Journal of Algebraic Systems
Vol. 10, No. 2, (2023), pp 309-321

ON THE PATH HYPEROPERATION AND ITS CONNECTIONS WITH HYPERGRAPH THEORY

R. BAYAT TAJVAR* AND M. LATIFI

ABSTRACT. In this paper, we introduce a path hyperoperation associated with a hypergraph, which is an extension of the Corsini's hyperoperation. We investigate some related properties and study relations between the path hyperoperation and hypergraph theory.

1. Introduction

Hypergraphs in the early 1960s as a generalization of graphs, became an independent theory. Hypergraphs are systems of finite sets and form the most general concept in discrete mathematics. This branch of mathematics has developed very rapidly during the twentieth century. In [2], there is a very good presentation of graph and hypergraph theory.

Algebraic hyperstructures, in particular hypergroups, were introduced in 1934 by Marty, at the 8th Congress of Scandinavian Mathematicians (see [23]) and then it was developed by many researchers. Since then, hundreds of papers and several books have been written on this topic. Nowadays, there are many connections between hyperstructures and other branches of mathematics, leading to applications in hypergraphs, binary relations, combinatorics, artificial intelligence, automata and fuzzy sets. One can find a survey of hyperstructure theory and their applications in [9, 10].

DOI: 10.22044/JAS.2022.11493.1580.
MSC(2010): Primary: 05C65; Secondary: 20N20, 14L17.
Keywords: Hypergraph; Path hyperoperation; Partial hyperoperation; Directed hypergraph.
Received: 15 December 2021, Accepted: 24 February 2022.
*Corresponding author.

The concept of hypergroupoids deriving from binary relations, namely *C-hypergroupoids* were delineated by Corsini in [7] (see also [24, 25, 26, 27]). Also, the connection between hyperstructures and binary relations in general, has been investigated in [10, 11, 12, 13, 14].

A new class of hyperoperations, namely path hyperoperations that are obtained from binary relations and their connections with graph theory, were introduced by Kalampakas et al. [18, 20]. In this paper, we introduce path hyperoperations by obtaining it from directed hypergraphs, which are an extension of the Corsini's hyperoperation. Also, we study some properties of these hyperoperations and their
correlations with hypergraph theory are investigated.

More exactly, for a given hypergraph $\Gamma = (V, E)$, we define a *path hyperoperation* on V by,

$$\circ_\Gamma : V \times V \to P^*(V) \ : \ (u, v) \mapsto u \circ_\Gamma v,$$

where

$$u \circ_\Gamma v := \{z \in V \mid z \text{ belongs to at least one directed path from } u \text{ to } v\}.$$

It is obvious that this definition is a natural extension of the Corsini's hyperoperation. In other words, for each binary relation $E \subseteq V \times V$ and elements $u, v \in V$, the Corsini product $u \bullet_E v$ is a subset of $u \circ_\Gamma v$ i.e., $u \bullet_E v \subseteq u \circ_\Gamma v$. The relationship between this concept and directed hypergraphs is presented by proving that: A directed hypergraph $\Gamma = (V, E)$, is strongly connected if and only if the associated path hyperoperation is non-partial.

This article is organized as follows. In Section 2, we review several basic concepts of hypergraphs and hypergroupoids. In Section 3 we introduce the path hyperoperation which is an extension of the Corsini's hyperoperation. Also we study the connection between this concept with hypergraph theory. In Section 4, we present some connections between commutative path hypergroupoids and hypergraphs. In Section 5, we introduce an induced equivalence relation on a path hypergroupoid associated with a given hypergraph and we prove that every commutative path hypergroupoid can be obtained as the disjoint union of a set of non-partial commutative hypergroupoids. Finally, we discuss applications of the path hyperoperations in Section 6.

2. Preliminary Results

In this section, we introduce some preliminary results and definitions which will be needed in the subsequent sections.

We provide some definitions from the theory of hypergraphs. The interested reader should refer to [2, 3] for more concepts of hypergraph theory.

