**Journal of Algebraic Systems**
**Vol. 10, No. 2, (2023), pp 361-374**

# WEAKLY BAER RINGS

S. MEHRALINEJADIAN, A. MOUSSAVI* AND SH. SAHEBI

Abstract. We say that a ring $R$ with unity is left weakly Baer if the left annihilator of any nonempty subset of $R$ is right s-unital by right semicentral idempotents, which implies that $R$ modulo the left annihilator of any nonempty subset is flat. It is shown that, unlike the Baer or right $PP$ conditions, the weakly Baer property is inherited by polynomial extensions. Examples are provided to explain the results.

## 1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unital. Recall that $R$ is a *Baer* ring if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. Kaplansky [17] introduced Baer rings to abstract various properties of $AW^{\star}$-algebras and von Neumann algebras. The class of Baer rings includes the von Neumann algebras.

A ring $R$ is called *quasi-Baer* if the right annihilator of every right ideal of $R$ is generated as a right ideal by an idempotent. It is easy to see that the quasi-Baer property is left-right symmetric for any ring. Quasi-Baer rings were initially considered by Clark [13] and used to characterize a finite dimensional algebra over an algebraically closed field as a twisted semigroup algebra of a matrix unit semigroup.

DOI: 10.22044/JAS.2022.11148.1555.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.
Keywords: Left weakly Baer ring; Weakly p.q.-Baer ring; APP ring; S-unital left (resp. right) ideal.
Received: 28 August 2021, Accepted: 23 April 2022.
*Corresponding author.

Closely related to Baer rings are $PP$ rings. A ring $R$ is called *right (left) PP* if every principal right (left) ideal is projective (equivalently, if the right (left) annihilator of any element of $R$ is generated (as a right (left) ideal) by an idempotent of $R$). A ring $R$ is called a *PP ring* (or *Rickart ring* [25, p. 18]), if it is both right and left $PP$. The concept of PP ring is not left-right symmetric by Chase [12]. A right $PP$ ring $R$ is Baer (so $PP$) when $R$ is orthogonally finite [27], and a right $PP$ ring $R$ is $PP$ when $R$ is *abelian* [14]. Also Von Neumann regular rings are right (left) $PP$ [15, Theorem 1.1].

Birkenmeier, Kim and Park [7] initiated the concept of principally quasi-Baer rings. A ring $R$ is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, $R$ is right p.q.-Baer if $R$ modulo the right annihilator of any principal right ideal is projective. Some examples were given [7] to show that the class of left p.q.-Baer rings is not contained in the class of right PP rings and the class of right $PP$ rings is not contained in the class of left p.q.-Baer rings. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian $PP$ rings. Further work on Baer and quasi-Baer rings appeared in [2, 5, 6, 8, 10, 22, 23, 24, 26].

Following Tominaga [29], a left ideal $I$ of a ring $R$ is said to be *right s-unital*, if for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. According to Liu and Zhao [19], a ring $R$ is called *left APP* if the left annihilator $l_R(Ra)$ is right $s$-unital as an ideal of $R$ for any element $a \in R$. As a generalization of p.q.-Baer rings, Majidinya et al. [20] introduced the concept of weakly p.q.-Baer rings. A ring $R$ with unity is *weakly p.q.-Baer* if for each $a \in R$ there exists a nonempty subset $E$ of right semicentral idempotents of $R$ such that $l_R(Ra) = \bigcup_{e\in E} Re$. The class of weakly p.q.-Baer rings is a natural subclass of the class of $APP$ rings and includes both left p.q.-Baer rings and right p.q.-Baer rings.

In this paper, we introduce and study the notion of left (resp. right) weakly Baer rings. A ring $R$ with unity is *left (resp. right) weakly Baer* if for each $A \subseteq R$ there exists a nonempty subset $E$ of right (resp. left) semicentral idempotents of $R$ such that $l_R(A) = \bigcup_{e\in E} Re$ (resp. $r_R(A) = \bigcup_{e\in E} eR$). This implies that $R$ modulo the left (resp. right) annihilator of any nonempty subset is flat. A ring $R$ is weakly Baer if it is both left and right weakly Baer. The class of left (resp. right) weakly Baer rings is a natural subclass of the class of $APP$ rings and weakly p.q.-Baer rings. Since the class of weakly p.q.-Baer rings, includes left (resp. right) weakly Baer rings, some results and their proofs are similar. It is proved that, the weakly Baer property is inherited by polynomial extensions and this class of rings is closed under direct products. Moreover, various classes of left (resp. right) weakly Baer rings which are neither Baer nor $PP$ nor p.q.-Baer are constructed.

