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A GRAPH ASSOCIATED TO FILTERS OF A LATTICE

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ABSTRACT. Let L be a lattice with the least element 0 and the greatest element 1. In this paper, we associate a graph to filters of L, in which the vertex set is being the set of all non-trivial filters of L, and two distinct vertices F and E are adjacent if and only if $F \cap E \neq \{1\}$. We denote this graph by $\mathcal{G}(L)$. The basic properties and possible structures of $\mathcal{G}(L)$ are studied. Moreover, we characterize the planarity of $\mathcal{G}(L)$.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade. There are many papers on assigning a graph to a ring, a semiring and a lattice, see for example [1, 2, 5, 6, 7, 9, 12, 11]. One of these graphs is the intersection graph. Bosak [5] in 1964 defined the intersection graph of semigroups. In 1969, Cs*á*kany and Poll*á*k studied the graph of subgroups of a finite group, in [7]. In 2009, the intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [6]. By using this idea, in [11], the authors investigated the intersection graph of co-ideals of a semiring. In this paper, we introduce *intersection graphs* of lattices with respect to filters. The intersection graph of filters of a lattice L, denoted by $\mathcal{G}(L)$, is a graph with all elements of

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$$\mathcal{V}(L) = \{F : \{1\} \neq F \text{ is a proper filter of } L\}$$

as vertices and two distinct vertices F_1 and F_2 are adjacent if and only if $F_1 \cap F_2 \neq \{1\}$. Let L be a distributive lattice with 1 and 0. In this paper, we are interested in investigating intersection graphs of filters of lattices and associate which exist in the literature as laid forth in [6]. Here is a brief outline of the article. Among many results in this paper, Section 2 lists some results, and it is proved that $\mathcal{G}(L)$ is empty if and only if $\mathcal{V}(L) = \mathrm{Max}(L) = \{P_1, P_2\}$ or $L = \{0, 1\}$ and we find independence number of $\mathcal{G}(L)$ by using minimal filters of L. Also, if $\mathcal{G}$ (L) is connected, then $\mathrm{diam}(\mathcal{G}(\mathrm{L})) \leq 2$ and $\mathrm{gr}(\mathcal{G}(\mathrm{L})) \in \{3, \infty\}$. It is shown that $\mathcal{G}$ (L) is finite if and only if $\omega(\mathcal{G}(\mathrm{L}))$ is finite. Moreover, we characterize the filters of L, when $\mathcal{G}$ (L) has a vertex with degree 1. Section 3 is devoted to investigate the planarity of $\mathcal{G}$ (L).

Now, we recall some definitions of graph theory from [4] which are needed in this paper. For a *graph* G by $\mathcal{E}$ (G) and $\mathcal{V}$ (G), we denote the set of all *edges* and *vertices*, respectively. A graph G is said to be *connected* if there exists a path between any two distinct vertices. Otherwise, G is called *disconnected*. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is equal to

$$\sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}.$$

A graph is *complete* if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. A *complete bipartite graph* with part sizes m an n is denoted by $K_{m,n}$. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. A *clique* of a graph is a complete subgraph of G and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. In a graph $G = (\mathcal{V}, \mathcal{E})$, a set $S \subseteq \mathcal{V}$ is an *independent set* if the subgraph induced by S is totally disconnected. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G. Note that a graph whose vertices-set is empty is a *null graph* and a graph whose edge-set is empty is an *empty graph*.

Let us recall some notions and notations of lattice theory from [3]. By a lattice L we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L

is a lattice with 0 and 1). A lattice L is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A lattice L is called 1-*distributive* (resp. 0-*distributive*) if $a \vee b = 1$ and $a \vee c = 1$ (resp. $a \wedge b = 0$ and $a \wedge c = 0$), then $a \vee (b \wedge c) = 1$ (resp. $a \wedge (b \vee c) = 0$) for all $a, b, c \in L$. A non-empty subset F of a lattice L is called a *filter*, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter F of L is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. If F is a filter of a lattice L with 0, then $0 \in F$ if and only if $F = L$. Let H be a subset of a lattice L. Then the filter generated by H, denoted by $T(H)$ is the intersection of all filters that is containing H. A lattice L with 1 is called L-*domain* if $a \vee b = 1$ $(a, b \in L)$, then $a = 1$ or $b = 1$. Let L be a lattice. L is called *semisimple*, if for each proper filter F of L, there exists a filter E of L such that $L = T(F \cup E)$ and $F \cap E = \{1\}$. A filter F of L is *minimal* (simple) if it has no filters besides the $\{1\}$ and itself. We show the set of all simple (minimal) filters of L by $\mathrm{Min}(L)$. A proper filter P of L is said to be *maximal* if E is a filter in L with $P \subsetneqq E$, then $E = L$. The set of all maximal filters in L is denoted by $\mathrm{Max}(L)$. If L is a lattice, then the *Jacobson radical* of L, denoted by $\mathrm{Jac}(L)$, is the intersection of all maximal filters of L. Let F, E be filters of L. Then we call E is a *complement* of F if $F \cap E = \{1\}$ and E is maximal with respect to this property. First we need the following lemma proved in [3, 13].

Lemma 1.1. *Let L be a lattice.*

(a) *A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

(b) *If L is 1-distributive and $x \in L$, then*
$$(\{1\} :_L x) = (1 : x) = \{a \in L : a \vee x = 1\}$$
is a filter of L.

Proposition 1.2. [10]

(i) *If F is a non-zero proper filter of a lattice L, then F is contained in a maximal filter of L.*

(ii) *Let P be a maximal filter of a distributive lattice L. If $T(P \cup F) = L$ and $P \cap F = \{1\}$ for some filter F of L, then F is a minimal filter of L.*

(iii) *Assume that L is a distributive lattice and let* $\mathrm{Jac}(L) = \{1\}$. *If* $\mathrm{Max}(L)$ *is finite, then L is semisimple.*

Proposition 1.3. [8]

(i) *If L is a distributive lattice and F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.*

(ii) *Let H be an arbitrary non-empty subset of a lattice L. Then $T(H) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in H$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and $A \subseteq F$, then $T(A) \subseteq F$ and $T(F) = F$.*

Let F be a proper filter of a lattice L with 0 and 1. The filter-based identity-summand graph of L with respect to F, denoted by $\Gamma_F(L)$, is the graph whose vertices are

$$I_F(L) = \{x \in L \setminus F : x \vee y \in F \text{ for some } y \in L \setminus F\},$$

and distinct vertices x and y are adjacent if and only if $x \vee y \in F$. If $F = \{1\}$, then we put $\Gamma_{\{1\}}(L) = \Gamma(L)$. We need the following proposition proved in [12, Proposition 2.3 and Theorem 3.14 (1)].

