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PERFECTNESS OF THE ANNIHILATOR GRAPH OF ARTINIAN COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring and $Z(R)$ be the set of its zero-divisors. The annihilator graph of R, denoted by $AG(R)$ is a simple undirected graph whose vertex set is $Z(R)^*$, the set of all nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$. In this paper, perfectness of the annihilator graph for some classes of rings is investigated. More precisely, we show that if R is an Artinian ring, then $AG(R)$ is perfect.

1. Introduction

One of the most important and active areas in algebraic combinatorics is study of graphs associated with rings. This field has attracted the attention of many researchers during the past 20 years. There are many papers on assigning a graph to a ring, see for instance [1, 2, 4, 5]. Let R be a commutative ring with nonzero identity. The annihilator graph of R, denoted by $AG(R)$ is a simple undirected graph whose vertex set is the set of all nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if

$$\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y).$$

The annihilator graph was first introduced in [4], and some of its properties have been studied. In [10], it was proved that if R is a finite direct product of integral domains, then $AG(R)$ is weakly

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perfect. Morover, in [8], for a nonreduced ring R it is shown that $AG(R)$ is perfect. In this article, we show that if R is a finite direct product of integral domains or if R is an Artinian ring, then $AG(R)$ is perfect.

We use the standard terminology for graphs following [12]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. We write $u \sim v$, to denote an edge with ends u, v. The open neighborhood of a vertex u is defined to be the set

$$N(u) = \{v \in V(G) \ : \ u \text{ is adjacent to } v\}$$

and the closed neighborhood of u is the set $N[u] = N(u) \cup \{u\}$. A graph $H = (V_0, E_0)$ is called a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an induced subgraph by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V(G)$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. For a graph G a subset $S \subseteq V(G)$ is called a clique if the subgraph induced on S is complete. The number of vertices in a largest clique of graph G is called the clique number of G and is often denoted by $\omega(G)$. For a graph G, let $\chi(G)$ denote the chromatic number of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be weakly perfect if $\omega(G) = \chi(G)$. A perfect graph G is a graph in which the chromatic number of every induced subgraph equals to the size of a largest clique of that subgraph.

Throughout this paper, all rings are assumed to be commutative with nonzero identity. We denote by $Z(R)$ the set of all zero-divisor elements of R. The set of nilpotent elements of R is denoted by $\mathrm{Nil}(R)$. For every element x of R, we denote the annihilator of x by $\mathrm{ann}_R(x) = \{r \in R : rx = 0\}$. For $A \subseteq R$ we let $A^* = A \setminus \{0\}$. Some more definitions, properties and notation about commutative rings can be found in [3, 9, 11].

2. The annihilator graph of Artinian rings is perfect

Let R be an Artinian ring, in this section we show that $AG(R)$ is perfect. We start with the following lemma, which has a fundamental role in proving the results of this section.

Lemma 2.1. *Let n be a positive integer and let $R = R_1 \times \cdots \times R_n$, where $R_i \cong \mathbb{Z}_4$, for every $1 \leq i \leq n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two nonzero zero-divisors of R. Then the following statements are true:*

(1) *If $Rx \nsubseteq Ry$ and $Ry \nsubseteq Rx$, then $x \sim y$ is an edge of $AG(R)$.*
(2) *If $x \sim y$ is an edge of $AG(R)$ and either $Rx \subseteq Ry$ or $Ry \subseteq Rx$, then for some $1 \leq i \leq n$, $x_i = y_i \in \mathrm{Nil}(R_i)^*$.*
(3) *If $Rx \subseteq Ry$ and $Rx \cap \mathrm{ann}_R(y) \neq 0$, then $x \sim y$ is an edge of $AG(R)$.*