## 2. Main results

It follows from Theorem 1 of [29] that an ideal $I$ of a ring $R$ is right $s$-unital if and only if for any finitely many elements $a_1, a_2, ..., a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $1 \leq i \leq n$. A submodule $N$ of a left $R$-module $M$ is called a pure submodule if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right $R$-module $L$. By [28, Proposition 11.3.13], an ideal $I$ is right $s$-unital if and only if $\frac{R}{I}$ is flat as a left $R$-module if and only if $I$ is pure as a left ideal of R. Note that if $I$ and $J$ are right $s$-unital ideals, then so is $I \cap J$. An idempotent $e \in R$ is called left (resp. right) semicentral if $xe = exe$ (resp. $ex = exe$), for all $x \in R$ [4]. The set of left (resp. right) semicentral idempotents of $R$ is denoted by $\mathcal{S}_l(R)$ (resp. $\mathcal{S}_r(R)$). The set of central idempotent elements of a ring $R$ is denoted by $\mathcal{B}(R)$. Observe that $\mathcal{S}_l(R) \cap \mathcal{S}_r(R) = \mathcal{B}(R)$, and if $R$ is semiprime or abelian, then $\mathcal{S}_l(R) = \mathcal{S}_r(R) = \mathcal{B}(R)$, by [7, Proposition 1.17].

**Definition 2.1.** A left ideal $I$ of a ring $R$ is said to be right $s$-unital by right semicentral idempotents if for every $a \in I$, $ae = a$ for some $e \in I \cap \mathcal{S}_r(R)$ or equivalently, $I = \cup_{e \in E} Re$ for some nonempty subset $E$ of $\mathcal{S}_r(R)$. The left $s$-unital right ideal by left semicentral idempotents may be defined analogously.

**Definition 2.2.** A ring $R$ is called left weakly Baer if $l_R(A)$ is right $s$-unital by right semicentral idempotents for all $A \subseteq R$. The right weakly Baer rings are defined similarly. A ring $R$ is weakly Baer if it is both left and right weakly Baer ring.

Note that if $R$ is a commutative Von Neuman regular ring then $R$ is weakly Baer. For this, let $A \subseteq R$. Then for $a \in l_R(A)$, there exists $x \in R$ where $a = axa$. Let $e = xa$. It is clear that $e$ is a right semicentral idempotent and $ae = a$. So $R$ is a weakly Baer ring.

**Example 2.3.**

(1) For a field $F$, let $R =< \oplus_{n=1}^{\infty} F_n, 1_{\prod_{n=1}^{\infty} F_n} >$ be the $F$-subalgebra of $\prod_{n=1}^{\infty} F_n$ generated by $\oplus_{n=1}^{\infty} F_n$ and $1_{\prod_{n=1}^{\infty} F_n}$, where $F_n = F$ for all $n$, defined in [18, Example 1. (2)]. Then $R$ is a commutative Von Neumann regular ring and as it is

mentioned above, it is weakly Baer. But, by [18, Example 1. (2)], $R$ is not Baer.

(2) An example of Cohn showed that the matrix ring $R = M_2(\mathbb{Z})$ is a Baer (and hence quasi-Baer) ring [2]. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ be an element of $R$. Then
$$l_R(A) = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} = Re$$
for the unique idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, that is not a right semicentral idempotent of $R$. So the ring $R$ is not left weakly Baer.

(3) The ring $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is quasi-Baer (hence left and right p.q.-Baer) by [24, Proposition 9], where $D$ is a domain which is not a division ring. Put $B = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, which is an element of $R$. Then
$$l_R(B) = \left\{ \begin{pmatrix} 0 & d_1 \\ 0 & d_2 \end{pmatrix} \mid d_1, d_2 \in D \right\},$$
where each element of $l_r(B)$ is right $s$-unital by the idempotent $e_1 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ or $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ only, but none of them is a right semicentral idempotent. So $R$ is not left weakly Baer. Note that this Example also shows that the class of weakly p.q.-Baer rings contains the class of left weakly Baer rings properly.