Proposition 1.4. (i) *If L is 1-distributive and $\{F_i\}_{i \in \Lambda}$ is the set of all prime filters of L, then $\cap_{i \in \Lambda} F_i = \{1\}$ (Take $F = \{1\}$).*

(ii) *If L is a lattice, then $\omega(\Gamma(L)) = |\mathrm{Min}(\{1\})| = |\mathrm{Min}(L)|$.*

2. Basic properties of $\mathcal{G}(L)$

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1 and 0. Our starting point is the following definition:

Definition 2.1. Let L be a lattice. The *intersection graph of filters* of L, denoted by $\mathcal{G}(L)$, is the graph with all elements of

$$\mathcal{V}(L) = \{\{1\} \neq F : F \text{ is a proper filter of } L\}$$

as vertices and two distinct vertices F_1 and F_2 are adjacent if and only if $F_1 \cap F_2 \neq \{1\}$.

Theorem 2.2. *Let L be a lattice. Then the following statements hold:*

(i) *$\mathcal{G}(L)$ is an empty graph if and only if $\mathcal{V}(L) = \mathrm{Max}(L) = \{P_1, P_2\}$ or $L = \{0, 1\}$.*

(ii) *$\mathcal{G}(L)$ is a complete graph if and only if L is L-domain.*

(iii) *If $\alpha(\mathcal{G}(L))$ is finite, then $\alpha(\mathcal{G}(L)) = |\mathrm{Min}(L)|$.*

Proof. (i) Let $\mathcal{G}(L)$ be an empty graph. If $\mathrm{Max}(L) = \{P\}$, then Lemma 1.2 (i) gives $F \subseteq P$ for each filter F of L; so $F \cap P \neq \{1\}$. Now since $\mathcal{G}(L)$ is an empty graph, P is the only filter of L. Hence by Proposition 1.4 (i), $P = \{1\}$. Let $1 \neq a \in L$ (so $a \notin P$). Since $P \subsetneqq T(\{1, a\}) \subseteq L$, $T(\{1, a\}) = L$ gives $a = (1 \wedge a) \leq 0$; hence $a = 0$, and so $L = \{0, 1\}$. Suppose that $|\mathrm{Max}(L)| \geq 2$. Since $\mathcal{G}(L)$ is empty, $P_i \cap P_j = \{1\}$ for each

$P_i, P_j \in \text{Max}(L)$. As $P_i \subsetneqq T(P_i \cup P_j) \subseteq L$, we get $L = T(P_i \cup P_j)$ which implies that P_i and P_j are minimal filters of L by Proposition 1.2 (ii). It is enough to show that $\text{Max}(L) = \{P_i, P_j\}$. Suppose to the contrary that $P_i, P_j \neq P_k \in \text{Max}(L)$. Therefore $P_k \cap P_i = P_k \cap P_j = \{1\}$. Let $a \in P_i$. If $x \in P_j$, then $x \vee a \in P_i \cap P_j = \{1\}$ which implies that $x \in (1 : a)$; so $P_j \subseteq (1 : a)$. Similarly, $P_k \subseteq (1 : a)$. It follows that $P_j = (1 : a) = P_k$, a contradiction. Thus $\text{Max}(L) = \{P_i, P_j\}$. As P_i and P_j are minimal, we get $\mathcal{V}(L) = \text{Max}(L)$. The other implication is clear.

(ii) At first we show that if $a, b \in L$ with $a \neq b$ and $a \vee b = 1$, then $T(\{a\}) \cap T(\{b\}) = \{1\}$ and $T(\{a\}) \neq T(\{b\})$. If $x \in T(\{a\}) \cap T(\{b\})$, then $a \leq x$ and $b \leq x$ which implies that $1 = a \vee b \leq x$; hence $x = 1$. If $T(\{a\}) = T(\{b\})$, then $a \in T(\{b\})$ and $b \in T(\{a\})$ gives $a \leq b$ and $b \leq a$, a contradiction. Hence $T(\{a\}) \neq T(\{b\})$. Assume that $\mathcal{G}\ (L)$ is a complete graph and let $a, b \in L$ such that $a \vee b = 1$. If $a = b$, then we are done. So we may assume that $a \neq b$. Let $a \neq 1$ and $b \neq 1$. Now $a \vee b = 1$ gives $T(\{a\}) \neq T(\{b\})$ and $T(\{a\}) \cap T(\{b\}) = \{1\}$ that is a contradiction. The other implication is clear.

(iii) By Proposition 1.4 (ii), $\omega(\Gamma(L)) = |\text{Min}(L)|$. It is enough to show that $\alpha(\mathcal{G}(L)) = \omega(\Gamma(L))$. Let $\{F_1, F_2, \ldots, F_n\}$ be an independent set in $\mathcal{G}(L)$; so for every i, j with $i \neq j$, $F_i \cap F_j = \{1\}$. Let $a_i \in F_i$ $(1 \leq i \leq n)$. Then $\{a_1, a_2, \ldots, a_n\}$ is a vertex set of complete subgraph in $\Gamma(L)$. So $\omega(\Gamma(L) \geq \alpha(\mathcal{G}(L))$. Now, let $\{a_1, a_2, ...\}$ be a clique in $\Gamma(L)$. Then $\{T(\{a_1\}), T(\{a_2\}), \ldots\}$ is an independent set in $\mathcal{G}(L)$. So $\alpha(\mathcal{G}(L)) \geq \omega(\Gamma(L))$. Hence $\alpha(\mathcal{G}(L)) = \omega(\Gamma(L))$. □

Example 2.3. Let $L = (P(T), \cup, \cap, \subseteq)$, where $P(T)$ is the power set of $T = \{t, z\}$. Then $\text{Max}(L) = \{P_1, P_2\}$, where $P_1 = \{T, \{t\}\}$ and $P_2 = \{T, \{z\}\}$. It is clear that $\mathcal{G}\ (L)$ is empty.