Proof. (1) Since $Rx \nsubseteq Ry$, we may assume that $R_1x_1 \nsubseteq R_1y_1$. Thus, if $x_1 \in U(R_1) = \{1, 3\}$, then

$$y_1 \in \mathrm{Nil}(R_1) = \{0, 2\}$$

and if $x_1 \in \mathrm{Nil}(R_1)^* = \{2\}$, then $y_1 = 0$. Hence, clearly

$$\mathrm{ann}_{R_1}(y_1) \nsubseteq \mathrm{ann}_{R_1}(x_1)$$

and so $\mathrm{ann}_R(y) \nsubseteq \mathrm{ann}_R(x)$. Similarly, since $Ry \nsubseteq Rx$ we can get $\mathrm{ann}_R(x) \nsubseteq \mathrm{ann}_R(y)$. Therefore, $x \sim y$ is an edge of $AG(R)$, by [10, Lemma 2.2(1)].

(2) Suppose that $Rx \subseteq Ry$. Since $x \sim y$ is an edge of $AG(R)$, by [10, Lemma 2.1], $Rx \cap \mathrm{ann}_R(y) \neq 0$ and $Ry \cap \mathrm{ann}_R(x) \neq 0$. Now, by $Rx \subseteq Ry$ and $Rx \cap \mathrm{ann}_R(y) \neq 0$, it follows that

$$Ry \cap \mathrm{ann}_R(y) \neq 0.$$

This implies that $y_i \in \mathrm{Nil}(R_i)^* = \{2\}$, for some $1 \leq i \leq n$. Without loss of generality, we may assume that $y_1 \in \mathrm{Nil}(R_1)^* = \{2\}$. If $x_1 \in \mathrm{Nil}(R_1)^* = \{2\}$, then there is nothing to prove. Otherwise, $x_1 = 0$ and $R_1x_1 \cap \mathrm{ann}_{R_1}(y_1) = 0$. For other components of x, $2 \leq j \leq n$, if $x_j \in U(R_j) = \{1, 3\}$, then $y_j \in U(R_j) = \{1, 3\}$ since $Rx \subseteq Ry$ thus $R_jx_j \cap \mathrm{ann}_{R_j}(y_j) = 0$. This means that $Rx \cap \mathrm{ann}_R(y) = 0$ which is a contradiction. Hence, $x_j \in \mathrm{Nil}(R_j)^* = \{2\}$, for some $2 \leq j \leq n$. Assume that $x_2 \in \mathrm{Nil}(R_2)^* = \{2\}$. Then

$$y_2 \in U(R_2) \cup \mathrm{Nil}(R_2)^* = \{1, 2, 3\}.$$

If $y_2 \in U(R_2) = \{1, 3\}$, then $R_2x_2 \cap \mathrm{ann}_{R_2}(y_2) = 0$. If we continue this procedure, then we obtain $x_i = y_i \in \mathrm{Nil}(R_i)^* = \{2\}$, for some $1 \leq i \leq n$.

(3) By [10, Lemma 2.1], we need only to show that $Ry \cap \mathrm{ann}_R(x) \neq 0$. Since $Rx \cap \mathrm{ann}_R(y) \neq 0$ and $Rx \subseteq Ry$ so $Ry \cap \mathrm{ann}_R(y) \neq 0$. On the other hand, since $Rx \subseteq Ry$ we have $\mathrm{ann}_R(y) \subseteq \mathrm{ann}_R(x)$. Hence, $Ry \cap \mathrm{ann}_R(x) \neq 0$. □

Let n be a positive integer, $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times) and x, y be distinct elements of $Z(R)^*$. By a similar argument to that of Lemma 2.1 we can show that $x \sim y$ is an edge of $AG(R)$ if and only if $Rx \nsubseteq Ry$

and $Ry \nsubseteq Rx$. Moreover, if $Rx \subseteq Ry$ and $Rx \cap \mathrm{ann}_R(y) \neq 0$, then $x \sim y$ is an edge of $AG(R)$.