**Theorem 2.4.** *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the formal upper triangular matrix ring and $A$ and $B$ be rings and $M$ be an $(A, B)$-bimodule. Then $R$ is left weakly Baer if and only if the following conditions are satisfied:*

(1) *$A$ and $B$ be left weakly Baer rings.*

(2) *For each $X \subseteq A$, $N \subseteq M$ and $Y \subseteq B$ there exist*
$$F \subseteq r_R\left(\begin{pmatrix} X & N \\ 0 & Y \end{pmatrix}\right) \bigcap \mathcal{S}_r(R),$$
$$F_1 = \{f_1 \mid \exists\, E_1 = \begin{pmatrix} f_1 & k_1 \\ 0 & f_2 \end{pmatrix} \in F\},$$
$$K = \{k \mid \exists\, E = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix} \in F\}$$

*and* $F_2 = \{h_1 \mid \exists\, E_2 = \begin{pmatrix} h_1 & k_2 \\ 0 & h_2 \end{pmatrix} \in F\}$ *such that*

(a) $l_B(Y) = \bigcup_{h_2 \in F_2} Bh_2$.

(b) *For every* $f_1 \in F_1$ *where*

$$\begin{pmatrix} f_1 & k_1 \\ 0 & f_2 \end{pmatrix} \in F,\ f_1 \in l_A(X) \bigcap l_A(N + k_1 Y).$$

(c) *If* $a \in A$ *and* $m \in M$ *such that* $ax = 0$ *and* $aN + mY = 0$ *then* $a \in \bigcup_{f_1 \in F_1} Af_1$ *and* $m \in AK + MF_2$.

*Proof.* Let $G = \begin{pmatrix} X & N \\ 0 & Y \end{pmatrix} \subseteq R$. Assume that $R$ is left weakly Baer. Then there exists $F \subseteq l_R(G) \bigcap \mathcal{S}_r(R)$, $l_R(G) = \cup_{E \in F} RE$. Let

$$F_1 = \{f_1 \mid \exists\, E_1 = \begin{pmatrix} f_1 & k_1 \\ 0 & f_2 \end{pmatrix} \in F\},$$

$$K = \{k \mid \exists\, E = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix} \in F\}$$

and

$$F_2 = \{h_2 \mid \exists\, E_2 = \begin{pmatrix} h_1 & k_2 \\ 0 & h_2 \end{pmatrix} \in F\}.$$

By [20, Lemma 2.18], $F_1 \subseteq \mathcal{S}_r(A)$ and $F_2 \subseteq \mathcal{S}_r(B)$. Taking $N = 0$ and $Y = 0$, $l_A(X) = \bigcup_{f_1 \in F_1} Af_1$. Thus $A$ is left weaky Baer. Similarly $B$ is left weakly Baer. Let $\begin{pmatrix} x & n \\ 0 & y \end{pmatrix} \in G$. Then for $\begin{pmatrix} \gamma & \lambda \\ 0 & \delta \end{pmatrix} \in F$,

$$\begin{pmatrix} \gamma x & \gamma n + \lambda y \\ 0 & \delta y \end{pmatrix} = 0.$$

Since $\lambda = \gamma\lambda + \lambda\delta$, $\lambda y = \gamma\lambda y$, then $\gamma \in l_A(N + \lambda Y)$. If $a \in A$ and $m \in M$ such that $aX = 0$ and $aN + mY = 0$ then $\begin{pmatrix} a & m \\ 0 & 0 \end{pmatrix} \in l_R(G)$. Thus there exists $\begin{pmatrix} \gamma' & \lambda' \\ 0 & \delta' \end{pmatrix} \in F$ such that $a = a\gamma'$ and $a\lambda' + m\delta' = m$. So $a \in \bigcup_{f_1 \in F_1} Af_1$ and $m \in Ak + MF_2$. Conversely, assume that $A$ and $B$ are left weakly Baer rings and for $G = \begin{pmatrix} X & N \\ 0 & Y \end{pmatrix} \subseteq R$ there exist $F \subseteq r_R(G) \bigcap \mathcal{S}_r(R), F_1 \subseteq \mathcal{S}_r(A)$, $F_2 \subseteq \mathcal{S}_r(B)$ and $K \subseteq M$ such that conditions $(a), (b)$ and $(c)$ hold. Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in l_R(G)$. Then $\begin{pmatrix} ax & an + my \\ 0 & by \end{pmatrix} = 0$ where $\begin{pmatrix} x & n \\ 0 & y \end{pmatrix} \in G$. Therefore $aX = 0$,

