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ON HOMOLOGICAL CLASSIFICATION OF MONOIDS BY CONDITION ($\mathbf{PWP_{sc}}$) OF RIGHT ACTS

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Abstract. In this paper, we introduce Condition (PWP_{sc}) as a generalization of Condition (PWP_E) of acts over monoids, and we observe that Condition (PWP_{sc}) does not imply Condition (PWP_E). In general, we show that Condition (PWP_{sc}) implies the property of being principally weakly flat, and that in left PSF monoids, the converse of this implication is also true. Moreover, we present some general properties and a homological classification of monoids by comparing Condition (PWP_{sc}) with some other properties. Finally, we describe left PSF monoids for which S_S^I satisfies Condition (PWP_{sc}) for any nonempty set I.

1. Introduction

Throughout this paper, we use S to denote a monoid. We refer the reader to [5, 7] for basic definitions and terminology related to semigroups and acts over monoids, and to [1, 8, 9] for definitions and results on flatness properties which are used in the paper.

A right S-act A_S satisfies *Condition* (P) if for all $a, a' \in A_S$ and $s, s' \in S$, $as = a's'$ implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. Many papers have been devoted to the investigation of this property. In 1987, Normak [10] studied Condition (P). According to the results obtained in [10], Condition

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(P) strictly implies flatness, and pullback flatness strictly implies this condition.

A right S-act A_S is said to satisfy *Condition* (E) if $as = as'$, with $a \in A_S$ and $s, s' \in S$, implies the existence of $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = us'$. It satisfies *Condition* (E') if for all $a \in A_S$ and $s, s', z \in S$, $as = as'$ and $sz = s'z$ imply the existence of $a' \in A_S$ and $u \in S$ such that $a = a'u$ and $us = us'$. It is obvious that Condition (E) implies Condition (E').

We say that A_S satisfies *Condition* (PWP) if $as = a's$, for $a, a' \in A_S$ and $s \in S$, implies the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. A right S-act A_S satisfies *Condition* (P') if for all $a, a' \in A_S$ and $s, t, z \in S$, $as = at$ and $sz = tz$ imply the existence of $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. It is obvious that Condition (P) implies Condition (P'), but Condition (P') does not imply Condition (P). See [2] for further details. Also, we say that A_S satisfies *Condition* (EP) if for all $a \in A_S$ and $s, t \in S$, $as = at$ implies the existence of $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$.

A_S satisfies *Condition* $(E'P)$ if for all $a \in A_S$ and $s, t, z \in S$, $as = at$ and $sz = tz$ imply the existence of $a' \in A_S$ and $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$. A_S satisfies *Condition* (PWP_E) if $as = a's$, with $a, a' \in A_S$ and $s \in S$, implies the existence of $a'' \in A_S$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue$, $a'f = a''vf$, $es = s = fs$, and $us = vs$.

A monoid S is called *left* PP if every principal left ideal of S is projective, or equivalently, for every $s \in S$ there exists an idempotent e of S such that $ker\rho_s = ker\rho_e$. The monoid S is said to be *left* PSF if every principal left ideal of S is strongly flat, or equivalently, it satisfies Condition (E). Therefore, S is left PSF if and only if $as = bs$ for $a, b, s \in S$ implies the existence of $u \in S$ such that $au = bu$ and $us = s$. We say that S is *left* PCP (see [4]) (or *left* $P(P)$; see [14]) if every principal left ideal of S satisfies Condition (P). It can be easily checked that a monoid S is left $P(P)$ if and only if $as = bs$ for $a, b, s \in S$ implies the existence of $u, v \in S$ such that $au = bv$ and $us = vs = s$. The monoid S is called *weakly left* $P(P)$ if the equalities $as = bs$ and $xb = yb$ imply the existence of $r \in S$ such that $xar = yar$ and $rs = s$.

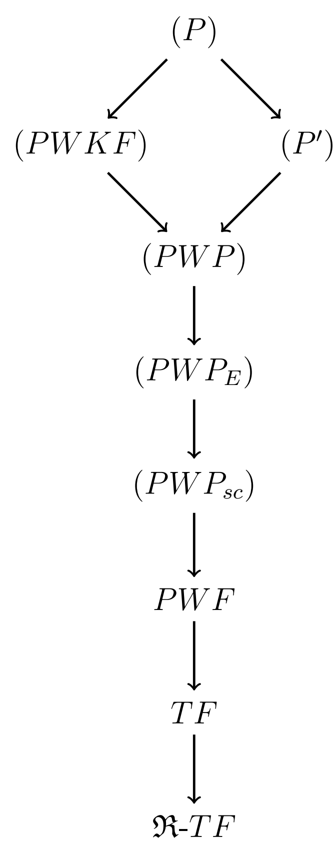
The above definitions and [14, Proposition 2.2] show that left $PP \Rightarrow$ left $PSF \Rightarrow$ left $PCP \Rightarrow$ weakly left $P(P)$. But, as shown in [7, 14], the converses of these implications are not true in general.

2. General Properties

In this section, we introduce *Condition* (PWP_{sc}) and present some of its general properties. Also, we provide characterizations of a monoid S in terms of Condition (PWP_{sc}) of right S-acts.

Definition 2.1. Let S be a monoid, and let A_S be a right S-act. We say that A_S satisfies *Condition* (PWP_{sc}) if $as = a's$, with $a, a' \in A_S$ and $s \in S$, implies the existence of $a'' \in A_S$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$ and $us = vs$.

In the following diagram, we see the relation between Condition (PWP_{sc}) and the properties already studied.



Here, the abbreviations stand for the following properties of S-acts. $PWKF$=principal weak kernel flatness, PWF=principal weak flatness, TF=being torsion-free, $\mathfrak{R}$-TF=being $\mathfrak{R}$-torsion free.

In Theorem 2.2, all statements are easy consequences of the definition.

Theorem 2.2. *The following statements are true.*

1. Θ_S *and* S_S *satisfy Condition* (PWP_{sc}).
2. *If* A_S *satisfies Condition* (PWP_E), *it also satisfies Condition* (PWP_{sc}).
3. *For an idempotent monoid, Condition* (PWP_E) *and Condition* (PWP_{sc}) *are equivalent.*
4. *Let* $A_S = \prod\limits_{i\in I} A_i$, *where each* A_i *is a right* S*-act. If* A_S *satisfies Condition* (PWP_{sc}), *then so does every* A_i.
5. *If* $A_S = \coprod\limits_{i\in I} A_i$, *where each* A_i *is a right* S*-act, then* A_S *satisfies Condition* (PWP_{sc}) *if and only if* A_i *satisfies Condition* (PWP_{sc}) *for every* $i \in I$.
6. *If* $\{B_i \mid i \in I\}$ *is a chain of subacts of* A_S *and every* B_i, $i \in I$, *satisfies Condition* (PWP_{sc}), *then* $\bigcup\limits_{i\in I} B_i$ *satisfies the condition.*
7. *If* A_S *satisfies Condition* (PWP_{sc}), *then every retract of* A_S *satisfies Condition* (PWP_{sc}).

In the following example, we show that Condition (PWP_{sc}) does not imply Condition (PWP_E). Note that for a proper right ideal I of S, $A(I)$ stands for the amalgamated coproduct of two copies of S over I.