A cycle of a graph is a path such that the start and end vertices are the same. For a graph G, it is well-known that if G contains a cycle, then $\text{gr}(G) \leq 2\text{diam}(G) + 1$.

Theorem 2.4. (i) *If L is a lattice such that $\mathcal{G}(L)$ is not empty, then $\mathcal{G}(L)$ is connected and* $\text{diam}(\mathcal{G}(L)) \leq 2$.

(ii) *If L is a lattice, then* $\text{gr}(\mathcal{G}(L)) \in \{3, \infty\}$.

Proof. (i) Let F_1 and F_2 be distinct elements of $\mathcal{V}(L)$. We need to show there is a path connects F_1 and F_2, if $F_1 \cap F_2 \neq \{1\}$, then we are done. So we may assume that $F_1 \cap F_2 = \{1\}$. By Proposition 1.2 (i), there exist maximal filters P_1, P_2 of L such that $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$. If $F_1 \cap P_2 \neq \{1\}$, then $F_1 - P_2 - F_2$ is a path between F_1 and F_2. If $F_2 \cap P_1 \neq \{1\}$, then $F_1 - P_1 - F_2$ is a path between F_1 and F_2. If

$F_1 \cap P_2 = \{1\}$ and $F_2 \cap P_1 = \{1\}$, then F_1 and F_2 are minimal filters of L by Proposition 1.2 (ii) since $T(F_1 \cup P_2) = L = T(F_2 \cup P_1)$. We show that $T(F_1 \cup F_2) \neq L$. Assume to the contrary, $T(F_1 \cup F_2) = L$. Then by Proposition 1.3 (i),

$$P_1 = P_1 \cap L = P_1 \cap T(F_1 \cup F_2) = T(F_1 \cup (P_1 \cap F_2)) = T(F_1) = F_1.$$

Similarly, $P_2 = F_2$. If $p \in P_1$, then $P_2 \subseteq (1 : p)$; thus $P_2 = (1 : p) = P_1$, a contradiction. So $T(F_1 \cup F_2)$ is a proper filter of L and

$$F_1 - T(F_1 \cup F_2) - F_2$$

is a path between F_1 and F_2. Hence $\mathrm{diam}(\mathcal{G}(L)) \leq 2$.

(ii) Suppose that $\mathcal{G}(L)$ contains a cycle. We may assume that $\mathrm{gr}(\mathcal{G}(L)) \leq 5$. Suppose that $\mathrm{gr}(\mathcal{G}(L)) = n$, where $n \in \{4, 5\}$ and let $F_1 - F_2 \ldots F_n - F_1$ be a cycle of minimum length in $\mathcal{G}(L)$. Since F_1 is not adjacent to F_3, $F_1 \cap F_3 = \{1\}$. We show that $F_1 \cap F_2 \neq F_2$. Otherwise, $F_2 \subseteq F_1$ gives $F_2 \cap F_3 \subseteq F_1 \cap F_3 = \{1\}$, a contradiction. If $F_1 \cap F_2 \neq F_1$, then $F_1 - F_1 \cap F_2 - F_2 - F_1$ is a cycle in $\mathcal{G}(L)$ that is a contradiction. So we may assume that $F_1 \cap F_2 = F_1$. Hence $F_1 \subseteq F_2$. Since F_2, F_4 are not adjacent, $F_2 \cap F_4 = \{1\}$. Clearly, $F_2 \cap F_3 \neq F_3$. If $F_2 \cap F_3 \neq F_2$, then $F_2 - F_2 \cap F_3 - F_3 - F_2$ is a cycle in $\mathcal{G}(L)$ which is a contradiction. So $F_2 \cap F_3 = F_2$; hence $F_2 \subseteq F_3$. It follows that $F_1 \cap F_3 = F_1 \neq \{1\}$, a contradiction. Therefore, there must be a shorter cycle in $\mathcal{G}(L)$ and $\mathrm{gr}(\mathcal{G}(L)) = 3$. □

The following example shows that the condition "distributive" is not superficial, in Theorem 2.4.

Example 2.5. Let L be the lattice as in Figure 1.

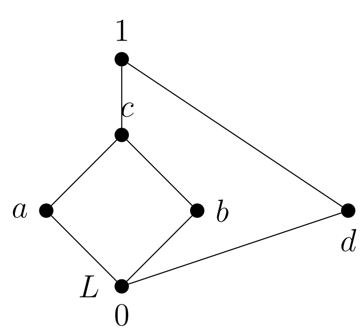

FIGURE 1.

Since $a \wedge (b \vee d) \neq (a \wedge b) \vee (a \wedge d)$, L is not distributive. Set $S_1 = \{a, c, 1\}$, $S_2 = \{b, c, 1\}$ and $S_3 = \{1, d\}$. Then S_1, S_2 and S_3 are maximal filters of L. It is clear that another filter of L is $S_4 = \{1, c\}$ and $\mathcal{G}(L)$ is not connected.

The degree of a vertex a in the graph G is the number of edges of G incident with a and denoted by $\deg(a)$.