$aN + mY = 0$ and $bY = 0$. Thus by $(a)$ and $(c)$, $a \in \bigcup_{f_1 \in F_1} Af_1$, $b \in \bigcup_{h_2 \in F_2} Bh_2$ and $m \in Ak + MF_2$. Hence $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \bigcup_{E \in F} RE$. So $R$ is left weakly Baer. □

**Corollary 2.5.** *Let $R = \begin{pmatrix} \overline{A} & \overline{A} \\ 0 & A \end{pmatrix}$ where $A$ is a left weakly Baer ring and $\overline{A} = \frac{A}{P}$ for a prime ideal $P$ such that $\overline{A}$ is a left weakly Baer ring and if $Y \subseteq P$ then $l_A(Y) \nsubseteq P$. Then $R$ is left weakly Baer. Morever if $P$ is not right s-unital by right semicentral idempotents of $R$ then $R$ is not right weakly Baer.*

*Proof.* Let $X, N \subseteq \overline{A}$ and $Y \subseteq A$. Then there exist $F_1 \subseteq \mathcal{S}_r(\overline{A})$ and $F_2 \subseteq \mathcal{S}_r(A)$ such that $l_{\overline{A}}(X + N + \overline{A}Y) = \bigcup_{e_1 \in F_1} \overline{A}e_1$ and $l_A(Y) = \bigcup_{e_2 \in F_2} Ae_2$. Consider the following cases:

Case 1. Assume that $Y \subseteq P$. Then $l_{\overline{A}}(Y) = \overline{A}$. Since $\overline{A}$ is a prime ring, by [7, Lemma 2.1] we have, $\mathcal{S}_r(\overline{A}) = \{\overline{0}, \overline{1}\}$. If $l_{\overline{A}}(X+N) = 0$, then in Theorem 2.4(ii) take $F_1 = \{\overline{0}\}$, $K = \{\overline{1}\}$, $F_2 = F_2$. If $l_{\overline{A}}(X+N) \neq 0$, then $F_1 = \{\overline{1}\}$. Now to satisfy Theorem 2.4(ii) take $F_1 = \{\overline{1}\}$, $K = \{\overline{0}\}$ and $F_2 = F_2$.

Case 2. Assume that $Y \nsubseteq P$. Then $l_{\overline{A}}(Y) = l_{\overline{A}}(\overline{A}Y) = \{\overline{0}\}$. Therefore $F_1 = \{\overline{0}\}$. Since $F_2Y = \{0\} \subseteq P$, then $F_2 \subseteq P$. Thus in Thorem 2.4(ii), take $F_1 = \{\overline{0}\}$, $K = \{\overline{0}\}$ and $F_2 = F_2$. Hence in all cases Theorem 2.4 yields thar $R$ is left weakly Baer. Now suppose that $P$ is not right s-unital by right semicentral idempotents and let $m = \begin{pmatrix} 0 & \overline{1} \\ 0 & 0 \end{pmatrix}$. Then $r_R(mR) = \begin{pmatrix} \overline{A} & \overline{A} \\ 0 & P \end{pmatrix}$ and so $R$ is not right weakly Baer. □

Note that for a ring $R$, $\mathcal{S}_r(R) = \mathcal{B}(R)$ if and only if $\mathcal{S}_l(R) = \mathcal{B}(R)$.