Example 2.3. *Let* $S_1 = \{a^i | i \in \mathbb{N}, a^i a^j = a^{ij}\}$,

$$S_2 = \{b^i | i \in \mathbb{N}, b^i b^j = b^{ij}\}, S_3 = \{d^i | i \in \mathbb{N} \setminus \{1\}, d^i d^j = d^{ij}\},$$

$(I, \leq)$ *be a totally ordered set which has neither the maximum nor the minimum element, and* $S_4 = \{h_i^m | i \in I, m \in \mathbb{N}\}$ *such that*

$$h_i^m h_j^n = \begin{cases} h_j^n & i < j \\ h_i^{m+n} & i = j. \end{cases}$$

Let $T = S_1 \cup S_2 \cup S_3 \cup S_4$ *such that*

$$a^n b^m = b^m a^n = a^n d^m = d^m a^n = b^m d^n = d^n b^m = b^{mn},$$

$a^n h_i^m = h_i^m a^n = b^n = b^n h_i^m = h_i^m b^n$ *and* $d^n h_i^m = h_i^m d^n = d^n$. *It is clear that* T *is a semigroup. Let* $S = T^1$ *and* $J = S_2$. *Obviously,* $A(J)$ *satisfies Condition* (PWP_{sc}). *Now, we show that* $A(J)$ *does not satisfy Condition* (PWP_E). *Since*

$$(a^2, x)d^3 = (b^6, z) = (a^2, y)d^3,$$

and $e = 1$ is the only idempotent such that $ed^3 = d^3$, there must be $a'' \in A(J)$ and $u, v \in S$ such that $(a^2, x) = a''u$, $(a^2, y) = a''v$ and $ud^3 = vd^3$. Now, note that $(a^2, x) = a''u$ implies $a'' = (1, x)$ and $u = a^2$, or $a'' = (a^2, x)$ and $u = 1$. But in either case, $(a^2, y) \neq a''v$ for every $v \in S$.

Theorem 2.4. *If the right S-act A_S satisfies Condition (PWP_{sc}), then A_S is principally weakly flat.*

Proof. Suppose that A satisfies Condition (PWP_{sc}). Also, assume that $as = a's$ for $a, a' \in A$ and $s \in S$. Then, there exist $a'' \in A$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$ and $us = vs$. Thus,

$$a \otimes s = a \otimes rs = ar \otimes s = a''ur \otimes s = a'' \otimes urs = a'' \otimes us$$

in $A \otimes Ss$. Similarly, $a' \otimes s = a'' \otimes vs$ in $A \otimes Ss$. Now, $us = vs$ implies $a \otimes s = a' \otimes s$ in $A \otimes Ss$, and so, by [7, Lemma 3.10.1], A_S is principally weakly flat. □

If S is a left PSF monoid, then the converse of Theorem 2.4 is true. This is the content of the following theorem.

Theorem 2.5. *For a left PSF monoid S, the right S-act A_S is principally weakly flat if and only if A_S satisfies Condition (PWP_{sc}).*

Proof. Let $as = a's$, for $a, a' \in A$ and $s \in S$. By our assumption, there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$, and $s_1, t_1, \ldots, s_n, t_n \in S$ such that

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 s &= t_1 s \\ a_2 t_2 &= a_3 s_3 & s_2 s &= t_2 s \\ &\vdots & &\vdots \\ a_n t_n &= a' & s_n s &= t_n s. \end{aligned}$$

Since S is left PSF, $s_1 s = t_1 s$ implies the existence of $v_1 \in S$ such that $v_1 s = s$ and $s_1 v_1 = t_1 v_1$. Then, $s_2 v_1 s = t_2 v_1 s$ implies the existence of $v_2 \in S$ such that $v_2 s = s$ and $s_2 v_1 v_2 = t_2 v_1 v_2$. If $v' = v_1 v_2$, then

$$vs = v_1 v_2 s = s,\ s_1 v = s_1 v_1 v_2 = t_1 v_1 v_2 = t_1 v,\ s_2 v = t_2 v.$$

Continuing this procedure, there exists $u' \in S$ such that $u's = s$ and $s_i u' = t_i u'$, for $1 \leq i \leq n$. Let $u = v = 1$, $r = r' = u'$ and $a'' = a'$. Then,

$$ar = au' = (a_1 s_1)u' = a_1(s_1 u') = a_1(t_1 u')$$

$$= (a_1 t_1)u' = \cdots = (a_n t_n)u' = a'u' = a''ur.$$

Also, $a'r' = a''vr', rs = r's = s, us = vs$. So, A_S satisfies Condition (PWP_{sc}), as required. □

Theorem 2.6. *Let S be a left PP monoid. Then for every right S-act,*

$$(PWP_E) \Leftrightarrow (PWP_{sc}) \Leftrightarrow \textit{principally weakly flat}.$$

Proof. This is a direct consequence of Theorem 2.2, Theorem 2.4 and [3, Theorem 2.5]. □

Theorem 2.7. *For a monoid S, the following statements are true.*

(1) *For every right S-act,*
Condition (PWP) $\Rightarrow$ Condition (PWP_E) $\Rightarrow$ Condition (PWP_{sc}) $\Rightarrow$ principally weakly flat $\Rightarrow$ torsion-free.

(2) *If S is left PSF, then for every right S-act,*
Condition (PWP) $\Rightarrow$ Condition (PWP_E) $\Rightarrow$ Condition (PWP_{sc}) $\Leftrightarrow$ principally weakly flat $\Rightarrow$ torsion-free.

(3) *If S is left PP, then for every right S-act,*
Condition (PWP) $\Rightarrow$ Condition (PWP_E) $\Leftrightarrow$ Condition (PWP_{sc}) $\Leftrightarrow$ principally weakly flat $\Rightarrow$ torsion-free.

(4) *If S is left almost regular, then for every right S-act, Condition (PWP) $\Rightarrow$ Condition (PWP_E) $\Leftrightarrow$ Condition (PWP_{sc}) $\Leftrightarrow$ principally weakly flat $\Leftrightarrow$ torsion-free.*

(5) *If S is right cancellative, then for every right S-act,*
Condition (PWP) $\Leftrightarrow$ Condition (PWP_E) $\Leftrightarrow$ Condition (PWP_{sc}) $\Leftrightarrow$ principally weakly flat $\Leftrightarrow$ torsion-free.

Proof. (1) This is obvious, by the definitions of Condition (PWP) and Condition (PWP_E), Theorem 2.2(2), Theorem 2.4 and [7, Proposition 3.10.3].

(2) This easily follows from (1) and Theorem 2.5.

(3) This is obvious by (1) and Theorem 2.6.

(4) The statement immediately follows from (1), Theorem 2.6 and [7, Theorem 4.6.5].

(5) Every right cancellative monoid is left almost regular. Thus, Condition (PWP_E) $\Leftrightarrow$ Condition (PWP_{sc}) $\Leftrightarrow$ principally weakly flat $\Leftrightarrow$ torsion-free, by (4). Since S is right cancellative, we obtain $E(S) = \{1\}$, and so Conditions (PWP) and (PWP_E) are equivalent. □

In what follows, we use Condition (PWP_{sc}) to find several equivalent formulations of the regularity of a monoid S.

Theorem 2.8. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition (PWP_{sc}).*
(2) *All finitely generated right S-acts satisfy Condition (PWP_{sc}).*
(3) *All cyclic right S-acts satisfy Condition (PWP_{sc}).*
(4) *All monocyclic right S-acts satisfy Condition (PWP_{sc}).*
(5) *All monocyclic right S-acts of the form $S/\rho(s, s^2)$ ($s \in S$) satisfy Condition (PWP_{sc}).*
(6) *All right Rees factor S-acts satisfy Condition (PWP_{sc}).*
(7) *All right Rees factor S-acts of the form S/sS ($s \in S$) satisfy Condition (PWP_{sc}).*
(8) *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) and (3) ⇒ (6) ⇒ (7) are obvious.

(5) ⇒ (8) All monocyclic right S-acts of the form $S/\rho(s, s^2)$ ($s \in S$) are principally weakly flat, by the assumption and Theorem 2.7(1). Thus, by [7, Theorem 4.6.6], S is regular.