Theorem 2.6. *Let L be a lattice. Then $\mathcal{G}(L)$ is finite if and only if $\deg(P)$ is finite for some maximal filter P of L.*

Proof. At first we show that there is at most one filter F of L such that P is not adjacent to F. Let F_1 and F_2 be filters of L such that $F_1 \cap P = F_2 \cap P = \{1\}$. Then $T(F_1 \cup P) = L = T(F_2 \cup P)$; so F_1, F_2 are minimal filters of L by Proposition 1.2 (ii). So there exist $a \in F_1$, $b \in F_2$ and $p_1, p_2 \in P$ such that $a \wedge p_1 \leq 0$ and $b \wedge p_2 \leq 0$; hence $a \wedge p_1 = 0$ and $b \wedge p_2 = 0$. Since $a \vee b \in F_1 \cap F_2 = \{1\}$, $a \vee b = 1$. By assumption, $(p_1 \wedge p_2) \wedge a = 0$ and $(p_1 \wedge p_2) \wedge b = 0$ gives $(p_1 \wedge p_2) \wedge (a \vee b) = p_1 \wedge p_2 = 0 \in P$ which is a contradiction. It follows that $\deg(P) = |\mathcal{G}(\mathrm{L})| - 1$ or $\deg(P) = |\mathcal{G}(\mathrm{L})| - 2$; hence $\mathcal{G}(L)$ is finite if and only if $\deg(P)$ is finite. □

Theorem 2.7. *Let L be a lattice. Then $\mathcal{G}$ (L) is finite if and only if $\omega(\mathcal{G}(L))$ is finite.*

Proof. By assumption, it suffices to show that if $\omega(\mathcal{G}(\mathrm{L}))$ is finite, then $\mathcal{G}$ (L) is finite. At first we show that if F_1, F_2 and F_3 are minimal filters of L, then $T(F_1 \cup F_2) \neq T(F_1 \cup F_3)$. Assume to the contrary, $T(F_1 \cup F_2) = T(F_1 \cup F_3)$. Let $1 \neq a \in F_2$. Then $a \in T(F_1 \cup F_3)$ gives $a = (b \wedge c) \vee a \leq a \vee b$ and $a = (b \wedge c) \vee a \leq a \vee c$ for some $b \in F_1$ and $c \in F_3$ which implies that $c \vee a, b \vee a \in F_2$ since F_2 is a filter; hence $c \vee a \in F_2 \cap F_3 = \{1\}$ and $b \vee a \in F_2 \cap F_1 = \{1\}$. Thus $b, c \in (1 : a)$ gives $b \wedge c \in (1 : a)$ since $(1 : a)$ is a filter; so $a = (b \wedge c) \vee a = 1$, a contradiction. Thus $T(F_1 \cup F_2) \neq T(F_1 \cup F_3)$. Now we claim that the number of minimal filters of L is finite. Assume to the contrary, let $\{F_i\}_{i \in \Lambda}$ be an infinite set of minimal filters of L. Clearly, $T(F_i \cup F_j) \neq T(F_i \cup F_k)$ for $i, j, k \in \Lambda$. Hence for minimal filter F_i of L we have the infinite complete subgraph $\{T(F_i \cup F_j)\}_{j \in \Lambda}$ which is a contradiction. Therefore L contains only finite number of minimal filters. Since $\omega(\mathcal{G}(\mathrm{L}))$ is finite, each filter of L contains a minimal filter. Now if $\mathcal{G}(L)$ is infinite, then there are infinite filters which contain common minimal filter which is a contradiction. □

Proposition 2.8. *Let L be a lattice. If $\mathrm{Max}(L) = \{P_1, P_2, \ldots, P_n\}$ with $\cap_{i=1}^{n} P_i = \{1\}$, then each filter of L is of the form $\cap_{i \in \Lambda} P_i$, where $\Lambda \subseteq \{1, 2, \ldots, n\}$.*

Proof. Let F be a filter of L. If there exists exactly one filter, say P_1, of L such that $F \nsubseteq P_1$, then $T(F \cup P_1) = L$ and $F \subseteq \cap_{i=2}^{n} P_i$. Therefore

$$\cap_{i=2}^{n} P_i = \cap_{i=2}^{n} P_i \cap T(F \cup P_1) = T(F \cup (\cap_{i=2}^{n} P_i \cap P_1)) = T(F) = F$$

by Proposition 1.3 (i). So we may assume that there exist at least two maximal filters P_i, P_j of L such that $F \nsubseteq P_i, P_j$. Let $F \subseteq \cap_{i\in\Lambda} P_i$ and $F \nsubseteq \cup_{\Lambda'} P_i$, where $\Lambda \subseteq \{1, 2, ..., n\}$ and $\Lambda' = \{1, 2, \ldots, n\} \setminus \Lambda$. At first we show $L = T(F \cup (\cap_{i\in\Lambda'} P_i))$. Clearly, $0 \in L = T(F \cup P_i)$ for each $i \in \Lambda'$. So for each $i \in \Lambda'$, there exist $a_i \in F$ and $p_i \in P_i$ such that $(a_i \wedge p_i) \le 0$; so $a_i \wedge p_i = 0$. If $\Lambda' = \{i_1, i_2, ..., i_t\}$, then

$$a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_t} \wedge p_{i_1} = 0, \ldots, a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_t} \wedge p_{i_t} = 0;$$

hence $(a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_t}) \wedge (p_{i_1} \vee p_{i_2} \vee \cdots \vee p_{i_t}) = 0$. This implies $0 \in T(F \cup (\cap_{i\in\Lambda'} P_i))$; thus $L = T(F \cup (\cap_{i\in\Lambda'} P_i))$. Then $F \subseteq \cap_{i\in\Lambda} P_i$ gives

$$\begin{aligned} \cap_{i\in\Lambda} P_i &= T(F \cup (\cap_{i\in\Lambda'} P_i)) \cap (\cap_{i\in\Lambda} P_i) \\ &= T(F \cup ((\cap_{i\in\Lambda} P_i) \cap (\cap_{i\in\Lambda'} P_i))) \\ &= T(F) \\ &= F \end{aligned}$$

by Proposition 1.3 (i). □

Theorem 2.9. *Let L be a lattice. If $\mathrm{Max}(L) = \{P_1, P_2, \ldots, P_n\}$ with $\cap_{i=1}^{n} P_i = \{1\}$, then $\omega(\mathcal{G}(L)) = 2^{n-1} - 1$.*