Lemma 2.2. *Let R be a ring and $x, y \in V(AG(R))$ such that $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. Then $N(x) = N(y)$.*

Proof. Suppose that $x \sim a$ is an edge of $AG(R)$. So for some $r \in R$, $rax = 0$, $ra \neq 0$ and $rx \neq 0$. Since $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, we deduce that $ry \neq 0$ so we have $ra \neq 0$, $ry \neq 0$ and $ray = 0$. This means that $y \sim a$ is an edge of $AG(R)$ and so $N(x) \subseteq N(y)$. Similarly, $N(y) \subseteq N(x)$ and hence $N(x) = N(y)$, as desired. Moreover, if $x \sim y$, then $N[x] = N[y]$. □

In 2006, M. Chudnovsky et al. settled a long standing conjecture regarding perfect graphs and provided a characterization of perfect graphs.

Theorem 2.3. [6, The Strong Perfect Graph Theorem] *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

Theorem 2.4. *Let m, n be positive integers and let*

$$R = R_1 \times \cdots \times R_n \times R_{n+1} \times \cdots \times R_{n+m},$$

where $R_i \cong \mathbb{Z}_4$, for every $1 \leq i \leq n$, and $R_i \cong \mathbb{Z}_2$, for every $n+1 \leq i \leq n+m$. Then $AG(R)$ is perfect.

Proof. In view of Theorem 2.3, it is enough to show that $AG(R)$ and $\overline{AG(R)}$ contain no induced odd cycle of length at least 5. Indeed, we have the following claims:

Claim 1. $AG(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$x_1 \sim x_2 \sim \cdots \sim x_k \sim x_1$$

is an induced odd cycle of length at least 5 in $AG(R)$. Since x_1 is not adjacent to x_3, by Lemma 2.1(1) and paragraph after it, we have either $Rx_1 \subseteq Rx_3$ or $Rx_3 \subseteq Rx_1$. Without loss of generality, we may assume that $Rx_1 \subseteq Rx_3$. We continue the proof in the following steps.

Step 1. For every $3 \leq i \leq k-1$, $Rx_1 \subseteq Rx_i$. Since $Rx_1 \subseteq Rx_3$, for $i = 3$ it is clear. Since x_1 is not adjacent to x_4, by Lemma 2.1(1) and paragraph after it, we have either $Rx_1 \subseteq Rx_4$ or $Rx_4 \subseteq Rx_1$. If $Rx_4 \subseteq Rx_1$, then since $Rx_1 \subseteq Rx_3$, we have $Rx_4 \subseteq Rx_3$. By $Rx_4 \cap \mathrm{ann}_R(x_3) \neq 0$ and $Rx_4 \subseteq Rx_1$ it follows that $Rx_1 \cap \mathrm{ann}_R(x_3) \neq 0$. This, together with Lemma 2.1(3) imply that x_1, x_3 are adjacent that is a contradiction. So $Rx_1 \subseteq Rx_4$ and thus the Step 1 is true for $i = 4$, also. Again, for $i = 5$ we have $Rx_1 \subseteq Rx_5$ or $Rx_5 \subseteq Rx_1$. If

$Rx_5 \subseteq Rx_1$, then we have $Rx_5 \subseteq Rx_4$ since $Rx_1 \subseteq Rx_4$. Now, from $Rx_5 \cap \mathrm{ann}_R(x_4) \neq 0$ and $Rx_5 \subseteq Rx_1$ it follows that $Rx_1 \cap \mathrm{ann}_R(x_4) \neq 0$. This fact together with Lemma 2.1(3) imply that x_1, x_4 are adjacent that is a contradiction. So $Rx_1 \subseteq Rx_5$. By a similar argument on can show that $Rx_1 \subseteq Rx_i$, for every $6 \leq i \leq k-1$.