A *hypergraph* Γ is a pair (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of discrete elements known as vertices (or nodes) and $E = \{e_1, e_2, \ldots, e_m\}$ is a collection of arbitrary non-void subsets of V such that $\bigcup_j e_j = V$, known as edges (or hyperedges). A hypergraph is a generalization of an ordinary undirected graph, such that a hyperedge does not need to contain exactly two vertices, but can instead contain an arbitrary non-zero number of nodes. Also, an ordinary undirected graph (without self-loops) is a hypergraph such that every edge has exactly two vertices. Two vertices u and v are *adjacent* in $\Gamma = (V, E)$ if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$, $e \cap f \neq \emptyset$, then we say that e and f are *adjacent*. A vertex v and an edge e are *incident* if $v \in e$. We denote by $\Gamma(v)$ the *neighborhood* of a vertex v, i.e. $\Gamma(v) = \{u \in V : \{u, v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or d*-regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph Γ is *simple* if there are no repeated edges and no edge properly contains another. A hypergraph is known as *uniform* or k*-uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertices is a 2-uniform hypergraph.

A *partial hypergraph* (*or subhypergraph*) $\Gamma^{'} = (V^{'}, E^{'})$ of a hypergraph $\Gamma = (V, E)$, denoted by $\Gamma^{'} \subseteq \Gamma$, is a hypergraph such that $V^{'} \subseteq V$ and $E^{'} \subseteq E$. The partial hypergraph $\Gamma^{'} = (V^{'}, E^{'})$ is *induced* if $E^{'} = \{e \in E \mid e \subseteq V^{'}\}$. Induced hypergraphs will be denoted by $\langle V^{'} \rangle$. A partial hypergraph of a simple hypergraph is always simple.

Let $\Gamma = (V, E)$ be a hypergraph. A *path* of length k in Γ is an alternating sequence

$$P_{v_1, v_{k+1}} = (v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1})$$

in which $v_i \in V$ for each $i = 1, 2, \ldots, k+1$, $e_i \in E$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, \ldots, k$ and $v_i \neq v_j$, $e_i \neq e_j$ for $i \neq j$. Also, a hypergraph is *connected* if there is a path between every pair of vertices. A *connected component* of a hypergraph is every maximal set of vertices such that are pairwise connected by a path. A *cycle* of length k is a sequence $(v_1, e_1, v_2, \ldots, v_k, e_k, v_1)$, such that P_{v_1, v_k} is a path. Also, a hypergraph is called *acyclic* if it does not contain any cycles.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two hypergraphs. A *homomorphism* from Γ_1 into Γ_2 is a mapping $\varphi : V_1 \to V_2$ such that

$\varphi(e) = \{\varphi(v_1), \ldots, \varphi(v_r)\}$ is an hyperedge in Γ_2, if $e = \{v_1, \ldots, v_r\}$ is a hyperedge in Γ_1. Note, a homomorphism from Γ_1 into Γ_2 implies also a mapping $\varphi_E : E_1 \to E_2$. A homomorphism φ that is bijective is called an *isomorphism* if $\varphi(e) \in E_2$ if and only if $e \in E_1$ holds. We say, Γ_1 and Γ_2 are *isomorphic*, in symbols $\Gamma_1 \cong \Gamma_2$ if there exists an isomorphism between them. An isomorphism from a hypergraph Γ onto itself is an *automorphism*. The *automorphism group* of Γ is denoted by $\mathrm{Aut}(\Gamma)$. A hypergraph is *vertex transitive* if its automorphism group acts transitively on the set of vertices. Such a hypergraph is necessarily regular, that is, each vertex is incident to the same number of edges. In the same way a hypergraph is *edge transitive* if its automorphism group acts transitively on the set of edges.

Let H be a non-void set and $P^*(H)$ be the set of all non-void subsets of H. A *hyperoperation* on H is a map $* : H^2 \to P^*(H)$ and the couple $(H, *)$ is called a *partial hypergroupoid*. The structure $(H, *)$ is called a *non-partial hypergroupoid* if for every $x, y \in H$ we have $x * y \neq \emptyset$. A hypergroupoid $(H, *)$ is called a *total hypergroupoid* if for each pair of x and y in H, $x * y = H$. A hypergroupoid $(H, *)$ is called *commutative* if for all $x, y \in H$ we have $x * y = y * x$. Also, $(H, *)$ is called *weakly commutative* if $\forall x, y \in H,\ x * y \cap y * x \neq \emptyset$.

If A and B are non-void subsets of H, then $A * B$ is defined by,

$$A * B = \bigcup_{a \in A, b \in B} a * b.$$

(i) A *semihypergroup* is a hypergroupoid $(H, *)$ which satisfies the associative axiom:

$$\forall x, y, z \in H,\ (x * y) * z = x * (y * z).$$

(ii) A *quasihypergroup* is a hypergroupoid $(H, *)$ which satisfies the reproductive axiom:

$$\forall x \in H,\ x * H = H = H * x.$$

(iii) A *hypergroup* is a semihypergroup which is also a quasihypergroup.