(7) ⇒ (8) All right Rees factor S-acts of the form S/sS ($s \in S$) are principally weakly flat, by Theorem 2.7(1) and the assumption. Thus, by [7, Theorem 4.6.6], S is regular.

(8) ⇒ (1) All right S-acts are principally weakly flat, by [7, Theorem 4.6.6]. Since every regular monoid is left PP, all right S-acts satisfy Condition (PWP_{sc}), by Theorem 2.6. □

Theorem 2.9. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition (PWP_{sc}).*
(2) *Every right S-act satisfying Condition ($E'P$) also satisfies Condition (PWP_{sc}).*
(3) *Every right S-act satisfying Condition (E') also satisfies Condition (PWP_{sc}).*
(4) *Every right S-act satisfying Condition (EP) also satisfies Condition (PWP_{sc}).*
(5) *Every right S-act satisfying Condition (E) also satisfies Condition (PWP_{sc}).*
(6) *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are obvious.

(5) $\Rightarrow$ (6). Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so, s is a regular element of S.
Let $sS \neq S$. Put

$$A_S = A(sS) = \big\{(l,x)|\ l \in S \setminus sS\big\} \dot{\cup}\ sS \dot{\cup} \{(t,y)|\ t \in S \setminus sS\}.$$

It is obvious that $B_S = \{(l,x)|\ l \in S \setminus sS\big\} \dot{\cup} sS$ and

$$C_S = \{(t,y)|\ t \in S \setminus sS\} \dot{\cup} sS$$

are subacts of A isomorphic to S. Since S satisfies Condition (E), B_S and C_S satisfy Condition (E), and so, A_S satisfies Condition (E). Hence, by the assumption, A_S satisfies Condition (PWP_{sc}). Since $(1,x)s = (1,y)s$, there exist $a \in A_S$ and $u, v, r, r' \in S$ such that $(1,x)r = aur$, $(1,y)r' = avr$, $rs = s = r's$ and $us = vs$. Now, $(1,x)r = aur$ and $(1,y)r' = avr'$ imply that either $r \in sS$ or $r' \in sS$. If $r \in sS$, then there exists $s' \in S$ such that $r = ss'$, and so, $s = rs = ss's$. Thus, s is a regular element of S. Similarly, $r' \in sS$ implies that s is a regular element of S, that is, S is regular.

(6) $\Rightarrow$ (1) The proof is straightforward by Theorem 2.8. □

Notice that if $sS \neq S$ for some $s \in S$, then $A(sS) = (1,x)S \cup (1,y)S$. So, Theorem 2.9 is also valid for finitely generated right S-acts as well as for right S-acts generated by exactly two elements.

Theorem 2.10. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition (PWP_{sc}).*
(2) *All generator right S-acts satisfy Condition (PWP_{sc}).*
(3) *All finitely generated generator right S-acts satisfy Condition (PWP_{sc}).*
(4) *All generator right S-acts generated by at most three elements satisfy Condition (PWP_{sc}).*
(5) *If A_S is any generator right S-act, then $S \times A_S$ satisfies Condition (PWP_{sc}).*
(6) *If A_S is any finitely generated generator right S-act, then $S \times A_S$ satisfies Condition (PWP_{sc}).*
(7) *If A_S is any generator right S-act generated by at most three elements, then $S \times A_S$ satisfies Condition (PWP_{sc}).*
(8) *If A_S is any right S-act, then $S \times A_S$ satisfies Condition (PWP_{sc}).*
(9) *If A_S is any finitely generated right S-act, then $S \times A_S$ satisfies Condition (PWP_{sc}).*
(10) *If A_S is any right S-act generated by at most two elements, then $S \times A_S$ satisfies Condition (PWP_{sc}).*

(11) *The right S-act A_S satisfies Condition (PWP_{sc}) if*

$$Hom(A_S, S_S) \neq \emptyset.$$

(12) *The finitely generated right S-act A_S satisfies Condition (PWP_{sc}) if $Hom(A_S, S_S) \neq \emptyset$.*

(13) *The right S-act A_S generated by at most two elements satisfies Condition (PWP_{sc}) if $Hom(A_S, S_S) \neq \emptyset$.*

(14) *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4), (1) ⇒ (5) ⇒ (6) ⇒ (7), (8) ⇒ (9) ⇒ (10) and (1) ⇒ (11) ⇒ (12) ⇒ (13) are obvious.

(1) ⇔ (14) This follows from Theorem 2.8.

(2) ⇒ (8) Let A_S be a right S-act. Since

$$\begin{cases} \pi : S \times A_S \to S_S \\ \quad (s, a) \mapsto s \end{cases}$$

is an epimorphism, the right S-act $S \times A_S$ is a generator, and so satisfies Condition (PWP_{sc}).

(10) ⇒ (1) Let A_S be an arbitrary right S-act, and $as = a's$, for $a, a' \in A_S$ and $s \in S$. Let $A_S^* = aS \cup a'S$. Then A_S^* is a subact of A_S which is generated by at most two elements, and so by the assumption, the right S-act $S \times A_S^*$ satisfies Condition (PWP_{sc}). Hence,

$$(1, a)s = (1, a')s$$

implies the existence of $(w, a'') \in S \times A_S^*$ and $u, v, r, r' \in S$ such that $(1, a)r = (w, a'')ur,\ (1, a')r' = (w, a'')vr',\ rs = s = r's$, and $us = vs$. Thus, $ar = a''ur,\ a'r' = a''vr',\ rs = s = r's$, and $us = vs$, that is, A_S satisfies Condition (PWP_{sc}).

(13) ⇒ (2) Let A_S be a generator right S-act, and $as = a's$, for $a, a' \in A_S$ and $s \in S$. Let $A_S^* = aS \cup a'S$. Then A_S^* is a subact of A_S which is generated by at most two elements. Since A_S is a generator, there exists an epimorphism $\pi : A_S \to S_S$, that is, $\pi|_{A_S^*} : A_S^* \to S_S$ is an S-homomorphism, in the sense that $Hom(A_S^*, S_S) \neq \emptyset$. So, by the assumption, A_S^* satisfies condition (PWP_{sc}). Now, the equality $as = a's$ in A_S^* implies the existence of $a'' \in A_S^* \subseteq A_S$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$, and $us = vs$. Hence, A_S satisfies Condition (PWP_{sc}).

(7) ⇒ (2) Let A_S be a generator right S-act and $as = a's$, for $s \in S$ and $a, a' \in A_S$. Since A_S is a generator, there exists an epimorphism $\pi : A_S \to S_S$. Let $\pi(z) = 1$. Put $A_S^* = aS \cup a'S \cup zS$. Then, A_S^* is a subact of A_S generated by at most three elements. Obviously, $\pi|_{A_S^*} : A_S^* \to S_S$ is an epimorphism and so, A_S^* is a generator. Thus, by the assumption, $S \times A_S^*$ satisfies Condition (PWP_{sc}). Now, $as = a's$

implies that $(1,a)s=(1,a')s$ in $S\times A^*_S$, and so by the definition, there exist $(w,a'')\in S\times A^*_S$ and $u,v,r,r'\in S$ such that $(1,a)r=(w,a'')ur$, $(1,a')r'=(w,a'')vr'$, $rs=s=r's$ and $us=vs$. Thus, $ar=a''ur$, $a'r'=a''vr'$, $rs=s=r's$ and $us=vs$, that is, A_S satisfies Condition (PWP_{sc}).