Step 2. For every $4 \leq i \leq k$, $Rx_2 \subseteq Rx_i$. By the Step 1 we have $Rx_1 \subseteq Rx_4$ and Lemma 2.1(1) shows that either $Rx_2 \subseteq Rx_4$ or $Rx_4 \subseteq Rx_2$. If $Rx_4 \subseteq Rx_2$, then we have $Rx_4 \cap \mathrm{ann}_R(x_2) \neq 0$ since $Rx_1 \subseteq Rx_4$ and $Rx_1 \cap \mathrm{ann}_R(x_2) \neq 0$. This fact together with Lemma 2.1(3), imply that x_2 is adjacent to x_4, a contradiction. So $Rx_2 \subseteq Rx_4$. Next, we show that $Rx_2 \subseteq Rx_5$. If $Rx_5 \subseteq Rx_2$, then $Rx_2 \cap \mathrm{ann}_R(x_4) \neq 0$ because $Rx_5 \cap \mathrm{ann}_R(x_4) \neq 0$ also by $Rx_2 \subseteq Rx_4$ it follows that x_2 is adjacent to x_4 that is a contradiction. Hence, $Rx_2 \subseteq Rx_5$. Similarly, $Rx_2 \subseteq Rx_i$, for every $4 \leq i \leq k$.

Step 3. $Rx_3 \subseteq Rx_1$. By Lemma 2.1(1), we have either $Rx_3 \subseteq Rx_5$ or $Rx_5 \subseteq Rx_3$. If $Rx_5 \subseteq Rx_3$, then $Rx_5 \cap \mathrm{ann}_R(x_3) \neq 0$ since $Rx_2 \cap \mathrm{ann}_R(x_3) \neq 0$ and by the Step 2, $Rx_2 \subseteq Rx_5$ this contradicts to Lemma 2.1(3). Hence, $Rx_3 \subseteq Rx_5$. Now, we show that $Rx_3 \subseteq Rx_6$. If $Rx_6 \subseteq Rx_3$, then $Rx_6 \cap \mathrm{ann}_R(x_3) \neq 0$ since $Rx_2 \cap \mathrm{ann}_R(x_3) \neq 0$ and $Rx_2 \subseteq Rx_6$ that is a contradiction. Hence, $Rx_3 \subseteq Rx_6$. Similarly, we can show that $Rx_3 \subseteq Rx_i$, for every $7 \leq i \leq k$. Now, suppose that $Rx_1 \subseteq Rx_3$. Then since for every $5 \leq i \leq k$, $Rx_3 \subseteq Rx_i$, we have $Rx_1 \subseteq Rx_3 \subseteq Rx_k$. Since $Rx_1 \cap \mathrm{ann}_R(x_k) \neq 0$ thus $Rx_3 \cap \mathrm{ann}_R(x_k) \neq 0$, a contradiction. So $Rx_3 \subseteq Rx_1$ and by the Step 1, $Rx_3 = Rx_1$. This implies that $\mathrm{ann}_R(x_3) = \mathrm{ann}_R(x_1)$ so by Lemma 2.2, $N(x_3) = N(x_1)$. Thus $x_4 \in N(x_3) = N(x_1)$ and $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1$ is a cycle of length 4 that is a contradiction. Therefore, $AG(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{AG(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary,

$$x_1 \sim x_2 \sim \cdots \sim x_k \sim x_1$$

is an induced odd cycle of length at least 5 in $\overline{AG(R)}$. In view of Lemma 2.1, we may assume that $Rx_1 \subseteq Rx_2$. If $Rx_2 \subseteq Rx_3$, then since $Rx_1 \cap \mathrm{ann}_R(x_3) \neq 0$, we have $Rx_2 \cap \mathrm{ann}_R(x_3) \neq 0$. Thus x_2 is adjacent to x_3 in $AG(R)$, a contradiction. So

$$Rx_1 \subseteq Rx_2 \text{ and } Rx_3 \subseteq Rx_2.$$

Assume that $Rx_4 \subseteq Rx_3$. Then since $Rx_4 \cap \mathrm{ann}_R(x_2) \neq 0$ we have $Rx_3 \cap \mathrm{ann}_R(x_2) \neq 0$. So by Lemma 2.1, x_2 is adjacent to x_3 in $AG(R)$, a contradiction. Thus $Rx_3 \subseteq Rx_4$. If $Rx_4 \subseteq Rx_5$, then since $Rx_3 \subseteq Rx_4$ and $Rx_3 \cap \mathrm{ann}_R(x_5) \neq 0$ we have $Rx_4 \cap \mathrm{ann}_R(x_5) \neq 0$ so x_4, x_5 are adjacent in $AG(R)$, a contradiction. Thus