A non-void subset K of a hypergroup $(H, *)$ is called a *subhypergroup* if it satisfies the reproductive axiom, i.e., for all $k \in K$,

$$k * K = K * k = K.$$

Let $(H_1, *_1)$ and $(H_2, *_2)$ be two hypergroupoids. A map $f : H_1 \to H_2$ is called a *homomorphism* if $\forall x, y \in H_1,\ f(x *_1 y) \subseteq f(x) *_2 f(y)$ and it is called a *good homomorphism* if for all $x, y \in H_1,\ f(x *_1 y) = f(x) *_2 f(y)$.

We say that the two hypergroups H_1 and H_2 are *isomorphic* if there is a good homomorphism between them which is also a bijection and we write $H_1 \cong H_2$.

The relationship between hyperstructure theory and hypergraph theory has been studied by many authors (see [1, 8, 15, 16, 17, 21, 22]).

Let $\Gamma = (H, \{E_i\}_i)$ be a hypergraph. The hypergroupoid $H_\Gamma = (H, \circ)$ such that the hyperoperation "$\circ$" on H is defined as follows:

$$\forall (x, y) \in H^2, \ x \circ y = E(x) \cup E(y),$$

is called a *hypergraph hypergroupoid*, where $E(x) = \bigcup_{x \in E_i} E_i$. The following results have been obtained by Corsini in [8].

Theorem 2.1. *Let $(H, \circ)$ be a hypergroupoid. Then for any $(x, y) \in H^2$, the following holds:*

(i) $x \circ y = x \circ x \cup y \circ y$,
(ii) $x \in x \circ x$,
(iii) $y \in x \circ x \Longleftrightarrow x \in y \circ y$.

Theorem 2.2. *Let $(H, \circ)$ be a hypergroupoid satisfying* (i), (ii) *and* (iii) *of Theorem 2.1. Then $(H, \circ)$ is a hypergroup if and only if the following condition is valid:*

$$\forall (x, y) \in H^2, \ y \circ y \circ y - y \circ y \subset x \circ x \circ x. \tag{2.1}$$

Corollary 2.3. *Let $(H, \circ)$ be a hypergroupoid which satisfies* (i), (ii) *and* (iii) *of Theorem 2.1 and the condition:*

$$\forall x \in H, \ x \circ x \circ x = x \circ x. \tag{2.2}$$

Then $(H, \circ)$ is a hypergroup.

Example 2.4. Let $\Gamma = \{\{1, 2\}, \{2, 3\}\}$ be a hypergraph. We have

$$1 \circ 1 = \{1, 2\} \neq 1 \circ 1 \circ 1 = \{1, 2, 3\}.$$

Then, clearly H_Γ does not satisfy (2.2), but satisfying (2.1). Thus by Theorem 2.2, H_Γ is a hypergroup.

Let $E \subseteq H \times H$ be a binary relation. The Corsini's hyperoperation $\bullet_E : H \times H \to P^*(H)$ is defined as follows:

$$(x, y) \mapsto x \bullet_E y := \{z \in H \mid (x, z) \in E \ and \ (z, y) \in E\}.$$

The hypergroupoid $(H, \bullet_E)$ is called *Corsini's partial hypergroupoid* or simply *partial C-hypergroupoid* associated with the binary relation on H (see [7, 19, 24, 25]). If $x \bullet_E y \neq \emptyset$ for all $x, y \in H$, then $(H, \bullet_E)$ is called C-hypergroupoid. Clearly, a partial C-hypergroupoid $(H, \bullet_E)$ is a C-hypergroupoid if and only if $E \circ E = H^2$, so that "$\circ$" is the usual relation composition.

In this paper we introduce a *path hyperoperation* which is an extension of the Corsini's hyperoperation, obtained from directed hypergraphs.

3. Path Hyperoperations and Hypergraph Theory

In this section, we introduce path hypergroupoids that are obtained from directed hypergraphs, which are an extension of the Corsini's hypergroupoids and investigate their connections with hypergraph theory.

Let $\Gamma = (V, E)$ be a hypergraph. We define the *path hyperoperation* $\circ_\Gamma : V \times V \to P^*(V)$ for all $x, y \in V$ as follows:

$$x \circ_\Gamma y := \{z \in V \mid z \text{ belongs to a path from } x \text{ to } y\}.$$

The (partial) hypergroupoid $(V, \circ_\Gamma)$ is called the (partial) *path hypergroupoid* corresponding with Γ.