(4) $\Rightarrow$ (2) Let A_S be a generator right S-act and $as=a's$, for $a,a'\in A_S$ and $s\in S$. Since A_S is a generator, there exists an epimorphism $\pi: A_S\to S_S$. Let $\pi(z)=1$ and $A^*_S=aS\cup a'S\cup zS$. Obviously, A^*_S is a subact of A_S generated by at most three elements, and $\pi|_{A^*_S}: A^*_S\to S_S$ is an epimorphism. Thus A^*_S is a generator, and so by the assumption, A^*_S satisfies Condition (PWP_{sc}). Hence, the equality $as=a's$ in A^*_S implies the existence of $a''\in A^*_S\subseteq A_S$ and $u,v,r,r'\in S$ such that $ar=a''ur$, $a'r'=a''vr'$, $rs=s=r's$ and $us=vs$. Therefore, A_S satisfies Condition (PWP_{sc}), as required. □

Theorem 2.11. *The following statements are equivalent.*

(1) *All torsion-free right S-acts satisfy Condition (PWP_{sc}).*
(2) *All torsion-free finitely generated right S-acts satisfy Condition (PWP_{sc}).*
(3) *All torsion-free cyclic right S-acts satisfy Condition (PWP_{sc}).*
(4) *All torsion-free Rees factor right S-acts satisfy Condition (PWP_{sc}).*
(5) *S is left almost regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By Theorem 2.4 and by the assumption, all torsion-free Rees factor right S-acts are principally weakly flat. So, S is left almost regular, by [7, Theorem 4.6.5].

(5) $\Rightarrow$ (1) By [3, Theorem 3.4], all torsion-free right S-acts satisfy Condition (PWP_E). Thus, all torsion-free right S-acts satisfy Condition (PWP_{sc}), by Theorem 2.2(2). □

Recall from [16] that A_S is called $\mathfrak{R}$-*torsion free*, if for every $a,b\in A_S$ and any right cancellable $c\in S$, $ac=bc$ and $a\mathfrak{R}b$ imply $a=b$, where $\mathfrak{R}$ is Green's equivalence.

Theorem 2.12. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition (PWP_{sc}).*
(2) *All $\mathfrak{R}$-torsion free right S-acts satisfy Condition (PWP_{sc}).*
(3) *All $\mathfrak{R}$-torsion free finitely generated right S-acts satisfy Condition (PWP_{sc}).*

(4) *All $\mathfrak{R}$-torsion free right S-acts generated by at most two elements satisfy Condition (PWP_{sc}).*
(5) *All $\mathfrak{R}$-torsion free right S-acts generated by exactly two elements satisfy Condition (PWP_{sc}).*
(6) *S is regular.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are obvious.

(5) ⇒ (6) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$, and so s is regular. Let $sS \neq S$. Put

$$A_S = A(sS) = \big\{(l,x)|\ l \in S \setminus sS\big\}\dot{\cup} sS \dot{\cup}\big\{(t,y)|\ t \in S \setminus sS\big\}.$$

Then,

$$B_S = \{(l,x)|l \in S\backslash sS\}\dot{\cup}sS \cong S_S \cong \{(t,y)|t \in S\backslash sS\}\dot{\cup}sS = C_S$$

and

$$A_S = \langle (1,x),(1,y)\rangle = (1,x)S \cup (1,y)S = B_S \cup C_S.$$

By the proof of (5) ⇒ (6) in Theorem 2.9, A_S is a right S-act that is generated by exactly two elements, namely, $(1,x)$ and $(1,y)$, and also satisfies Condition (E). Every right S-act satisfying Condition (E) is $\mathfrak{R}$-torsion free, by [16, Proposition 1.2]. Thus, A_S is $\mathfrak{R}$-torsion free. So, it satisfies Condition (PWP_{sc}) by the assumption. Hence, by the proof of (5) ⇒ (6) in Theorem 2.9, s is regular. Therefore, S is regular, as required.

(6) ⇒ (1). This follows from Theorem 2.8. □

We recall from [7] that the right S-act A_S is (*strongly*) *faithful* if for $s, t \in S$, the validity of $as = at$ for (some) all $a \in A_S$ implies the equality $s = t$.

Notation 2.13. *We use C_l (C_r) to denote the set of all left (right) cancellable elements of S.*

Lemma 2.14. [6, Lemma 3.7] *The following statements are equivalent.*

(1) *There exists at least one strongly faithful right S-act.*
(2) *As an S-act, sS is strongly faithful, for every $s \in S$.*
(3) *As an S-act, S_S is strongly faithful.*
(4) *For every $s \in S$, $sS \subseteq C_l$.*
(5) *S is left cancellative.*

Theorem 2.15. *The following statements are equivalent.*

(1) *All strongly faithful right S-acts satisfy Condition (PWP_{sc}).*
(2) *All finitely generated strongly faithful right S-acts satisfy Condition (PWP_{sc}).*

(3) *All strongly faithful right S-acts generated by at most two elements satisfy Condition* (PWP_{sc}).
(4) *All strongly faithful right S-acts generated by exactly two elements satisfy Condition* (PWP_{sc}).
(5) *S is not left cancellative or it is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) Suppose that S is left cancellative, and that $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Since S is left cancellative, $xs = 1$, that is, s is left invertible. Now, let $sS \neq S$. Put

$$A_S = A(sS) = \{(l,x) |\ l \in S \setminus sS\} \dot{\cup} sS \dot{\cup} \{(t,y) |\ t \in S \setminus sS\}.$$

Then,

$$B_S = \{(l,x)|l \in S \backslash sS\} \dot{\cup} sS \cong S_S \cong \{(t,y)|t \in S \backslash sS\} \dot{\cup} sS = C_S$$

and

$$A_S = \langle (1,x),(1,y) \rangle = (1,x)S \cup (1,y)S = B_S \cup C_S.$$

Since S is left cancellative, Lemma 2.14 shows that S_S is strongly faithful. By the above isomorphisms, B_S and C_S are strongly faithful as subacts of A_S. So, A_S is strongly faithful. Since A_S is generated by exactly two elements, namely, $(1,x)$ and $(1,y)$, by the assumption, A_S satisfies Condition (PWP_{sc}). By the proof of (5) $\Rightarrow$ (6) in Theorem 2.9, s is regular. Thus, there exists $x \in S$ such that $sxs = s$. Since S is left cancellative, $xs = 1$. Hence, every element in S has a left inverse and so, S is a group.

(5) $\Rightarrow$ (1) If S is not left cancellative, then by Lemma 2.14, no strongly faithful right S-act exists. Thus, (1) is satisfied. If S is a group, then S is regular and so, (1) is satisfied by Theorem 2.8. $\square$

Theorem 2.16. *The following statements are equivalent.*

(1) *All right S-acts satisfy Condition* (PWP_{sc}).
(2) *All faithful right S-acts satisfy Condition* (PWP_{sc}).
(3) *All finitely generated faithful right S-acts satisfy Condition* (PWP_{sc}).
(4) *All faithful right S-acts generated by at most two elements satisfy Condition* (PWP_{sc}).
(5) *All faithful right S-acts generated by exactly two elements satisfy Condition* (PWP_{sc}).
(6) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so, s is regular. Now, let $sS \neq S$. Put

$$A_S = A(sS) = \big\{(l,x)|\ l \in S \setminus sS\big\}\dot{\cup} sS \dot{\cup} \{(t,y)|\ t \in S \setminus sS\}.$$

Then,

$$B_S = \{(l,x)|l \in S\backslash sS\}\dot{\cup} sS \cong S_S \cong \{(t,y)|t \in S\backslash sS\}\dot{\cup} sS = C_S$$

and

$$A_S = \langle (1,x),(1,y)\rangle = (1,x)S \cup (1,y)S = B_S \cup C_S.$$

Since S_S is faithful, B_S and C_S are faithful as subacts of A_S. So, A_S is faithful. Since A_S is generated by exactly two elements, namely, $(1,x)$ and $(1,y)$, by the assumption, A_S satisfies Condition (PWP_{sc}). Using an argument similar to the one utilized in the proof of

(5) $\Rightarrow$ (6) in Theorem 2.9, we conclude that s is regular. Therefore, S is regular, as required.