Proof. Let $A_i = \{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\}$ and $P(A_i)$, the power set of A_i ($1 \le i \le n$). For each $D_i \in P(A_i)$, set $S_{D_i} = \vee_{B\in D_i} B$ (so it is a filter of L). Then the subgraph of $\mathcal{G}$ (L) with vertex set $\{S_{D_i}\}_{D_i \in P(A_i)}$ is a complete subgraph of $\mathcal{G}(L)$ (if S_X and S_Y are two non-adjacent filter of L for $X, Y \in P(A_i)$, then there is a maximal filter which is not adjacent to more than one filter of L that is a contradiction). Since $|P(A_i) \setminus \{\}| = 2^{n-1} - 1$, $\omega(\mathcal{G}(L)) \ge 2^{n-1} - 1$. By Proposition 2.8, L has $2^n - 2$ proper filter. An inspection will show that all filters of L has complement. Now, let

$$\Omega = \{F_1, F_2, \ldots\}$$

be a complete subgraph of $\mathcal{G}(L)$. We partition the filters of L in parts $V_1, V_2, \ldots, V_{2^{n-1}-1}$ such that each part contains the filter F and its complement. Now if $|\Omega| > 2^{n-1} - 1$, then at least two of the elements of Ω are in the same part which is a contradiction. So

$$\omega(\mathcal{G}(\mathrm{L})) = 2^{n-1} - 1.$$

□

Theorem 2.10. *Let L be a lattice. Then the following hold:*

(i) *If $\mathcal{G}(L)$ contains a vertex F with degree 1, then F is maximal if and only if $|\ \mathcal{V}(L)| = 2$.*

(ii) *If $\mathcal{G}(L)$ contains a vertex F with degree 1, then F is not maximal and $\mathrm{Max}(L) = \{P\}$ if and only if $\mathcal{V}(L) = \{F, P\}$ or $\mathcal{V}(L) = \{F, E, P\}$, where $P \in \mathrm{Max}(L)$ and $F, E \in \mathrm{Min}(L)$.*

(iii) *If $\mathcal{G}(L)$ contains a vertex F with degree 1, then F is not maximal and $|\mathrm{Max}(L)| \neq 1$ if and only if $\mathcal{V}(L) = \{F, E, P, P'\}$, where $P, P' \in \mathrm{Max}(L)$ and $F, E \in \mathrm{Min}(L)$.*

Proof. (i) Let F be a vertex of L with degree 1. At first we show that $|\mathrm{Max}(L)| \leq 2$. Suppose to the contrary that $F, P_1, P_2 \in \mathrm{Max}(L)$. Since F is a maximal filter, there is at most one filter E of L such that $E \cap F = \{1\}$. If E is maximal, then E and F are minimal filters by Proposition 1.2 (ii); hence $\mathcal{G}(L)$ is an empty graph which is a contradiction. So we may assume that E is not maximal. So $F \cap P_1 \neq \{1\}$ and $F \cap P_2 \neq \{1\}$ which makes the degree of F more than 1 and it is a contradiction. Thus $|\mathrm{Max}(L)| \leq 2$. If $|\mathrm{Max}(L)| = 2$, then $F \cap P \neq \{1\}$ for some maximal filter P of L; so F is adjacent to P and $P \cap F$ which is a contradiction. Thus $\mathrm{Max}(L) = \{F\}$. Now $\deg(F) = 1$ gives $|\ \mathcal{V}(L)| = 2$. The other implication is clear.

(ii) Clearly, $F \subseteq P$ (so $F \cap P \neq \{1\}$). Since $\deg(F) = 1$, F is a minimal filter of L. We claim that $|\mathrm{Min}(L)| \leq 2$. If $F, E, G \in \mathrm{Min}(L)$, then $F \cap E = \{1\}$ and $F \cap G = \{1\}$. Now $F \subseteq T(F \cup E)$ and $F \subseteq T(F \cup G)$ gives a contradiction since $\deg(F) = 1$. Thus $|\mathrm{Min}(L)| \leq 2$. If $\mathrm{Min}(L) = 1$ (so $\mathrm{Min}(L) = \{F\}$), then we show that the graph $\mathcal{G}(L)$ has two vertices F and P. Suppose G is another filter of L. If $F \subseteq G$, then $G = F$ or $G = P$ since $\deg(F) = 1$; hence $\mathcal{V}(L) = \{F, P\}$. If $F \nsubseteq G$, then $\mathrm{Min}(L) = \{F\}$ implies $E \subsetneqq G$ for some filter E of L. Since F is minimal, $E \vee F = \{1\}$; so there is an element $x \in E$ such that $x \notin F$. So $x \in F \cup E$ and

$$F \subsetneqq F \cup E \subseteq T(F \cup E) \subseteq P$$

gives $T(F \cup E) = P$ since $\deg(F) = 1$. As $E \subseteq G$,

$$G = G \cap P = G \cap T(E \cup F) = T(E \cup (G \vee F)) = T(E) = E$$

by Proposition 1.3 (i), a contradiction. Therefore $F \subseteq G$ and $\mathcal{V}(L) = \{F, P\}$. Now suppose that $\mathrm{Min}(L) = \{F, E\}$. Clearly, $T(E \cup F) = P$. We claim that for each filter H of L, H adjacent to F or E. If $H \vee F = \{1\}$, then $F \subsetneqq H \cup F \subseteq T(H \cup F)$ which implies that $\deg(F) \neq 1$, a contradiction. Thus $F \vee H \neq \{1\}$ or $E \cap H \neq \{1\}$.

Since $\deg(F) = 1$ and F is minimal, we get $H \vee F = \{1\}$; hence $E \vee H \neq \{1\}$. Since $E \subseteq H$, Proposition 1.3 (i) gives

$$H = H \cap P = H \cap T(E \cup F) = T(E \cup (F \cap H)) = E;$$

hence $\mathcal{V}(L) = \{F, E, P\}$. Conversely, if $\mathcal{V}(L) = \{F, P\}$, then $F \subseteq P$; so $\deg(F) = 1$. If $\mathcal{V}(L) = \{F, E, P\}$, then $F, E \subseteq P$ and $E \vee F = \{1\}$; so $\deg(F) = 1 = \deg(E)$.