**Proposition 2.6.** *A left (resp. right) weakly Baer ring $R$ is semiprime if and only if $\mathcal{S}_r(R) = \mathcal{B}(R)$.*

*Proof.* Let $R$ be a semiprime left weakly Baer ring and $e \in \mathcal{S}_r(R)$. Since $e \in \mathcal{S}_r(R)$, then $eR(1-e)$ is an ideal by [7, Lemma 1.1]. Note that $eR(1-e)$ is nilpotent and so $eR(1-e) = 0$ by semiprimeness of $R$. Also since $e \in \mathcal{S}_r(R)$, $(1-e)Re = 0$. So $e \in \mathcal{B}(R)$. Conversely, assume that $\mathcal{S}_r(R) = \mathcal{B}(R)$ and $aRa = 0$ for some element $a \in R$. Then $a \in l_R(Ra)$. Hence there exists a cenral idempotent $e \in l_R(Ra)$ such that $a = ae$. Therefore $a = ae = ea = 0$. So $R$ is semiprime. The right case follows similarly. □

**Corollary 2.7.** *Commutative weakly Baer rings are reduced.*

Example 2.4 of [19] was given to show that quasi-Armendariz rings need not to be APP. In fact there exists a commutative reduced ring which is not weakly Baer. Regarding to [3], a ring $R$ satisfies the insertion of factors property (*IFP*) if and only if $r_R(x)$ is an ideal of R for all $x \in R$. Note that every ring with *IFP* is abelian. By Proposition 2.6 every abelian left (resp. right) weakly Baer ring is semiprime. Hence every left (resp. right) weakly Baer ring with *IFP* is semiprime. Also every abelian Baer ring is left (resp. right) weakly Baer and so semiprime. It is clear that every left (right) weakly Baer ring is weakly p.q-Baer and so a left APP-ring. By [16], $R$ is called quasi-Armendariz if whenever

$$f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$$

satisfy $f(x)R[x]g(x) = 0$, then $a_i R b_j = 0$ for every $i$ and $j$. Since every left $APP$ ring is quasi-Armendariz by [19, Proposition 2.3], then every left weakly Baer ring is quasi-Armendariz. On the other hand, Example 2.4 of [19] shows that there exists a quasi-Armendariz ring that is no left $APP$ and so it is not left weakly Baer.

The following example shows that there exists a class of $APP$ rings which are not left weakly Baer, so the class of $APP$ rings contains the class of left weakly Baer rings properly.

**Example 2.8.** For a field $F$, let $F_n = F$ for $n = 1, 2, ...$ and $S = M_2(\prod_{n=1}^{\infty} F_i)$. Let

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & < \oplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix},$$

which is a subring of $S$, where $(\oplus_{n=1}^{\infty} F_n,\ 1)$ is the $F$-algebra generated by $\oplus_{n=1}^{\infty} F_n$ and $1_{\prod_{n=1}^{\infty} F_n}$. Then by [7, Example 1.6], $R$ is a semiprime PP ring. So by [19, Proposition 2.3] $R$ is an APP ring. We show that $R$ is not left weakly Baer (note that by Proposition 2.6, in a semiprime left weakly Baer ring every right or left semicentral idempotent is cenral). Let $a = (a_n) \in \prod_{n=1}^{\infty} F_n$ with

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

and $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Let also $b = (b_n) \in \prod_{n=1}^{\infty} F_n$ with

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

and $B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $A \in l_R(B)$. If $R$ is a left weakly Baer ring, then there exists a central idempotent

$$E = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} \in l_R(B),$$

such that $A = AE$. Hence $ae_1 = a$ and $e_1 b = 0$. Let

$$e_1 = (x_1, x_2, ..., x_n, x, x, ...).$$

Since

$$e_1 b = (x_1, x_2, ..., x_n, x, x, ...)(0, 1, 0, 1, ...) = 0,$$

we have $x = x_{2i} = 0$ for each $i$. Therefore $e_1 = (x_1, 0, ..., x_{2k+1}, 0, 0, ...)$, where $2k+1$ is the biggest odd number which is not greater than $n$. On the other hand

$$\begin{aligned} a &= ae_1 \\ &= (1, 0, 1, 0, ...)(x_1, 0, x_3, 0, ..., x_{2k+1}, 0, 0, ...) \\ &= (x_1, 0, x_3, 0, ..., x_{2k+1}, 0, 0, ...) \end{aligned}$$

which is a contradiction. This shows that $R$ is not left weakly Baer and the result follows.

If a left (resp. right) ideal $J$ of a ring $R$ is a left (resp. right) direct summand of $R$, then $J = Re$ (resp. $J = eR$) for some idempotent $e \in R$. We say an ideal $J$ of a ring $R$ is a left (resp. right) ring direct summand of $R$, if $J = Re$ (resp. $J = eR$ ) for some $e \in \mathcal{S}_r(R)$ (resp. $e \in \mathcal{S}_l(R)$).