The hyperoperation "$\circ_\Gamma$" on V is called a *non-partial hyperoperation* if for all $x, y \in V$ we have $x \circ_\Gamma y \neq \emptyset$. In this case, the path hypergroupoid associated with Γ is called non-partial.

It is easy to check that, for any hypergraph $\Gamma = (V, E)$ and for all $x, y \in V$, the Corsini product $x \bullet_E y$ is a subset of $x \circ_\Gamma y$ i.e., $x \bullet_E y \subseteq x \circ_\Gamma y$.

By the definition of the path in a hypergraph we obtain the following results.

Proposition 3.1. *Let $\Gamma = (V, E)$ be a hypergraph. Then for any $x, y \in V$, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y.*

Proposition 3.2. *([19]) Let $G = (V, E)$ be a graph, then the Corsini hyperoperation "$\bullet_E$" associated with G, is non-partial if and only if there exists a path with length 2 between any pair of vertices of G.*

This result allows us to prove the following corollary.

Corollary 3.3. *Let $\Gamma = (V, E)$ be a hypergraph and let "$\circ_\Gamma$" be the associated path hyperoperation with Γ. Then "$\circ_\Gamma$" is a non-partial hyperoperation if and only if for any $x, y \in V$, there exists a path from x to y.*

Proof. The path hyperoperation associated with Γ is a partial hyperoperation if and only if $x \circ_\Gamma y \neq \emptyset$ for all $x, y \in V$. Therefore, by Proposition 3.1, $x \circ_\Gamma y \neq \emptyset$ if and only if there exists a path from x to y. □

Directed hypergraphs is a generalization of directed graphs (digraphs). Directed hypergraphs modelling can be very useful in formal language theory, relational database theory, scheduling and many other fields. A directed hypergraph is defined as follows.

A *directed hypergraph* Γ (or *dihypergraph*) is a pair (V, E), where V is a finite set of vertices and E is a set of *hyperarcs*. A hyperarc E is an ordered pair (T, H) of disjoint subsets of V. The set T is the *tail set* of the hyperarc, while H is called the *head set* of the hyperarc.

The *size* of a dihypergraph Γ is defined as $|\Gamma| = \Sigma_{e \in E}|tail(e)|$. An example of a directed hypergraph is illustrated in Figure 1.

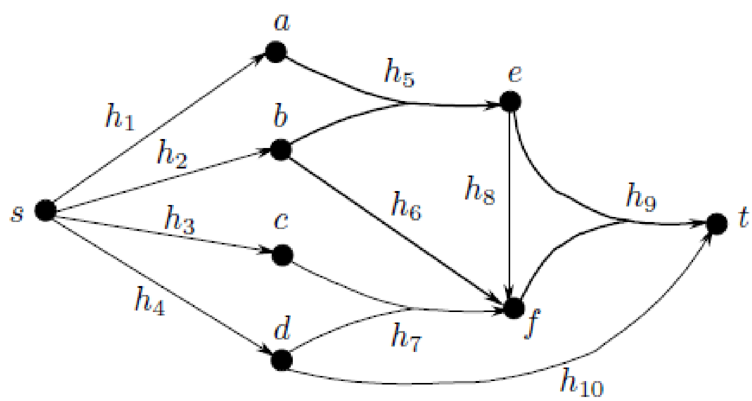

FIGURE 1. A directed hypergraph

In the following definition we state the notion of strong connectivity of hypergraphs. This concept is a generalization of strong connectivity of graphs. Hypergraph connectivity can be used in networking, mobile communication systems, shortest path, database theory, image processing and numerous other applications.

Definition 3.4. A dihypergraph $\Gamma = (V, E)$ is called *strongly connected* if for every $x, y \in V$ there exists at least one directed path between x and y.

In the sequel we study the connection between the path hyperoperation and hypergraph connectivity.

Theorem 3.5. *Let $\Gamma = (V, E)$ be a dihypergraph. Then Γ is strongly connected if and only if the associated path hyperoperation is non-partial.*

Proof. Suppose that Γ is a strongly connected hypergraph. Then for any $x, y \in V$ there exists a path between x and y. By Proposition

3.1 we have $x \circ_\Gamma y \neq \emptyset$ for every $x, y \in V$. Thus the associated path hyperoperation with Γ is non-partial.