(iii) At first we show that if F is a minimal filter of a lattice L, then there is at most one maximal filter P such that F is not adjacent to P. Suppose the result is false. Assume that there are two maximal filters P_1 and P_2 such that $P_1 \cap F = \{1\}$ and $P_2 \cap F = \{1\}$; so

$$T(F \cup P_1) = L = T(F \cup P_2).$$

Then there exist $a, b \in F$, $p_1 \in P_1$ and $p_2 \in P_2$ such that $a \wedge p_1 \leq 0$ and $b \wedge p_2 \leq 0$ which implies that $a \wedge p_1 = 0 = b \wedge p_2$. Therefore $a \wedge b \wedge p_1 = 0$ and $a \wedge b \wedge p_2 = 0$ gives

$$(a \wedge b) \wedge (p_1 \vee p_2) = 0 \in T(F \cap (P_1 \cap P_2);$$

hence $T(F \cap (P_1 \cap P_2) = L$. By Proposition 1.3 (i), since $P_1 \cap P_2 \subseteq P_1$, we have

$$\begin{aligned} P_1 &= P_1 \cap T(F \cup (P_1 \vee P_2)) \\ &= T((P_1 \cap P_2) \cup (P_1 \cap F)) \\ &= T(P_1 \cap P_2) \\ &= P_1 \cap P_2 \end{aligned}$$

which is a contradiction. Hence $|\mathrm{Max}(L)| = 2$. Let $\mathrm{Max}(L) = \{P_1, P_2\}$ and $F \subseteq P_1$. Clearly, $F \cap P_2 = \{1\}$. We claim that for every non-maximal filter G of L, $T(G \cup F) \neq L$. Assume to the contrary, let $T(G \cup F) = L$. Then $F \subseteq P_1$ gives $P_1 = P_1 \cap T(G \cup F) = T(F \cup (G \cap P_1))$. If $G \subseteq P_1$, then $P_1 = L$ which is a contradiction. If $G \subseteq P_2$, then

$$P_2 = P_2 \cap T(F \cup G) = T(G \cup (F \cap P_2)) = T(G) = G,$$

a contradiction. Thus $T(G \cup G) \neq L$. Now since $\deg(F) = 1$, $F \subseteq P_1$ and $F \subseteq T(F \cup G)$, we get $T(F \cup G) = P_1$ for each non-maximal filter G of L. Take $G \subseteq P_2$. Again $G \subseteq P_2$ gives

$$P_1 \cap P_2 = P_2 \cap T(P_1 \cup G) = T(G \cup (P_2 \cap G)) = G;$$

hence $\mathcal{V}(L) = \{F, P_1, P_2, P_1 \cap P_2\}$. Conversely, let $\mathcal{V}(L) = \{F, E, P', P\}$. If $P \cap P' = \{1\}$, then P and P' are minimal filters of L; hence $\mathcal{G}(L)$ is an empty graph (since E and F are minimal filters), a contradiction. Thus $P \cap P' \neq \{1\}$ is a filter of L such that it is either F or E. We may assume that $P \cap P' = F$; so $F \subseteq P, P'$. On the other hand $E \subseteq P$, so $E \nsubseteq P'$. Therefore $E \cap P' = \{1\}$; hence $\deg(E) = 1$. □

Theorem 2.11. *Assume that L is a lattice and let $\mathcal{G}(L)$ be a complete r-partite graph. Then at most one part has more than two vertex. In particular, $|\mathcal{V}(L)| = r$ or $r+1$.*

Proof. Suppose $\mathrm{Min}(L) = \{F_i\}_{i\in\Lambda}$. As $F_i \cap F_j = \{1\}$, all minimal filters of L are in the same part, say V_1. We claim that there is at most two minimal filters in this part. Assume that F_i, F_j and F_k are distinct minimal filters of L and let $c \in T(F_i \cup F_j) \cap F_k$. Then

$$(a \wedge b) \vee c = c = (a \vee c) \wedge (b \vee c) \in F_k$$

for some $a \in F_i$ and $b \in F_j$. By Lemma 1.1 (a), $a \vee c \in F_i \cap F_k = \{1\}$ and $b \vee c \in F_j \cap F_k = \{1\}$; hence $c = 1$. Thus $T(F_i \cup F_j) \cap F_k = \{1\}$. But $\mathcal{G}(L)$ is complete r-partite implies $T(F_i \cup F_j) \cap F_i = \{1\}$ which is a contradiction. Hence there is at most two filters in the part V_1. Now we show that other parts contain only one filter. Let E be a non-minimal filter of L. Since $\mathcal{G}(L)$ is complete r-partite, E contains a minimal filter, say E_1. If there exists a minimal filter E_2 such that $E_2 \nsubseteq E$, then $E \cap E_2 = \{1\}$ implies $E \in V_1$ which is a contradiction. Hence all non-minimal filters contain all minimal filters in the part V_1. Therefore for all filters E, F which are not minimal $E \cap F \neq \{1\}$. Hence the only part which has more than one vertex is V_1. The in particular statement is clear. □

3. Planarity of $\mathcal{G}$ (L)

In this section, we characterize all planar graph $\mathcal{G}(L)$. Recall that a planar graph is a graph that can be embedded on the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Kuratowski provided a nice characterization of planar graphs, which now is known as Kuratowski's Theorem: A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proposition 3.1. *Assume that L is a lattice and let $\mathcal{G}(L)$ is a planar graph. Then $|\max(L)| \leq 3$. Moreover, if $|\max(L)| = 3$, then L is semi-simple with $\mid \mathcal{V}(L) \mid = 6$.*

Proof. Suppose on the contrary, $P_1, P_2, P_3, P_4 \in \mathrm{Max}(L)$. If for each filter F of L, $F \cap P_1 \neq \{1\}$, then $P_1 \cap P_2 \cap P_3 \neq \{1\}$ and $P_i \cap P_j \neq \{1\}$ for each $P_i, P_j \in \mathrm{Max}(L)$; so the induced subgraph $\mathcal{G}(L)$ on $\{P_1, P_2, P_3, P_1 \cap P_2, P_1 \cap P_2 \cap P_3\}$ is isomorphic to K_5, by Kuratowski's Theorem $\mathcal{G}(L)$ is not planar which is impossible. If there exists a filter

F such that $F \cap P_1 = \{1\}$, then $T(F \cup P_1) = L$ gives F is minimal. As a minimal filter, F is not adjacent to at most one maximal filter, so we may assume that $F \cap P_1 = \{1\}$. Thus $\{F, P_2, P_3, P_4, P_2 \cap P_3 \cap P_4\}$ makes K_5 in $\mathcal{G}(L)$ that is a contradiction.