*Remark* 2.9. By [7, Lemma 1.1], for every idempotent $e \in R$, $Re$ is an ideal of $R$ if and only if $e \in \mathcal{S}_r(R)$. So by Definition 2.2, it is clear that $R$ is left weakly Baer if and only if $l_R(A)$ is a union of left ring direct summands of $R$, for every $A \subseteq R$.

**Theorem 2.10.** *Let $R$ be a left weakly Baer ring. Then for every subset $A$ of $R$, there exists a subset $E \subseteq \mathcal{S}_l(R)$ such that $A \subseteq \bigcap_{e\in E} eR$ and*

$$(\cap_{e\in E} eR) \cap l_R(A) = \cup_{f\in E}(\cap_{e\in E} eR(1-f)).$$

*Proof.* The proof is similar Theorem 2.8 of [20]. □

The following example shows that there is another large class of (even commutative) weakly Baer rings which are neither Baer nor PP.

**Example 2.11.** Let $A$ be a commutative Baer ring and $P$ be a nonzero prime ideal of $A$. Let $a_0 \in P$ be an element of $A$ such that $l_A(a_0) = 0$. Let

$$R = \{(a, \bar{b}) \mid a \in A, \, \bar{b} \in \oplus_{i=1}^{\infty} Q_i\},$$

where $Q_i = A/P$ for each $i$, $\overline{b} = (\overline{b_i})_{i=1}^{\infty}$ and $\overline{b_i} = b_i + P \in Q_i$. Then $R$ is a commutative ring with addition pointwise and multiplication defined by $(z, \overline{y})(t, \overline{x}) = (zt, \overline{zx} + \overline{ty} + \overline{xy})$, for every $z, t \in A$ and $\overline{x}, \overline{y} \in \oplus_{i=1}^{\infty} Q_i$. We claim that $R$ is a commutative weakly Baer ring which is neither p.q.-Baer nor $PP$.

*Proof.* The proof is similar Example 2.4 of [21]. □

Note that every domain which is not a field satisfies conditions of Example 2.11.

**Lemma 2.12.** *Let $J$ be a left ideal of a ring $R$, $a_1, \ldots, a_n \in J$. Assume that there exist $e_1, \ldots, e_n \in \mathcal{S}_r(R) \cap J$, such that $a_i e_i = a_i$ for each $i = 1, \ldots, n$. Then there exists $e \in \mathcal{S}_r(R)$ such that $a_i e = a_i$, for each $i = 1, \ldots, n$. In particular, a left ideal $J$ is right $s$-unital by right semicentral idempotents if and only if for any $a_1, \ldots, a_n \in J$, there exists an idempotent $e \in J \cap \mathcal{S}_r(R)$ such that $a_i e = a_i$, for each $i$.*

*Proof.* It is enough to prove for $n = 2$. Let $a, b \in J$ and for some $e, f \in \mathcal{S}_r(R) \cap J$, $ae = a$ and $bf = b$. It is clear that

$$(e + f - ef) \in \mathcal{S}_r(R) \cap J.$$

Then $a(e + f - ef) = ae + af - aef = a + af - af = a$ and

$$\begin{aligned} b(e + f - ef) &= be + bf - bef \\ &= bfe + b - bef \\ &= bfef + b - bfef \\ &= b. \end{aligned}$$

In particular, let $J$ be a left ideal which is right $s$-unital by right semicentral idempotents and assume that $a_1, ..., a_n \in J$. Then for each $1 \le i \le n$, there exists an $e_i \in J \cap \mathcal{S}_r(R)$, such that $a_i = a_i e_i$, and so there exists $e \in J \cap \mathcal{S}_r(R)$ such that $a_i = a_i e$. The converse is clear, and the proof is complete. □

*Remark* 2.13. Note that if $e_1, e_2 \in \mathcal{S}_r(R)$, then $e_1 e_2$, $e_2 e_1 \in \mathcal{S}_r(R)$ too. Also if the left ideals $I$ and $J$ are right $s$-unital by right semicentral idempotents, then so is $I \cap J$. To see this, let $x \in I \cap J$. Then $xe_1 = x$ and $xe_2 = x$, for some right semicentral idempotents $e_1 \in I$ and $e_2 \in J$. Let $e = e_1 e_2 = e_1 e_2 e_1$, then $xe = x$ and $e \in I \cap J \cap \mathcal{S}_r(R)$. Hence if the left ideals $I_1, \ldots, I_n$ are right $s$-unital by right semicentral idempotents, for some positive integer $n$, then so is $I = \cap_{i=1}^{n} I_i$.