Conversely, suppose that the associated path hyperoperation with Γ, "$\circ_\Gamma$" be non-partial. So, for any $x, y \in V$ we have $x \circ_\Gamma y \neq \emptyset$. Therefore, by Proposition 3.1 there exists a path between x and y. Thus Γ is strongly connected. □

For a given dihypergraph $\Gamma = (V, E)$, the strongly connected component of Γ is called the *strongly connected partial hypergraph.* More exactly, a strongly connected component of a hypergraph $\Gamma = (V, E)$ is a maximal partial hypergraph $\Gamma' = (V', E')$ such that there exists a path between every two vertices of V'. Thus we have the following result.

Proposition 3.6. *Let $\Gamma = (V, E)$ be a dihypergraph and $x, y \in V$. Then $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$ if and only if x and y belong to the common strongly connected component of Γ.*

Proof. By Theorem 3.5, the proof is obvious. □

4. Commutative Path Hyperoperations and Hypergraph Theory

In this section, we analyse some connections between commutative path hypergroupoids and hypergraphs.

In the following proposition, we state the relation between a path hyperoperation and cycles of a hypergraph.

Theorem 4.1. *Let $\Gamma = (V, E)$ be a hypergraph and $x, y \in V$. Then $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$ if and only if x and y belong to at least one same cycle in Γ.*

Proof. Suppose that $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$, then by Proposition 3.1, there exists a path from x to y and a path from y to x, for any $x, y \in V$. Therefore, there exists at least one cycle passing through x and y.

Conversely, suppose that x and y belong to one common cycle in Γ. Clearly, this cycle can be separated into two paths one from x to y and other from y to x. Thus, by Proposition 3.1, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$. □

Now, we obtain the following result by Theorem 4.1 .

Corollary 4.2. *Let $\Gamma = (V, E)$ be a hypergraph and $x, y \in V$. Then Γ is acyclic if and only if either the path hyperoperation is non-commutative or $x \circ_\Gamma y = y \circ_\Gamma x = \emptyset$ holds.*

This result allows us to prove the following theorem.

Theorem 4.3. *Let $\Gamma = (V, E)$ be a hypergraph such that for all $x, y \in V$, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$. Then the associated path hypergroupoid "$\circ_\Gamma$" is commutative.*

Proof. We must prove that $x \circ_\Gamma y = y \circ_\Gamma x$. To prove this, we first need to show that $x \circ_\Gamma y \subseteq y \circ_\Gamma x$. Since $x \circ_\Gamma y \neq \emptyset$, it follows that there exists a vertex $z \in x \circ_\Gamma y$. Thus, there exists a path from x to y in Γ as follows:

$$x, e_0, x_1, e_1, \ldots, z, e_z, \ldots, x_n, e_n, y.$$

Also, since $y \circ_\Gamma x \neq \emptyset$, it follows that there exists a path from y to x in Γ as follows:

$$y, e_0^{'}, y_1, e_1^{'}, \ldots, y_n, e_n^{'}, x.$$

Thus, there exists a path from y to x, passing from z as follows:

$$y, e_0^{'}, y_1, \ldots, y_n, e_n^{'}, x, e_0, x_1, \ldots, z, e_z, \ldots, x_n, e_n, y, e_0^{'}, y_1, \ldots, y_n, e_n^{'}, x.$$

Therefore, $z \in y \circ_\Gamma x$ holds. Similarly, we can prove that $y \circ_\Gamma x \subseteq x \circ_\Gamma y$. This completes the proof. □

From Theorem 4.3, it is easy to show the following corollary.

Corollary 4.4. *Let $\Gamma = (V, E)$ be a hypergraph. Then any non-partial path hypergroupoid associated with Γ is commutative.*

Theorem 4.5. *Let $\Gamma = (V, E)$ be a hypergraph such that for all $x, y, z \in V$, $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma z \neq \emptyset$. If the associated path hyperoperation, "$\circ_\Gamma$" is commutative, then $x \circ_\Gamma y = y \circ_\Gamma z$.*