$$Rx_3 \subsetneq Rx_4 \text{ and } Rx_5 \subsetneq Rx_4.$$

Since k is odd, if we continue this procedure, then we obtain

$$Rx_{k-2} \subsetneq Rx_{k-1} \text{ and } Rx_k \subsetneq Rx_{k-1}.$$

Now, assume that $Rx_1 \subsetneq Rx_k$. Then we get $Rx_k \cap \text{ann}_R(x_{k-1}) \neq 0$ since $Rx_1 \cap \text{ann}_R(x_{k-1}) \neq 0$. Thus by Lemma 2.1, x_k is adjacent to x_{k-1} in $AG(R)$, a contradiction. So $Rx_k \subsetneq Rx_1$. But in this case from $Rx_k \cap \text{ann}_R(x_2) \neq 0$ it follows that $Rx_1 \cap \text{ann}_R(x_2) \neq 0$. Hence, x_1 is adjacent to x_2 in $AG(R)$, a contradiction. Therefore, $\overline{AG(R)}$ contains no induced odd cycle of length at least 5. □

Let G be a graph and x be a vertex of G and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbors of x. We say that G' is obtained from G by expanding the vertex x to an edge $x \sim x'$. Hence, $V(G') = V(G) \cup \{x'\}$ and

$$E(G') = E(G) \cup \{x' \sim y : y \in N[x]\}.$$

Lemma 2.5. ([7, Lemma 5.5.5]) *Any graph obtained from a perfect graph by expanding a vertex is again perfect.*

Lemma 2.6. *Let G be a graph $x, y \in V(G)$ such that $N(x) = N(y)$. Then G is perfect if and only if $G \setminus \{x\}$ is perfect.*

Proof. Let G be a graph and $x, y \in V(G)$ such that $N(x) = N(y)$. We show that, G is perfect if and only if $G \setminus \{x\}$ is perfect. One side is obvious. So we may assume that $G \setminus \{x\}$ is perfect and show that G is perfect. Suppose that G is not perfect and look a contradiction. By Theorem 2.3, there is an induced odd cycle of length at least 5 in G such as

$$x_1 \sim x_2 \sim \cdots \sim x_n \sim x_1.$$

If $x_i \neq x$, for all $1 \leq i \leq n$, then

$$x_1 \sim x_2 \sim \cdots \sim x_n \sim x_1$$

is an induced odd cycle of length at least 5 in $G \setminus \{x\}$, a contradiction. So we may assume that $x_1 = x$. This implies that

$$x_2, x_n \in N(x) = N(y)$$

and hence we get

$$y \sim x_2 \sim \cdots \sim x_n \sim y$$

is an induced odd cycle of length at least 5 in $G \setminus \{x\}$, again a contradiction. Note that $y \neq x_i$, for all $2 \leq i \leq n$, otherwise we get a cycle of length less than n. So G contains no induced odd cycle of length at least 5. As above, by a similar argument one can show that $\overline{G}$ contains no induced odd cycle of length at least 5. Therefore, G is perfect. Now, let $N[x] = N[y]$. In this case G is obtained from G' by expanding the vertex y to an edge $x \sim y$. So by Lemma 2.3, G is perfect if and only if $G' = G \setminus \{x\}$ is perfect. □

Remark 2.7. Let G be a graph $x_1, y_1 \in V(G)$ such that either $N(x_1) = N(y_1)$ or $N[x_1] = N[y_1]$. Then, according to Lemmas 2.5, 2.6, G is perfect whenever $G \setminus \{x_1\}$ is perfect. Also, for $x_2, y_2 \in V(G) \setminus \{x_1\}$, if either $N(x_2) = N(y_2)$ or $N[x_2] = N[y_2]$, then $G \setminus \{x_1\}$ is perfect whenever $G \setminus \{x_1, x_2\}$ is perfect. So for $y \in V(G)$, $A \subseteq V(G)$ and $x \in A$. If either $N(x) = N(y)$ or $N[x] = N[y]$, then $G \setminus A$ is perfect whenever $G \setminus (A \setminus \{x\})$ is perfect. Hence, G is perfect whenever $G \setminus (A \setminus \{x\})$ is perfect.