**Proposition 2.14.** *If $R$ is a left weakly Baer then $eRe$ is a left weakly Baer ring, for every nonzero idempotent $e$ of $R$.*

*Proof.* Assume that $R$ is a left weakly Baer ring. Let $B \subseteq eRe$ and $a \in l_{eRe}(B)$. So there exists a right semicentral idempotent $z \in l_R(B)$ such that $az = a$. Let $e' = eze$. Then

$$ae' = a(eze) = (ae)(ze) = a(ze) = (az)e = ae = a.$$

Since $zB = 0$, we have $e'B = ezeB = ezB = 0$. Thus $e' \in l_{eRe}(B)$. Clearly $e'$ is right semicentral in $eRe$. So, the ring $eRe$ is left weakly Baer. □

Now we show that, unlike the Baer or right $PP$ conditions, the weakly Baer property is inherited by polynomial extensions.

**Theorem 2.15.** *A ring $R$ is left weakly Baer ring if and only if $R[x]$ is left weakly Baer ring.*

*Proof.* Let $R$ be a left weakly Baer ring and $B = A[x] \subseteq R[x]$ such that

$$A = \{a_i \mid \exists f = \textstyle\sum_{i=0}^{n} a_i x^i \in B\}.$$

Let $g(x) = \sum_{j=0}^{m} b_j x^j \in l_{R[x]}(B)$. By [19, Proposition 2.3], $R$ is quasi-Armendariz and then $R$ is a reduced ring. Hence by [2, Corollary 2], $l_{R[x]}(B) = l_R(A)[x]$. So $b_j \in l_R(A)$ for every $0 \leqslant j \leqslant n$. Since $R$ is left weakly Baer there exists $e_j \in l_R(A) \cap \mathcal{S}_r(R)$ such that $b_j e_j = b_j$. Then by Lemma 2.12, there exists $e \in l_R(A) \cap \mathcal{S}_r(R)$ and $b_j e = b_j$. Therefore $g(x)e = g(x)$ for $e \in l_{R[x]}(B) \cap \mathcal{S}_r(R[x])$ and so $R[x]$ is left weakly Baer. Conversely, Suppose that $R[x]$ is a left weakly Baer ring and $b \in l_R(A)$ for some subset $A \subseteq R$. Then $bf(x) = 0$ for every $f(x) \in A[x]$ and hence $b \in l_{R[x]}(A[x])$. Since $R[x]$ is left weakly Baer, $be(x) = b$ for some

$$e(x) = \textstyle\sum_{i=0}^{n} e_i x^i \in l_{R[x]}(A[x]) \cap \mathcal{S}_r(R[x]).$$

So $be_0 = b$ and $e_0 \in r_R(A)$. Since $e(x) \in \mathcal{S}_r(R[x])$,

$$e(x)R[x](1 - e(x)) = 0.$$

Hence $e_0 R(1 - e_0) = 0$ and therefore $e_0 \in \mathcal{S}_r(R)$. So $R$ is left weakly Baer and the proof is complete. □

There exists a commutative von Neumann regular ring $R$ (hence weakly Baer), such that $R[[x]]$ is not weakly Baer. For example, let $R$ be the ring that defined in [19, Example 2.4]. Then $R$ is a commutative von Neumann regular ring. But by [19, Example 2.4], $R[[x]]$ is not weakly Baer.

**Theorem 2.16.** *Let $I$ be an index set and $R_i$ is a ring, for each $i \in I$. Then, the ring $\prod_{i \in I} R_i$ is left weakly Baer if and only if $R_i$ is left weakly Baer for each $i \in I$.*

*Proof.* The proof is similar to that of [19, Proposition 3.1]. □

By the following example, we can obtain a rich class of weakly Baer rings which are not Baer.