Let $\text{Max}(L) = \{P_1, P_2, P_3\}$. If $\text{Jac}(L) \neq \{1\}$, then

$$\{P_1, P_2, P_3, P_1 \cap P_2, P_1 \cap P_2 \cap P_3\}$$

makes K_5 in $\mathcal{G}(L)$ which is impossible. So we may assume that $\text{Jac}(L) = \{1\}$; hence L is a semi-simple lattice by Proposition 1.2 (iii). Now for each i $(1 \leq i \leq 3)$, there exists a filter F_i such that $T(P_i \cup F_i) = L$ and $P_i \cap F_i = \{1\}$; thus F_i is simple for $i = 1, 2, 3$. As $T(F_1 \cup F_2 \cup F_3) \nsubseteq P_i$ for each $P_i \in \text{Max}(L)$, we get $T(F_1 \cup F_2 \cup F_3) = L$ (because every filter must be contained in a maximal filter). We can assume that $F_1 \subseteq P_2$, $F_2 \subseteq P_3$ and $F_3 \subseteq P_1$. Since $F_1 \cap P_1 = \{1\}$, $P_1 \cap F_2 \neq \{1\}$. Now F_2 is simple gives, $F_2 \subseteq P_1$. By Proposition 1.3 (i), $F_3 \subseteq P_1$ gives

$$\begin{aligned} P_1 &= P_1 \cap T(F_3 \cup (F_1 \cup F_2)) \\ &= T(F_3 \cup ((F_1 \cup F_2) \cap P_1)) \\ &= T(F_3 \cup ((P_1 \cap F_1) \cup (P_1 \cap F_2))) \\ &= T(F_3 \cup F_2). \end{aligned}$$

So $P_1 = T(F_3 \cup F_2)$. Similarly, $P_i = T(F_j \cup F_k)$ for $k, j \neq i$. Now let E be a filter of L which is not minimal and maximal. Since $\mathcal{G}(L)$ is planar, E contains a simple filter, say F_1. Clearly if $F_2 \subseteq E$, then $F_1 \cup F_2 \subseteq E$ gives $T(F_1 \cup F_2) \subseteq E$. But $T(F_1 \cup F_2) = P_3$ implies $E = M_3$ which is a contradiction. Similarly, if $F_3 \subseteq E$, $E = M_2$, a contradiction. So $F_2 \cap E = F_3 \cap E = \{1\}$. Let $x \in E \cap T(F_2 \cup F_3)$. Then

$$x = (a \wedge b) \vee x = (x \vee a) \wedge (x \vee b) \in E$$

for some $a \in F_2$ and $b \in F_3$. It follows that $x \vee a, x \vee b \in E$ which implies that $x \vee a = 1 = x \vee b$; hence $x = 1$. Thus $E \cap T(F_2 \cup F_3) = E \cap P_1 = \{1\}$. Now by Proposition 1.2 (ii), E is simple which is a contradiction. Thus $\mathcal{V}(L) = \{F_1, F_2, F_3, P_1, P_2, P_3\}$. □

Theorem 3.2. *Assume that L is a lattice and let $\mathcal{G}(L)$ be a planar graph. Then $|\mathcal{V}(L)| \leq 7$.*

Proof. Since $\mathcal{G}(L)$ is a planar, $|\text{Max}(L)| \leq 3$ and if $|\text{Max}(L)| = 3$, then $|\mathcal{V}(L)| = 6$ by Proposition 3.1. So we may assume that $\text{Max}(L)| \leq 2$. Now we split the proof into two cases. .

Case 1. $|\text{Max}(L)| = 2$. At first we show that $|\text{Min}(L)| \leq 2$. Suppose the result is false and let $\text{Min}(L) = \{F, E, G\}$. Then $T(F \cup E)$, $T(F \cup G)$ and $T(E \cup G)$ are proper filters of L and

$$T(F \cup E) \neq T(F \cup G) \neq T(E \cup G)$$

(see Theorem 2.7). Let $\text{Max}(L) = \{P_1, P_2\}$. Since every proper filter of L is contained in a maximal filter, without lose of generality, Suppose $T(F \cup E)$ and $T(F \cup G)$ contained in P_1; so $F, E, G \subseteq P_1$. Also, we know that for a maximal filter P_2, there is at most one minimal filter which is not contained in P_2. Let $F, E \subseteq P_2$. Then

$$\{P_1, T(F \cup E), P_2, E, F, T(F \cup K)\}$$

makes $K_{3,3}$ as a subgraph of $\mathcal{G}(L)$, which is impossible. Thus $|\text{Min}(L)| \leq 2$. Now we show that $|\mathcal{V}(L)| \leq 5$. Assume to the contrary, $|\mathcal{V}(L)| \geq 6$. If $\text{Min}(L) = \{F\}$, then $\mathcal{G}(L)$ is a planar gives $F \subseteq H$ for each filter H of L; hence $\mathcal{G}(L)$ is a complete graph, which is a contradiction. So we may assume that $\text{Min}(L) = \{F, E\}$.