(6) $\Rightarrow$ (1) This follows from Theorem 2.8. □

For fixed elements $u,v \in S$, define a binary relation $P_{u,v}$ on S by

$$(x,y) \in P_{u,v} \Leftrightarrow ux = vy\ (x,y \in S).$$

Recall that an act is called *cofree* whenever it is isomorphic to the act $X^S = \{f|f$ is a mapping from S into $X\}$, for some nonempty set X, where fs is defined by $fs(t) = f(st)$ for $f \in X^S$ and $s,t \in S$.

Theorem 2.17. *The following statements are equivalent.*

(1) *All fg-weakly injective right S-acts satisfy Condition (PWP_{sc}).*
(2) *All weakly injective right S-acts satisfy Condition (PWP_{sc}).*
(3) *All injective right S-acts satisfy Condition (PWP_{sc}).*
(4) *All cofree right S-acts satisfy Condition (PWP_{sc}).*
(5) *For every $s \in S$, there exist $u,v,r,r' \in S$ such that $rs = s = r's$, $us = vs$, and the following conditions are satisfied.*
  (*i*) $P_{ur,vr'} \subseteq P_{r,s} \circ \ker\lambda_s \circ P_{s,r'}$.
  (*ii*) $\ker\lambda_u \cap (rS \times rS) \subseteq \Delta_S$.
  (*iii*) $\ker\lambda_v \cap (r'S \times r'S) \subseteq \Delta_S$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) Suppose that $s \in S$. Also, let S_1 and S_2 be two sets such that $|S_1| = |S_2| = |S|$. Assume that $\alpha : S \to S_1$ and

$$\beta : S \to S_2$$

are bijections. Let $X = S/\ker\lambda_s \dot{\cup} S_1 \dot{\cup} S_2$. Define the mappings $f,g : S \to X$ by

$$f(x) = \begin{cases} [y]_{ker\lambda_s} & \text{if there exists } y \in S;\ x = sy \\ \alpha(x) & \text{if } x \in S \setminus sS \end{cases}$$

and

$$g(x) = \begin{cases} [y]_{ker\lambda_s} & \text{if there exists } y \in S;\ x = sy \\ \\ \beta(x) & \text{if } x \in S \setminus sS. \end{cases}$$

If there exist $y_1, y_2 \in S$ such that $sy_1 = sy_2$, then $(y_1, y_2) \in \ker \lambda_s$, which implies that $f(sy_1) = f(sy_2)$. So, f is well-defined. Similarly, it follows that g is well-defined. The right S-act X^S is cofree and so, it satisfies Condition (PWP_{sc}). According to our definition of f and g, $fs = gs$. Thus, there exist $u, v, r, r' \in S$ and a map $h : S \to X$ such that $fr = hur$, $gr' = hvr'$, $rs = s = r's$ and $us = vs$. Now, we show that the statements (i), (ii) and (iii) are true.

(i) Let $(l_1, l_2) \in P_{ur,vr'}$, $l_1, l_2 \in S$. Then $url_1 = vr'l_2$ and so,

$$\begin{aligned} f(rl_1) &= (fr)(l_1) = (hur)(l_1) = h(url_1) = h(vr'l_2) \\ &= (hvr')(l_2) = (gr')(l_2) = g(r'l_2). \end{aligned}$$

Our definition of f and g gives us $y_1, y_2 \in S$ such that $rl_1 = sy_1$ and $r'l_2 = sy_2$. Thus,

$$[y_1]_{\ker \lambda_s} = f(rl_1) = g(r'l_2) = [y_2]_{\ker \lambda_s},$$

that is, $sy_1 = sy_2$. Now, $rl_1 = sy_1$, $sy_1 = sy_2$ and $sy_2 = r'l_2$ imply $(l_1, y_1) \in P_{r,s}$, $(y_1, y_2) \in \ker \lambda_s$ and $(y_2, l_2) \in P_{s,r'}$, respectively. Therefore, $(l_1, l_2) \in P_{r,s} \circ \ker \lambda_s \circ P_{s,r'}$. Thus $P_{ur,vr'} \subseteq P_{r,s} \circ \ker \lambda_s \circ P_{s,r'}$ and so, (i) is satisfied.

(ii) Let $(t_1, t_2) \in \ker \lambda_u \cap (rS \times rS)$, $t_1, t_2 \in S$. Then $ut_1 = ut_2$ and there exist $w_1, w_2 \in S$ such that $t_1 = rw_1$ and $t_2 = rw_2$. Thus

$$urw_1 = ut_1 = ut_2 = urw_2,$$

which implies

$$\begin{aligned} f(rw_1) &= (fr)(w_1) = (hur)(w_1) = h(urw_1) = h(urw_2) \\ &= (hur)(w_2) = (fr)(w_2) = f(rw_2). \end{aligned}$$

Having in mind the definition of f, we consider two cases as follows.

Case 1. If $rw_1, rw_2 \in S \setminus sS$, then $f(rw_1) = f(rw_2)$ implies

$$\alpha(rw_1) = \alpha(rw_2).$$

Thus, $t_1 = rw_1 = rw_2 = t_2$.

Case 2. If $rw_1, rw_2 \in sS$, then there exist $y_1, y_2 \in S$ such that $rw_1 = sy_1$ and $rw_2 = sy_2$. Thus

$$[y_1]_{\ker \lambda_s} = f(rw_1) = f(rw_2) = [y_2]_{\ker \lambda_s},$$

which implies $(y_1, y_2) \in \ker \lambda_s$. Hence

$$t_1 = rw_1 = sy_1 = sy_2 = rw_2 = t_2,$$

that is, $\ker \lambda_u \cap (rS \times rS) \subseteq \Delta_S$, as required. The proof of (iii) is similar to that of (ii).

(5) $\Rightarrow$ (1) Suppose that A_S is fg-weakly injective, and that $as = a's$ for $a, a' \in A_S$ and $s \in S$. By the assumption, there exist $u, v, r, r' \in S$ such that $rs = s = r's, us = vs$ and statements $(i), (ii), (iii)$ are true. Define a mapping $\varphi : urS \cup vr'S \to A_S$ by

$$\varphi(x) = \begin{cases} arp & \exists p \in S : \ x = urp \\ \\ a'r'q & \exists q \in S : \ x = vr'q. \end{cases}$$

First, we show that φ is well-defined. If there exist $p, q \in S$ such that $urp = vr'q$, then $(p, q) \in P_{ur,vr'}$. By (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{r,s}$, $(y_1, y_2) \in \ker \lambda_s$ and $(y_2, q) \in P_{s,r'}$. Thus, $rp = sy_1$, $sy_1 = sy_2$ and $sy_2 = r'q$. Hence, $arp = asy_1 = a'sy_1 = a'sy_2 = a'r'q$.

If there exist $p_1, p_2 \in S$ such that $urp_1 = urp_2$, then

$$(rp_1, rp_2) \in \ker \lambda_u \cap (rS \times rS).$$

Now, by (ii), $rp_1 = rp_2$ and so $arp_1 = arp_2$. If there exist $q_1, q_2 \in S$ such that $vr'q_1 = vr'q_2$, then by (iii),

$$a'r'q_1 = a'r'q_2.$$

So, φ is well-defined. It is clear that φ is an S-homomorphism. Since A_S is fg-weakly injective, there exists an S-homomorphism $\psi : S_S \to A_S$ such that $\psi|_{urS \cup vr'S} = \varphi$. Put $a'' = \psi(1)$. Then

$$ar = \varphi(ur) = \psi(ur) = \psi(1)ur = a''ur$$

and $a'r' = \varphi(vr') = \psi(vr') = \psi(1)vr' = a''vr'$, that is, A_S satisfies Condition (PWP_{sc}). □

Lemma 2.18. *The following statements are true.*

(1) $(\forall s \in S)(P_{1,s} \circ \ker \lambda_s \circ P_{s,1} = (sS \times sS) \cap \Delta_S)$.

