathfrak{c}_i \in \phi$ such that $(\mathfrak{c}_i H^{i}_{\phi}(M,N))_{\mathfrak{p}_i} = 0$ for all $i = 1, \ldots, s$. It follows that, there is $\mathfrak{b} \in \phi$ such that $\mathfrak{b} \subseteq \mathfrak{c}_i$ and $(\mathfrak{b} H^{i}_{\phi}(M,N))_{\mathfrak{p}_i} = 0$ for all $i = 1, \ldots, s$. Therefore $\operatorname{Ass} \mathfrak{b} H^{i}_{\phi}(M,N) \subseteq \{\mathfrak{m}\}$ and $\operatorname{Supp} \mathfrak{b} H^{i}_{\phi}(M,N) \subseteq \{\mathfrak{m}\}$. Hence in view of Lemma 2.9,

$$\operatorname{Supp} H^{t-1}_{\phi}(M,N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M,N)) \subseteq \{\mathfrak{m}\}.$$

We may consider the exact sequence

$$\begin{aligned} 0 \longrightarrow (0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M,N)) &\longrightarrow H^{t-1}_{\phi}(M,N) \\ &\longrightarrow H^{t-1}_{\phi}(M,N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M,N)) \longrightarrow 0, \end{aligned} \tag{2.6}$$

to obtain the exact sequence

$$\begin{aligned} 0 &\longrightarrow (0 :_{\mathfrak{b}} (0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N))) \longrightarrow (0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N)) \\ &\longrightarrow (0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N)) \\ &\longrightarrow \mathrm{Ext}^1_R \Big(R/\mathfrak{b}, (0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N))). \end{aligned} \tag{2.7}$$

It follows from the exact sequence (2.5) that

$$(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N))$$

is a finitely generated R-module. Thus

$$(0 :_{\mathfrak{m}} H^{t-1}_{\phi}(M, N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N)))$$

is a finitely generated R-module. Therefore, in view of Remark 2.7, $H^{t-1}_{\phi}(M, N)/(0 :_{\mathfrak{b}} H^{t-1}_{\phi}(M, N))$ is an Artinian R-module. Now, by virtue of the exact sequence (2.6) the result follows. □

Acknowledgments

The authors would like to thank the referee for his/her useful comments.

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Journal of Algebraic Systems

FALTINGS' LOCAL-GLOBAL PRINCIPLE FOR THE MINIMAXNESS OF LOCAL COHOMOLOGY MODULES DEFINED BY A SYSTEM OF IDEALS

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اصل موضعی-سرتاسری فالتینگز برای مینیماکس بودن مدول‌های کوهمولوژی موضعی تعریف شده نسبت به دستگاه ایده‌آلی

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فرض کنید R یک حلقه نوتری و جابه‌جایی و ϕ یک دستگاه ایده‌آلی از R باشد. در این مقاله نشان می‌دهیم که در برخی حالات خاص، شرایط اصل موضعی-سرتاسری فالتینگز برای متناهی بودن و مینیماکس بودن مدول‌های کوهمولوژی موضعی $H^i_\phi(M,N)$ برقرار است.

کلمات کلیدی: مدول‌های کوهمولوژی موضعی تعمیم‌یافته، اصل موضعی-سرتاسری، مدول‌های مینیماکس.