Using these results, we show that if R is an Artinian ring, then $AG(R)$ is perfect.

Theorem 2.8. *Let R be an Artinian ring. Then $AG(R)$ is perfect.*

Proof. If R is local, then in view of [4, Theorem 3.10], $AG(R)$ is complete and so is perfect. Now, assume that R is not local. Thus

$$R = R_1 \times \cdots \times R_n \times R_{n+1} \times \cdots \times R_{n+m},$$

where R_i is a non-reduced Artinian local ring, for every $1 \leq i \leq n$, and is a field, for every $n+1 \leq i \leq n+m$, see [3, Theorem 8.7]. Note that for

$$x = (x_1, \ldots, x_n, x_{n+1} \ldots, x_{n+m}) \in R,$$

$x_i \in \mathrm{Nil}(R_i) \cup U(R_i)$, for all $1 \leq i \leq n$ and $x_i \in \{0\} \cup U(R_i)$, for every $n+1 \leq i \leq n+m$. Define the relation $\simeq$ on $V(AG(R))$ as follows: for

$$\begin{aligned} x &= (x_1, \ldots, x_n, x_{n+1} \ldots, x_{n+m}), \\ y &= (y_1, \ldots, y_n, y_{n+1} \ldots, y_{n+m}) \in V(AG(R)) \end{aligned}$$

we say $x \simeq y$ whenever the following three conditions hold:

(1) $x_i = 0$ if and only if $y_i = 0$, for every $1 \leq i \leq n+m$.
(2) $x_i \in \mathrm{Nil}(R_i)^*$ if and only if $y_i \in \mathrm{Nil}(R_i)^*$, for every $1 \leq i \leq n$.
(3) $x_i \in U(R_i)$ if and only if $y_i \in U(R_i)$, for every $1 \leq i \leq n+m$.

It is easy to see that $\simeq$ is an equivalence relation on $V(AG(R))$. Let $[x]$ denote the equivalence class of x and let x' and x'' be two elements of $[x]$. Since $x' \simeq x''$ we have $\mathrm{ann}_R(x') = \mathrm{ann}_R(x'')$ so Lemma 2.2 implies that $N(x') = N(x'')$ on $V(AG(R)) \setminus \{x', x''\}$. Now, if x' is not adjacent to x'', then by Lemma 2.6, $AG(R)$ is perfect if and only if $AG(R) \setminus \{x'\}$ is perfect. If x' is adjacent to x'', then by Lemma 2.5, $AG(R)$ is perfect if and only if $AG(R) \setminus \{x'\}$ is perfect. We continue this procedure and we obtain $AG(R)$ is perfect if and only if $AG(R)\setminus\{[x]\setminus\{x'\}\}$ is perfect. We do this for all equivalence classes and get $AG(R)$ is perfect if and only if $AG(R)[A]$ is perfect, where A is a subset of $V(AG(R))$ such that for every equivalence class $[x]$, $|A \cap [x]| = 1$. Hence, $AG(R)[A]$ is an annihilator graph with $3^n 2^m - 2$ vertices.

Assume that $S = S_1 \times \cdots \times S_n \times S_{n+1} \times \cdots \times S_{n+m}$, where $S_i \cong \mathbb{Z}_4$, for every $1 \leq i \leq n$, $S_i \cong \mathbb{Z}_2$, for every $n+1 \leq i \leq n+m$. By a similar argument as above one can show that $AG(S)$ is perfect if and only if $AG(S)[B]$ is perfect. Here

$$\begin{aligned} B &= \Big\{(x_1, \ldots, x_n, x_{n+1} \ldots, x_{n+m}) \in V(AG(S)) \mid x_i \in \{0,1,2\} \\ &\qquad for\ 1 \leq i \leq n \text{ and } x_i \in \{0,1\} \text{ for } n+1 \leq i \leq n+m\Big\} \\ &\subseteq Z(S)^* \end{aligned}$$