(2) $(\forall u, v, s \in S)$
$[(us = vs, P_{u,v} \subseteq (sS \times sS) \cap \Delta_S) \Leftrightarrow (P_{u,v} = (sS \times sS) \cap \Delta_S)]$.

Proof. (1) Let $s \in S$. For $l_1, l_2 \in S$,

$$\begin{aligned}[(l_1, l_2) \in P_{1,s} \circ \ker \lambda_s \circ P_{s,1}] &\Longleftrightarrow [(\exists y_1, y_2 \in S)(l_1, y_1) \in P_{1,s}, \\ &\qquad (y_1, y_2) \in \ker \lambda_s, (y_2, l_2) \in P_{s,1}] \\ &\Longleftrightarrow [(\exists y_1, y_2 \in S) \\ &\qquad l_1 = sy_1 = sy_2 = l_2] \\ &\Longleftrightarrow (l_1, l_2) \in (sS \times sS) \cap \Delta_S].\end{aligned}$$

So, statement (1) is true.

(2) Let $us = vs$ and $P_{u,v} \subseteq (sS \times sS) \cap \Delta_S$, for $u, v, s \in S$. Now, let $(l_1, l_2) \in (sS \times sS) \cap \Delta_S$. Then, there exist $y_1, y_2 \in S$ such that $sy_1 = l_1 = l_2 = sy_2$. Thus

$$ul_1 = usy_1 = vsy_1 = vsy_2 = vl_2,$$

which implies $(l_1, l_2) \in P_{u,v}$ and so, $(sS \times sS) \cap \Delta_S \subseteq P_{u,v}$. Therefore, $P_{u,v} = (sS \times sS) \cap \Delta_S$.

Conversely, suppose that

$$P_{u,v} = (sS \times sS) \cap \Delta_S.$$

Since $(s, s) \in (sS \times sS) \cap \Delta_S = P_{u,v}$, $us = vs$, statement (2) is also true. □

Theorem 2.19. *The following statements are equivalent.*

(1) *All fg-weakly injective right S-acts satisfy Condition (PWP).*
(2) *All weakly injective right S-acts satisfy Condition (PWP).*
(3) *All injective right S-acts satisfy Condition (PWP).*
(4) *All cofree right S-acts satisfy Condition (PWP).*
(5) *For every $s \in S$, there exist $u, v \in S$ such that $us = vs$ and* $\ker \lambda_u = \ker \lambda_v = \Delta_S, P_{u,v} \subseteq P_{1,s} \circ \ker \lambda_s \circ P_{s,1}$.
(6) *For every $s \in S$, there exist $u, v \in S$ such that*
$$\ker \lambda_u = \ker \lambda_v = \Delta_S, P_{u,v} = P_{1,s} \circ \ker \lambda_s \circ P_{s,1}.$$
(7) *For every $s \in S$, there exist $u, v \in S$ such that $us = vs$ and* $\ker \lambda_u = \ker \lambda_v = \Delta_S, P_{u,v} \subseteq (sS \times sS) \cap \Delta_S$.
(8) *For every $s \in S$, there exist $u, v \in S$ such that*
$$\ker \lambda_u = \ker \lambda_v = \Delta_S, P_{u,v} = (sS \times sS) \cap \Delta_S.$$

Proof. Letting $r = r' = 1$ in Theorem 2.17, we find that statements $(1) - (5)$ are equivalent. Also, Lemma 2.18 shows that statements $(5) - (8)$ are equivalent. □

In the following theorems, we present classifications of monoids when Condition (PWP_{sc}) of acts implies other properties.

Theorem 2.20. *The following statements are equivalent.*

(1) *Any right S-act satisfying Condition (PWP_E) is a generator.*
(2) *Any finitely generated right S-act satisfying Condition (PWP_E) is a generator.*
(3) *Any cyclic right S-act satisfying Condition (PWP_E) is a generator.*
(4) *Any Rees factor right S-act satisfying Condition (PWP_E) is a generator.*

(5) $S = \{1\}$.

Proof. Implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) By [3, Theorem 2.2(2)], $\Theta_S \cong S/S_S$ satisfies Condition (PWP_E). Hence, by the assumption, $\Theta_S \cong S/S_S$ is a generator. Then, there exists an epimorphism $\pi : \Theta_S \to S_S$, which implies $S = \{1\}$. □

Corollary 2.21. *The following statements are equivalent.*

(1) *Any right S-act satisfying Condition (PWP_{sc}) is a generator.*
(2) *Any finitely generated right S-act satisfying Condition (PWP_{sc}) is a generator.*
(3) *Any cyclic right S-act satisfying Condition (PWP_{sc}) is a generator.*
(4) *Any right Rees factor act of S satisfying Condition (PWP_{sc}) is a generator.*
(5) $S = \{1\}$.

Proof. Implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) This follows from Theorem 2.2(2) and Theorem 2.20. □

Corollary 2.22. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_{sc}) are free.*
(2) *Any right S-act satisfying Condition (PWP_{sc}) is a projective generator.*
(3) *All finitely generated right S-acts satisfying Condition (PWP_{sc}) are free.*
(4) *Any finitely generated right S-act satisfying Condition (PWP_{sc}) is a projective generator.*
(5) *All cyclic right S-acts satisfying Condition (PWP_{sc}) are free.*
(6) *Any cyclic right S-act satisfying Condition (PWP_{sc}) is a projective generator.*
(7) *All right Rees factor S-acts satisfying Condition (PWP_{sc}) are free.*
(8) *Any right Rees factor S-act satisfying Condition (PWP_{sc}) is a projective generator.*
(9) $S = \{1\}$.

Proof. Since free $\Rightarrow$ projective generator $\Rightarrow$ generator, the proof is straightforward by Corollary 2.21. □

Recall from [7] that a right ideal K_S of S satisfies *Condition (LU)* if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

Lemma 2.23. [6, Lemma 3.12] *Let S be a monoid such that $S \neq C_r$. Then, the following statements are true.*

(1) *$I = S \setminus C_r$ is a proper right ideal of S.*
(2) *S/I $(I = S \setminus C_r)$ is a torsion-free right S-act.*
(3) *If S is left PSF, then the right ideal $I = S \setminus C_r$ satisfies Condition (LU).*

We recall from [12] that A_S is called *GP-flat* if for every $s \in S$ and $a, a' \in A_S$, $a \otimes s = a' \otimes s$ in $A_S \otimes S$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes Ss^n$. Also, we recall from [15] that an act A_S is called *strongly torsion-free* if for any $a, b \in A_S$ and any $s \in S$, the equality $as = bs$ implies $a = b$.

Remark 2.24. Note that in **Act-**S, strongly torsion-free $\Rightarrow$ Condition (PWP) $\Rightarrow$ Condition (PWP_E) $\Rightarrow$ Condition (PWP_{sc}) $\Rightarrow$ principally weakly flat. Hence, we can add Condition (PWP_{sc}) to [6, Lemma 3.13]. Also, by Theorem 2.5, the property $(*)$ in [6, Theorem 3.14] can be considered as Condition (PWP_{sc}).

We recall from [6] that the right S-act A_S satisfies *Condition* (PWP_{ssc}) if $as = a's$, for $a, a' \in A_S$ and $s \in S$, implies the existence of $r \in S$ such that $ar = a'r$ and $rs = s$. It is easy to see that,

$$\text{Condition}(PWP_{ssc}) \Rightarrow \text{Condition}(PWP_{sc})$$

Also, S_S satisfies Condition (PWP_{ssc}) if and only if S is left PSF (right semi-cancellative), by [6, Theorem 2.2].