Proof. We first need to show that $x \circ_\Gamma y \subseteq y \circ_\Gamma z$. Since $x \circ_\Gamma y \neq \emptyset$, it follows that there is a vertex $w \in x \circ_\Gamma y$. Thus, there exists a path from x to y in Γ which contains w as follows:

$$x, e_0, x_1, e_1, \ldots, w, e_w, \ldots, x_n, e_n, y.$$

By hypothesis, since "$\circ_\Gamma$" is commutative, so $y \circ_\Gamma x \neq \emptyset$, thus there exists at least one path from y to x as follows:

$$y, e_0^{'}, y_1, e_1^{'}, \ldots, y_n, e_n^{'}, x.$$

Also, since $y \circ_\Gamma z \neq \emptyset$, thus there exists a path from y to z as follows:

$$y, e_0^{''}, z_1, e_1^{''}, \ldots, z_n, e_n^{''}, z.$$

Thus, there exists a path from y to z, passing from w as follows:

$$y, e_0^{'}, y_1, \ldots, y_n, e_n^{'}, x, e_0, x_1, \ldots, w, e_w, \ldots, x_n, e_n, y, e_0^{''}, z_1, \ldots, z_n, e_n^{''}, z.$$

Therefore, $w \in y \circ_\Gamma z$ and thus $x \circ_\Gamma y \subseteq y \circ_\Gamma z$. Similarly, we can prove that $y \circ_\Gamma z \subseteq x \circ_\Gamma y$. This completes the proof. □

5. Equivalence Relation on a Path Hypergroupoid

In this section, we introduce an induced equivalence relation on a path hypergroupoid associated with a given hypergraph and we prove that every commutative path hypergroupoid can be obtained as the disjoint union of a set of non-partial commutative hypergroupoids.

For a given hypergraph $\Gamma = (V, E)$, we define a relation $\sim_V$ on the associated path hypergroupoid $(V, \circ_\Gamma)$ as follows: for all $x, y \in V$,

$$x \sim_V y \Longleftrightarrow x \circ_\Gamma y \neq \emptyset.$$

We have the following proposition:

Proposition 5.1. *Let $\Gamma = (V, E)$ be a hypergraph and "$\circ_\Gamma$" be the associated path hyperoperation with Γ. If "$\circ_\Gamma$" is commutative, then $\sim_V$ is an equivalence relation on V.*

Proof. By Theorem 4.3 clearly, $\sim_V$ satisfies the symmetric relation. Also, it is easy to see that the transitive relation holds by Theorem 4.5. To see that $\sim_V$ is reflexive, without loss of generality, we may suppose that Γ be a hypergraph without isolated vertices. It follows that for any vertex in V, there is at least one incoming hyperedge to it or one outgoing hyperedge from it. Thus, for any $x \in V$, there is at least one $y \in V$ such that $x \circ_\Gamma y \neq \emptyset$ or $y \circ_\Gamma x \neq \emptyset$. By hypothesis, since "$\circ_\Gamma$" is commutative, we have: $x \circ_\Gamma y = y \circ_\Gamma x \neq \emptyset$. It follows that there exists at least one path from x to y and at least one path from y to x. Therefore, there exists a path from x to x and thus we have $x \circ_\Gamma x \neq \emptyset$. Hence, $\sim_V$ is reflexive and thus $\sim_V$ is an equivalence relation on V. □

The following theorem gives a necessary and sufficient condition for a characterization of hypergraphs with commutative associated path hyperoperation.

Theorem 5.2. *Let $\Gamma = (V, E)$ be a hypergraph. Then the associated path hyperoperation with Γ is commutative, if and only if Γ consists of a disjoint union of strongly connected hypergraphs.*

Proof. Suppose that "$\circ_\Gamma$" is the associated path hyperoperation with Γ and let "$\circ_\Gamma$" be commutative. Then $\sim_V$ is an equivalence relation on V, by Proposition 5.1. Let $V_1, V_2, \ldots, V_n$ be a partition of V induced by $\sim_V$ i.e.,

$$V = V_1 \cup V_2 \cup \ldots \cup V_n \text{ such that } V_i \cap V_j = \emptyset,$$

for all $1 \leq i, j \leq n$, $i \neq j$. We have to show that $\Gamma = (V, E)$ is constructed by the disjoint union of strongly connected hypergraphs as follows:

$$\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2), \ldots, \Gamma_n = (V_n, E_n).$$

Assume that $x, y \in V_i,\ 1 \leq i \leq n$, then $x \sim_V y$ and thus, there exists a path from x to y. Therefore, the partial hypergraph $\Gamma_i = (V_i, E_i)$ is strongly connected for any $1 \leq i \leq n$. Now we prove that for any $x \in V_i$ and $y \in V_j,\ 1 \leq i, j \leq n,\ i \neq j$, there does not exist a path from x to y. Suppose that such a path exists. Thus $x \circ_\Gamma y \neq \emptyset$ and so $x \sim_V y$ which leads to a contradiction because V_i and V_j are different partition classes of $\sim_V$. Hence, Γ is constructed by the disjoint union of Γ_i where $1 \leq i \leq n$.