and $AG(S)[B]$ is an annihilator graph with $3^n 2^m - 2$ vertices. In view of Theorem 2.4, $AG(S)$ is perfect so $AG(S)[B]$ is perfect. Now, we can easily get the graph homomorphism $\phi : AG(R)[A] \longrightarrow AG(S)[B]$ by the rule $\phi((x_1, \cdots, x_n, x_{n+1} \cdots, x_{n+m})) = (y_1, \cdots, y_n, y_{n+1} \cdots, y_{n+m})$, where $x_i = 0$ if and only if $y_i = 0$ and $x_i \in U(R_i)$ if and only if $y_i \in U(S_i) = \{1\}$, for every $1 \leq i \leq n+m$, $x_i \in \mathrm{Nil}(R_i)^*$ if and only if $y_i \in \mathrm{Nil}(S_i)^* = \{2\}$, for every $1 \leq i \leq n$, is an isomorphism. Hence, $AG(S)[B] \cong AG(R)[A]$. Thus $AG(R)[A]$ is perfect and so $AG(R)$ is perfect. This completes the proof. □

Theorem 2.9. *Let n be a positive integer and let $R = D_1 \times \cdots \times D_n$, where D_i is an integral domain, for every $1 \leq i \leq n$. Then $AG(R)$ is perfect.*

Proof. Assume that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two vertices of $AG(R)$. Define the relation $\simeq$ on $V(AG(R))$ as follows: $x \simeq y$ whenever

$$x_i = 0 \text{ if and only if } y_i = 0,$$

for every $1 \leq i \leq n$. It is easily seen that $\simeq$ is an equivalence relation on $V(AG(R))$ so $V(AG(R))$ is a union of $(2^n - 2)$ distinct equivalence classes. Let $[x]$ denote the equivalence class of

$$x \in V(AG(R))$$

and $a, b \in [x]$. Then it is easy to see that $\mathrm{ann}_R(a) = \mathrm{ann}_R(b)$ so $N(a) = N(b)$, by Lemma 2.2. This fact together with a not being adjacent to b, implies that $AG(R)$ is perfect whenever $AG(R)\setminus([x]\setminus\{a\})$

is perfect, see Remark 2.7. We do this for all equivalence classes and get $AG(R)$ is perfect if and only if $AG(R)[A]$ is perfect, where

$$A = \{(x_1, \ldots, x_n) \in V(AG(R)) | x_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n\} \subseteq Z(R)^*.$$

In view of Theorem 2.4, $AG(S)$ for $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n times), is perfect. Furthermore, it is easy to see that $AG(S) \cong AG(R)[A]$. Hence, $AG(R)[A]$ is perfect and so $AG(R)$ is perfect. □

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PERFECTNESS OF THE ANNIHILATOR GRAPH OF ARTINIAN COMMUTATIVE RINGS

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تام بودن گراف پوچ‌ساز حلقه‌های جابه‌جایی آرتینی

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فرض کنید R یک حلقه جابه‌جایی و $Z(R)$ مجموعه مقسوم علیه‌های صفر R باشد. گراف پوچ‌ساز R با نماد $AG(R)$ نشان داده می‌شود و گرافی ساده و غیرجهت‌دار است که مجموعه رئوس آن $Z(R)^*$، مجموعه مقسوم علیه‌های صفر ناصفر R، است و دو راس متمایز x و y از آن مجاورند هرگاه و فقط هرگاه $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$. در این مقاله، تام بودن گراف پوچ‌ساز برای برخی کلاس‌ها از حلقه‌ها مورد بررسی قرار می‌گیرد. به‌طور دقیق‌تر، نشان می‌دهیم که اگر R یک حلقه آرتینی باشد، آن‌گاه $AG(R)$ یک گراف تام است.

کلمات کلیدی: حلقه آرتینی، گراف پوچ‌ساز، تام بودن.