Theorem 2.25. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_{sc}) are principally weakly kernel flat and satisfy Condition (PWP_{ssc}).*
(2) *All right S-acts satisfying Condition (PWP_{sc}) are translation kernel flat and satisfy Condition (PWP_{ssc}).*
(3) *All right S-acts satisfying Condition (PWP_{sc}) satisfy Conditions (PWP) and (PWP_{ssc}).*
(4) *All right S-acts satisfying Condition (PWP_{sc}) satisfy Conditions (P') and (PWP_{ssc}).*
(5) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (5) By Theorem 2.2(1), S_S satisfies Condition (PWP_{sc}). Thus, by the assumption, S_S satisfies Condition (PWP_{ssc}) and so, S is left PSF. Also, by the assumption, all right S-acts satisfying Condition (PWP_{sc}) satisfy Condition (PWP). Thus, S is right cancellative, by [6, Theorem 3.14].

(5) $\Rightarrow$ (4) All right S-acts satisfying Condition (PWP_{sc}) satisfy Condition (P'), by [6, Theorem 3.14]. Now, we show that all right

S-acts satisfying Condition (PWP_{sc}) satisfy Conditions (PWP_{ssc}). Suppose that A_S satisfies Condition (PWP_{sc}) and $as = a's$, for $a, a' \in A_S$ and $s \in S$. Since A_S satisfies Condition (PWP_{sc}), there exist $a'' \in A_S$ and $u, v, r, r' \in S$ such that $ar = a''ur$, $a'r' = a''vr'$, $rs = s = r's$, and $us = vs$. Since S is right cancellative, $r = r' = 1$ and $u = v$. Hence $a = a'$ and so, A_S satisfies Condition (PWP_{ssc}).
(5) $\Rightarrow$ (1). All right S-acts satisfying Condition (PWP_{sc}) are principally weakly kernel flat, by [6, Theorem 3.14]. Also, by the proof of (5) $\Rightarrow$ (4), all right S-acts satisfying Condition (PWP_{sc}) satisfy Condition (PWP_{ssc}), and so, we are done. □

Theorem 2.26. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_E) are (strongly) faithful.*
(2) *All finitely generated right S-acts satisfying Condition (PWP_E) are (strongly) faithful.*
(3) *All cyclic right S-acts satisfying Condition (PWP_E) are (strongly) faithful.*
(4) *All right Rees factor S-acts satisfying Condition (PWP_E) are (strongly) faithful.*
(5) *$S = \{1\}$.*

Proof. Implications (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) The right Rees factor $S/S_S \cong \Theta_S$ satisfies Condition (PWP_E). Thus, by the assumption, Θ_S is (strongly) faithful. So, $S = \{1\}$. □

Using an argument similar to the one utilized in the proof of the above theorem, we obtain the following result.

Theorem 2.27. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_{sc}) are (strongly) faithful.*
(2) *All finitely generated right S-acts satisfying Condition (PWP_{sc}) are (strongly) faithful.*
(3) *All cyclic right S-acts satisfying Condition (PWP_{sc}) are (strongly) faithful.*
(4) *All right Rees factor S-acts satisfying Condition (PWP_{sc}) are (strongly) faithful.*
(5) *$S = \{1\}$.*

Theorem 2.28. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_E) are strongly torsion-free.*

(2) *All finitely generated right S-acts satisfying Condition* (PWP_E) *are strongly torsion-free.*
(3) *All cyclic right S-acts satisfying Condition* (PWP_E) *are strongly torsion-free.*
(4) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4) The cyclic right S-act S_S satisfies Condition (PWP_E), by [3, Theorem 2.2]. So, it is strongly torsion-free, by the assumption. Thus, S is right cancellative by [15, Proposition 2.1].
(4) $\Rightarrow$ (1) This follows from Remark 2.24. □

Theorem 2.29. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition* (PWP_{sc}) *are strongly torsion-free.*
(2) *All finitely generated right S-acts satisfying Condition* (PWP_{sc}) *are strongly torsion-free.*
(3) *All cyclic right S-acts satisfying Condition* (PWP_{sc}) *are strongly torsion-free.*
(4) *S is right cancellative.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4) All cyclic right S-acts satisfying Condition (PWP_E) are strongly torsion-free, by Theorem 2.2(2). Thus, S is right cancellative by Theorem 2.28.
(4) $\Rightarrow$ (1) This follows from Remark 2.24. □

Theorem 2.30. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition* (PWP_{sc}) *satisfy Condition* (PWP_{ssc}).
(2) *All finitely generated right S-acts satisfying Condition* (PWP_{sc}) *satisfy Condition* (PWP_{ssc}).
(3) *All cyclic right S-acts satisfying Condition* (PWP_{sc}) *satisfy Condition* (PWP_{ssc}).
(4) *All monocyclic right S-acts satisfying Condition* (PWP_{sc}) *satisfy Condition* (PWP_{ssc}).
(5) *S is left PSF.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5) The monocyclic right S-act $S_S \cong S/\triangle_S = S/\rho(s,s)$, $s \in S$, satisfies Condition (PWP_{sc}). So, it satisfies Condition (PWP_{ssc}). Thus, S is left PSF by [6, Theorem 2.2].

(5) $\Rightarrow$ (1) Suppose that A_S satisfies Condition (PWP_{sc}). Then, A_S is principally weakly flat by Theorem 2.4. So, A_S satisfies Condition (PWP_{ssc}) by [6, Theorem 2.8]. □

Theorem 2.31. *The following statements are equivalent.*

(1) *All right S-acts satisfying Condition (PWP_{sc}) are divisible.*
(2) *All finitely generated right S-acts satisfying Condition (PWP_{sc}) are divisible.*
(3) *All cyclic right S-acts satisfying Condition (PWP_{sc}) are divisible.*
(4) *All monocyclic right S-acts satisfying Condition (PWP_{sc}) are divisible.*
(5) *All right S-acts are divisible.*
(6) *For every $c \in C_l$, $Sc = S$.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5) The monocyclic right S-act $S_S \cong S/\triangle_S = S/\rho(s,s)$, $s \in S$, satisfies Condition (PWP_{sc}). So, S_S is divisible, by the assumption. Hence, all right S-acts are divisible by [7, Proposition 4.2.2].

(5) $\Rightarrow$ (6) Every left cancellable element of S is left invertible, by [7, Proposition 4.2.2]. So, $Sc = S$ for every $c \in C_l$.

(6) $\Rightarrow$ (1). All right S-acts are divisible, by [7, Proposition 4.2.2]. So, all right S-acts satisfying Condition (PWP_{sc}) are divisible. □

Recall from [14] that for S, the Cartesian product $S \times S$, equipped with the right S-action $(s,t)u = (su,tu)$, for $s,t,u \in S$, is called the *diagonal act* of S. It is denoted by $D(S)$.

Theorem 2.32. *The following statements are equivalent.*

(1) *S is left PSF.*
(2) *S is left $P(P)$, and S_S^n satisfies Condition (PWP_{sc}) for every $n \in \mathbb{N}$.*
(3) *S is weakly left $P(P)$, and S_S^n satisfies Condition (PWP_{sc}) for every $n \in \mathbb{N}$.*
(4) *S is left $P(P)$ and $D(S)$ satisfies Condition (PWP_{sc}).*
(5) *S is weakly left $P(P)$ and $D(S)$ satisfies Condition (PWP_{sc}).*

Proof. Implications (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are obvious.

(1) $\Rightarrow$ (2) Every left PSF monoid is left $P(P)$. By [11, Corollary 2.16], S_S^n is principally weakly flat, for every $n \in \mathbb{N}$. So, Theorem 2.5 shows that S_S^n satisfies Condition (PWP_{sc}), for every $n \in \mathbb{N}$.