Conversely, suppose that Γ consists of the disjoint union of n strongly connected hypergraphs as follows:

$$\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2), \ldots, \Gamma_n = (V_n, E_n).$$

where,

$$V = V_1 \cup V_2 \cup \ldots \cup V_n \text{ such that } V_i \cap V_j = \emptyset,$$

for all $1 \leq i, j \leq n,\ i \neq j$. Now we have to show that "$\circ_\Gamma$" is commutative i.e., for all $x, y \in V$, $x \circ_\Gamma y = y \circ_\Gamma x$. Let $x, y \in V_i,\ 1 \leq i \leq n$. Since $\Gamma_i = (V_i, E_i)$ is strongly connected, by Proposition 3.6 we have $x \circ_\Gamma y \neq \emptyset$ and $y \circ_\Gamma x \neq \emptyset$, also by Theorem 4.3, we obtain $x \circ_\Gamma y = y \circ_\Gamma x$. Now we consider that x and y belong to different strongly connected components Γ_i and Γ_j. Since Γ consists of the disjoint union of n strongly connected hypergraphs $\Gamma_1, \ldots, \Gamma_n$, therefore, there exists no path from x to y or from y to x. It follows that $x \circ_\Gamma y = y \circ_\Gamma x = \emptyset$. Hence, "$\circ_\Gamma$" is commutative. □

6. Applications of Path Hyperoperations

In this section, we describe application of the path hyperoperations in designing *mixed-model assembly lines*.

Nowadays, mixed-model assembly lines are found in many industrial environments. With the grown trend towards product variability and shorter life cycle, mixed-model assembly lines are replacing the traditional single-model assembly lines. In marketplaces where product life cycles are short, and product variety requests are high, many different models of a product should be constructed in relatively small lot sizes and reach the customer in a short lead-time. Also, assembly lines must still attain high productivity, similar quality and low assembly cost [6].

In a mixed-model assembly line, setup times and costs are decreased sufficiently enough to be ignored, so that various products are jointly produced in intermixed product sequences on the same line. As a result, usually, all models are variations of the same base product and only differ in special customizable product attributes, that are called *options*. During the configuration sketching of an assembly line, that is generally known as the *assembly line balancing problem* (ALB-P) has to be solved which specifies the assignment of tasks and all their required resources to the workstations of the line [5]. The design of such a system requires the assignment of special tasks to workstations with regard to a given set of precedence constraints. The so obtained assembly sequence, which can be visualized by digraphs, are usually called *precedence graphs* [20].

The introduced path hyperoperation is utilized in order to construct the precedence graph of a special product with respect to a known set of precedence relations. Furthermore, a similar procedure is also employed in order to design the *joint precedence graph* of two or more precedence graphs which encompass all their characteristics.

Acknowledgments

The authors are highly thankful to the Editor-in-Chief and the referees for their valuable comments and suggestions for improving the paper.

References

1. M. I. Ali, Hypergraphs, hypergroupoid and hypergroups, *Ital. J. Pure Appl. Math.*, **8** (2000), 45–48.
2. C. Berge, *Graphs and Hypergraphs*, North-Holland, New York, 1976.
3. C. Berge, *Hypergraphs*, North-Holland, Amsterdam, 1989.
4. N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, New York, 1974.
5. N. Boysen, M. Fliedner, A. Scholl, Assembly line balancing: Joint precedence graphs under high product variety, *IIE Transactions*, **41** (2009), 183–193.
6. J. Bukchin, E. M. Dar-El, J. Rubinovitz, Mixed model assembly line design in a make-to-order environment, *Comput. Ind. Eng.*, **41** (2001), 405–421.
7. P. Corsini, Binary relations and hypergroupoids, *Ital. J. Pure Appl. Math.*, **7** (2000), 11–18.
8. P. Corsini, Hypergraphs and hypergroups, *Algebra Universalis*, **35** (1996), 548–555.
9. P. Corsini, *Prolegomena of Hypergroups Theory*, Aviani Editore, Italy, 1993.
10. P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.
11. P. Corsini, V. Leoreanu, Hypergroups and binary relations, *Algebra Universalis*, **43** (2000), 321–330.
12. I. Cristea, M. Stefanescu, Binary relations and reduced hypergroups, *Discrete Math.*, **308** (2008), 3537–3544.
13. I. Cristea, M. Stefanescu, Hypergroups and n-ary relations, *European J. Combin.*, **31** (2010), 780–789.