If $P_1 \cap P_2$ is a minimal filter of L, we put $P_1 \cap P_2 = F$. Then $E \cap F = \{1\}$ gives either $E \not\subseteq P_1$ or $E \not\subseteq P_2$. Let $E \not\subseteq P_2$ (so $E \subseteq P_1$). Then $P_2 \subsetneqq T(E \cup P_2)$ gives $T(E \cup P_2) = L$. Since $E \subseteq P_1$, we get

$$P_1 = P_1 \cap T(E \cup P_2) = T(E \cup (P_1 \cap P_2)) = T(E \cup F).$$

Let H be a filter of L which is not minimal and maximal. We claim that $E \not\subseteq H$. Assume to the contrary, $E \subseteq H$. Then $H \not\subseteq P_2$; hence $H \subseteq P_1$ and $T(P_2 \cup H) = L$. If $P_2 \cap H = \{1\}$, then H is minimal by Proposition 1.2 (ii), a contradiction. Thus $P_2 \cap H \neq \{1\}$. Also $H \cap P_2 = (H \cap P_1) \cap P_2 = H \cap F \neq \{1\}$ which implies that $F \subseteq H$. Then $E \cup F \subseteq H$ gives $P_1 = T(E \cup F) \subseteq H$; hence $H = P_1$, which is impossible. Thus $E \not\subseteq H$. since $\mathcal{G}$ (L) is a planar graph and H is not minimal, H contains minimal filter F. We show that $T(E \cup H) \neq P_1, L$. If $T(E \cup H) = P_1$, then $H \subseteq P_1$ gives

$$P_1 = T(H \cup (P_1 \cap P_2)) = T(H \cup F) = T(H) = H,$$

a contradiction. If $T(E \cup H) = L$, then $H \subsetneqq P_1$ (for if $H \subseteq P_1$, then $E \cup H \subseteq P_1$; so $T(E \cup H) = L \subseteq P_1$, a contradiction). Thus $H \subseteq P_2$ and $T(H \cup P_1 = L$. As $H \subseteq P_2$,

$$P_2 = P_2 \cap T(H \cup P_1) = T(H \cap (P_1 \cap P_2)) = T(H \cap F) = H,$$

which is impossible. Therefore $T(E \cup H) \neq P_1, L$. Hence

$$\mathcal{V}(L) = \{F, H, F_3, T(H \cup E), P_1, P_2\}$$

makes K_5 in $\mathcal{G}(L)$, which is a contradiction.

So we may assume that $P_1 \cap P_2$ is not a minimal filter. Then there is a simple filter F such that $F \subseteq P_1 \cap P_2$. Let G be another filter of L. Let $E \subseteq P_1 \cap P_2$. Since G is not simple, it contains a simple filter. If $F \subseteq G$, then $\{F, G, P_1 \cap P_2, P_1, P_2\}$ makes K_5, which is a contradiction. If $E \subseteq G$, then $\{E, G, P_1 \cap P_2, P_1, P_2\}$ makes K_5, which is a contradiction. So we may assume that $E \subsetneqq P_1 \cap P_2$. Then $E \subsetneqq P_1$ or $E \subsetneqq P_2$. We may assume that $E \subsetneqq P_2$; hence $E \subseteq P_1$. As $E \subsetneqq P_2$, $T(F \cup E) \neq P_1 \cap P_2$. Also, $T(F \cup E) \neq P_1$ (if $T(F \cup E) = P_1$, then $F \subseteq P_2$ gives

$$P_1 \cap P_2 = P_2 \cap T(E \cup F) = T(F \cup (E \cap P_2)) = T(F) = F,$$

a contradiction. Hence $\{F, P_1, P_2, T(E \cup F), P_1 \cap P_2\}$ makes K_5 in $\mathcal{G}(L)$, which is a contradiction. Thus $|\mathcal{V}(L)| \leq 5$.

Case 2. $\mathrm{Max}(L) = \{P\}$. If $\mathrm{Min}(L) = \{F, E\}$, then we show that $|\mathcal{V}(L)\,| \leq 5$. If $T(F \cup E) = P$, then $\mathcal{V}(L) = \{F, E, P\}$ and we are done. So we may assume that $T(F \cup E) \neq P$. Let G, H be another filters of L. If $F \subseteq G, H$, then $\{F, G, H, T(F \cup E), P\}$ makes K_5 in $\mathcal{G}(L)$, a contradiction. Suppose $E \nsubseteq G$, $F \nsubseteq H$. So $F \subseteq G$, $E \subseteq H$. Clearly, $T(E \cup G) \neq T(F \cup H) \neq P$. Hence $\{F, G, T(F \cup H), T(F \cup E), P\}$ makes K_5, a contradiction. If $\mathrm{Min}(L) = \{F, E, G\}$, then show that $|\mathcal{V}(L)\,| \leq 7$. If $T(F \cup E \cup G) \neq P$, then

$$\{T(F \cup E), T(F \cup G), T(F \cup E \cup G), P, F\}$$

makes K_5 in $\mathcal{G}(L)$ which is a contradiction. So we may assume that $T(F \cup E \cup G) = P$. Let H be a filter of L. Since $\mathcal{G}(L)$ is a planar, H contains a minimal filter, say F. If $H \cap E = \{1\} = H \cap G$, Then

$$\begin{aligned} H &= H \cap P \\ &= H \cap T(F \cup E \cup G) \\ &= T(F \cup (H \cap (E \cup G))) \\ &= T(F) \\ &= F. \end{aligned}$$

If $F, E \subseteq H$ with $H \cap G = \{1\}$, then by the similar way $H = T(F \cup E)$. Similarly, if $F, E, G \subseteq H$, then $H = P$. Hence

$$\mathcal{V}(L) = \{F, E, G, T(F \cup E), T(F \cup G), T(E \cup G), P\}.$$

□

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A GRAPH ASSOCIATED TO FILTERS OF A LATTICE

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گراف مرتبط با فیلترهای یک مشبکه

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فرض کنید L یک مشبکه باشد که دارای کوچکترین عضو ۰ و بزرگترین عضو ۱ می‌باشد. در این مقاله، گرافی را به فیلترهای مشبکه L مرتبط می‌کنیم که مجموعه رئوس آن، مجموعه همه فیلترهای غیربدیهی از L است و دو رأس F و E مجاورند هرگاه $F \cap E \neq \{۱\}$. این گراف را با نماد $\mathcal{G}(L)$ نمایش می‌دهیم. خواص اساسی و ساختار این گراف را مورد مطالعه قرار می‌دهیم. علاوه براین، مسطح بودن این گراف را بررسی می‌کنیم.

کلمات کلیدی: مشبکه، فیلتر، گراف اشتراکی.