(5) $\Rightarrow$ (1) $D(S)$ is principally weakly flat, by Theorem 2.4. Thus, by [14, Theorem 2.5], S is left PSF. □

Theorem 2.33. *Let S be a commutative monoid. Then, the following statements are equivalent.*

(1) *S is left PSF.*
(2) *S_S^n satisfies Condition (PWP_{sc}), for every $n \in \mathbb{N}$.*
(3) *$D(S)$ satisfies Condition (PWP_{sc}).*

Proof. Implication (1) ⇒ (2) follows from Theorem 2.32.

Implication (2) ⇒ (3) is obvious.

(3) ⇒ (1) This follows from Theorem 2.4 and [14, Proposition 3.2]. □

In the following theorem, we present some conditions for a monoid which are equivalent to the property of being left PP.

Theorem 2.34. *The following statements are equivalent.*

(1) *S is left PP.*
(2) *S is left PSF and the submonoid $[1]_{\ker \rho_s}$ of S, $s \in S$, contains a right zero.*
(3) *S is left $P(P)$ and the submonoid $[1]_{\ker \rho_s}$ of S, $s \in S$, contains a right zero.*
(4) *S is left PSF and S_S^I satisfies Condition (PWP_{sc}), for any nonempty set I.*
(5) *S is left PSF and $S_S^{S\times S}$ satisfies Condition (PWP_{sc}).*
(6) *S is left P(P) and S_S^I satisfies Condition (PWP_{sc}), for any nonempty set I.*
(7) *S is left P(P) and $S_S^{S\times S}$ satisfies Condition (PWP_{sc}).*
(8) *S is weakly left P(P) and S_S^I satisfies Condition (PWP_{sc}), for any nonempty set I.*
(9) *S is weakly left P(P) and $S_S^{S\times S}$ satisfies Condition (PWP_{sc}).*

Proof. Implications (2) ⇒ (3), (4) ⇒ (5) ⇒ (7) ⇒ (9) and (4) ⇒ (6) ⇒ (8) ⇒ (9) are obvious, because left PSF ⇒ left $P(P)$ ⇒ weakly left $P(P)$.

(1) ⇒ (2) It is clear that every left PP monoid is left PSF. Now, we show that $[1]_{\ker \rho_s}$, $s \in S$, contains a right zero. By the assumption, there exists $e \in E(S)$ such that $\ker \rho_s = \ker \rho_e$. Now,

$$(1, e) \in \ker \rho_e = \ker \rho_s$$

and thus, $e \in [1]_{\ker \rho_s}$. If $t \in [1]_{\ker \rho_s}$, then $(1, t) \in \ker \rho_e$, which implies $te = e$, that is, e is a right zero of the submonoid $[1]_{\ker \rho_s}$.

(3) ⇒ (1) Let $s \in S$ and e be a right zero of $[1]_{\ker \rho_s}$. We claim that $\ker \rho_s = \ker \rho_e$. Let $(l_1, l_2) \in \ker \rho_e$. Since $e \in [1]_{\ker \rho_s}$, $s = es$. Thus,

$$l_1 s = l_1 es = l_2 es = l_2 s$$

and so, $(l_1, l_2) \in \ker \rho_s$. Hence, $\ker \rho_e \subseteq \ker \rho_s$. Now, let $(x, y) \in \ker \rho_s$. Since S is left $P(P)$, there exist $u, v \in S$ such that $s = us = vs$ and $xu = yv$. From $s = us = vs$ we deduce that $(1, u), (1, v) \in \ker \rho_s$. So, $u, v \in [1]_{\ker \rho_s}$. Since e is a right zero of $[1]_{\ker \rho_s}$, $ue = e$ and $ve = e$. Then, $xu = yv$ implies that $xe = xue = yve = ye$, that is, $(x, y) \in \ker \rho_e$. Thus $\ker \rho_s = \ker \rho_e$, which implies that S is left PP.

(1) ⇒ (4) By [14, Corollary 2.6], S_S^I is principally weakly flat. Thus, by Theorem 2.6, S_S^I satisfies Condition (PWP_{sc}) for any nonempty set I.

(9) ⇒ (1) By Theorem 2.4, S is weakly left P(P) and $S_S^{S\times S}$ is principally weakly flat. So, for any nonempty set I, S_S^I is principally weakly flat by [13, Proposition 2.2.]. Thus, S is left PP, by [14, Corollary 2.6]. □

Now, we investigate the previous theorem for a commutative monoid S.

Theorem 2.35. *Let S be a commutative monoid. Then, the following statements are equivalent.*

(1) *S is left PP.*
(2) *S_S^I satisfies Condition (PWP_{sc}), for any nonempty set I.*
(3) *$S_S^{S\times S}$ satisfies Condition (PWP_{sc}).*

Proof. (1) ⇒ (2) The proof is straightforward by Theorem 2.34.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) By Theorem 2.4, $S_S^{S\times S}$ is principally weakly flat. So, for any nonempty set I, S_S^I is principally weakly flat, by [13, Proposition 2.2.]. Thus, S is left PP, by [14, Proposition 3.2]. □

We summarize our results in the commutative and non-commutative cases in Table 1.

Comparing the tables in the commutative and non-commutative cases, we find that for commutative monoids, the first condition is removed.

Table 1. Classification of commutative and non-commutative monoids

First condition		Second condition		Equivalent condition
For non-commutative monoids				
S is left $P(P)$		S_S^n satisfies Condition (PWP_{sc}), for every $n \in \mathbb{N}$	$\Longleftrightarrow$	S is left PSF
S is weakly left $P(P)$		S_S^n satisfies Condition (PWP_{sc}), for every $n \in \mathbb{N}$	$\Longleftrightarrow$	S is left PSF
S is left PSF		S_S^I satisfies Condition (PWP_{sc}), for every nonempty set I	$\Longleftrightarrow$	S is left PP
S is left $P(P)$		S_S^I satisfies Condition (PWP_{sc}), for every nonempty set I	$\Longleftrightarrow$	S is left PP
S is weakly left $P(P)$		S_S^I satisfies Condition (PWP_{sc}), for every nonempty set I	$\Longleftrightarrow$	S is left PP
For commutative monoids				
S is left PSF	$\Longleftrightarrow$	S_S^n satisfies Condition (PWP_{sc}), for every $n \in \mathbb{N}$		
S is left PP	$\Longleftrightarrow$	S_S^I satisfies Condition (PWP_{sc}), for every nonempty set I		

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ON HOMOLOGICAL CLASSIFICATION OF MONOIDS BY CONDITION $(\mathbf{PWP_{sc}})$ OF RIGHT ACTS

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دسته‌بندی همولوژیکی تکواره‌ها بر اساس شرط (PWP_{sc}) از سیستم‌های راست

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در این مقاله شرط (PWP_{sc}) را به عنوان تعمیمی از شرط (PWP_E) سیستم‌ها روی تکواره‌ها معرفی می‌کنیم و مشاهده می‌کنیم که شرط (PWP_{sc})، شرط (PWP_E) را نتیجه نمی‌دهد. به طور کلی نشان می‌دهیم که شرط (PWP_{sc})، خاصیت به طور اساسی ضعیف هموار را نتیجه می‌دهد و این‌که در تکواره‌های PSF چپ، عکس این نتیجه‌گیری نیز همواره درست است. به علاوه، بعضی از خواص کلی و یک دسته‌بندی از تکواره‌ها را با مقایسه شرط (PWP_{sc}) با ویژگی‌های دیگر ارائه می‌دهیم. در پایان، تکواره‌های PSF چپ را برای وقتی که S_S^I، برای هر مجموعه ناتهی I، در شرط (PWP_{sc}) صدق کند توصیف می‌کنیم.

کلمات کلیدی: S–سیستم، شرط (PWP_{sc})، همواری.