14. I. Cristea, M. Stefanescu, C. Anghluta, About the fundamental relations defined on the hypergroupoids associated with binary relations, *European J. Combin.*, **32** (2011), 72–81.
15. M. De Salvo, G. Lo Faro, A new class of hypergroupoids associated to binary relations, *J. Mult.-Valued Logic Soft Comput.*, **9** (2003), 361–375.
16. M. Farshi, B. Davvaz, S. Mirvakili, Degree hypergroupoids associated with hypergraphs, *J. Faculty of Sciences and Math, University of Nis, Serbia.*, **28** (2014), 119–129.
17. M. N. Iradmusa, A. Iranmanesh, The combinatorial and algebraic structure of the hypergroup associated to a hypergraph, *J. Mult.-Valued Logic Soft Comput.*, **11** (2005), 127–136.
18. A. Kalampakas, S. Spartalis, Path hypergroupoids: Commutativity and graph connectivity, *European J. Combin.*, **44** (2015), 257–264.
19. A. Kalampakas, S. Spartalis, K. Skoulariki, Directed graphs representing isomorphism classes of C-hypergroupoids, *Ratio Math.*, **23** (2012), 51–64.
20. A. Kalampakas, S. Spartalis, A. Tsigkas, The path hyperoperation, *J. Anal. Univ. Ovidius Constanta, Ser. Mat.*, **22** (2014), 141–153.
21. V. Leoreanu and L. Leoreanu, Hypergroups associated with hypergraphs, *Ital. J. Pure Appl. Math.*, **4** (1998), 119–126.
22. V. Leoreanu-Fotea, P. Corsini, A. Iranmanesh, On the sequence of hypergroups and membership functions determined by a hypergraph, *J. Mult.-Valued Logic Soft Comput.*, **14** (2008), 565–577.
23. F. Marty, Sur une généralization de la notion de group, *8th Congress Math. Scandenaves, Stockholm*, (1934), 45–49.
24. S. Spartalis, Hypergroupoids obtained from groupoids with binary relations, *Ital. J. Pure Appl. Math.*, **16** (2004), 201–210.
25. S. Spartalis, The hyperoperation relation and the Corsini's partial or not-partial hypergroupoids (A classification), *Ital. J. Pure Appl. Math.*, **24** (2008), 97–112.
26. S. Spartalis, M. Konstantinidou, A. Taouktsoglou, C-hypergroupoids obtained by special binary relations, *Comput. Math. Appl.*, **59** (2010), 2628–2635.
27. S. Spartalis, C. Mamaloukas, On hyperstructures associated with binary relations, *Comput. Math. Appl.*, **51** (2006), 41–50.

Reza Bayat Tajvar
Mathematics Department, Faculty of Basic Science, Khatam-ol-Anbia (PBU) University, Tehran, Iran.
Email: r.bayat.tajvar@gmail.com

Mehdi Latifi
Mathematics Department, Faculty of Basic Science, Khatam-ol-Anbia (PBU) University, Tehran, Iran.
Email: latifi.mmali@gmail.com

Journal of Algebraic Systems

ON THE PATH HYPEROPERATION AND ITS CONNECTIONS WITH HYPERGRAPH THEORY

R. BAYAT AND M. LATIFI

ابرعمل‌های مسیر و ارتباط آن‌ها با نظریه ابرگراف

رضا بیات[۱] و مهدی لطیفی[۲]

[۱,۲] گروه ریاضی، دانشکده علوم پایه، دانشگاه خاتم‌الانبیاء (ص)، تهران، ایران

در این مقاله یک ابرعمل مسیر، متناظر با یک ابرگراف را معرفی می‌کنیم که تعمیمی از ابرعمل معرفی شده توسط کرسینی است. همچنین رابطه بین ابرعمل مسیر و نظریه ابرگراف و برخی از خصوصیات مرتبط، مورد مطالعه قرار خواهند گرفت.

کلمات کلیدی: ابرگراف، ابرعمل مسیر، ابرعمل جزئی، ابرگراف جهت‌دار.