**Example 2.17.** Let $R$ be a left weakly Baer ring which is not Baer and $I$ a nonempty index set. Then by Theorem 2.15 $R[x]$ is left weakly Baer. But $R[x]$ is not Baer by [2, Theorem B]. Also, by Theorem 2.16, $\prod_{i\in I} Ri$ is left weakly Baer which is not Baer by [11, Proposition 3.1.5], where $R_i = R$ for each $i \in I$.

**Lemma 2.18.** *A left ideal $J$ of a ring $R$ is right $s$-unital if and only if for any $a_1, a_2, \ldots, a_n \in J$, there exists an element $e \in J$ such that $a_i = a_i e$ for each $1 \leq i \leq n$.*

*Proof.* The proof follows from [29, Theorem 1]. □

**Proposition 2.19.** *Let $I$ be a left ideal of a ring $R$. If $I$ is right $s$-unital and finitely generated as a left ideal, then $I = Re$ for an idempotent $e \in R$.*

*Proof.* Let $I$ be finitely generated as a left ideal by elements

$$a_1, a_2, \ldots, a_n \in R.$$

Since $I$ is right $s$-unital by Lemma 2.18, $a_i = a_i a$ for $1 \leq i \leq n$ and some $a \in I$. It is clear that $I = Ra$. Since $I$ is right $s$-unital, $ara = a$ for some $r \in R$. Then $(ra)^2 = rara = ra$. Hence $ra$ is an idempotent element. On the other hand $Ra = Rara \subseteq Rra \subseteq I$. So $I = Rra$ and the proof is complete. □

**Corollary 2.20.** *Let $R$ be a left weakly Baer ring. If $l_R(A)$ is finitely generated as a left ideal for all $A \subseteq R$, then $R$ is a Baer ring.*

**Proposition 2.21.** *(1) A ring $R$ is left weakly Baer if and only if for each $A \subseteq R$, $l_R(A) = \sum_{e\in E} Re$ for some $E \subseteq \mathcal{S}_r(R)$.*

*(2) Suppose that $R$ is left weakly Baer. Then $R$ is not Baer if and only if there exists $A \subseteq R$ and $E \subseteq \mathcal{S}_r(R)$ where*

$$l_R(A) = \sum_{e\in E} Re$$

*but for each $E' \subseteq E$ which is finite, $l_R(A) \neq \sum_{e\in E'} Re$.*

*Proof.* (1) Suppose that $R$ ia left weakly Baer. The result follows from definitions 2.1 and 2.2. Conversely suppose that $A \subseteq R$. Then $l_R(A) = \sum_{e\in E} Re$ for some $E \subseteq \mathcal{S}_r(R)$. Let $y \in l_R(A)$. Then $y \in \sum_{e\in F} Re$ for some finite subset $F \subseteq E$. From [9, Proposition 1.3], $\sum_{e\in F} Re = Rf$ for some $f \in \mathcal{S}_r(R)$. Hence $l_R(A) = \bigcup_{e\in H} Re$ for $H \subseteq \mathcal{S}_r(R)$. So $R$ is left weakly Baer.

(2) Suppose that $R$ is not Baer. Then for some $A \subseteq R$, $l_R(A) \neq Re$ where $e$ is an idempotent. Since $R$ is left weakly Baer, $l_R(A) = \sum_{e\in E} Re$ for some $E \subseteq \mathcal{S}_r(R)$, where $E$ can not be a finite set by Proposition 2.19. Conversely, suppose that there exist $A \subseteq R$ and $E \subseteq \mathcal{S}_r(R)$ such that $l_R(A) = \sum_{e\in E} Re$ but $l_R(A) \neq \sum_{e\in E'} Re$ for each finite subet $E'$ of E. If $R$ is Baer, then $l_R(A) = Rf$ for an idempotent $f \in I(R)$. Hence $f \in \sum_{e\in F} Re$ for finite subset $F$ of $E$. This is a contradiction and the result follows.

□

**Corollary 2.22.** *Let $R$ be a left weakly Baer ring and assume that $R$ satisfies the ACC on left (resp. right) ring direct summands. Then $R$ is a Baer ring.*

*Proof.* It follows from Propositions 2.19 and 2.21. □

Note that Example 2.11 shows that in Corollary 2.22 the condition ACC on right or left ring direct summands is not redundant. Let $R$ be a ring and $M$ a left (resp. right) $R$-module. Denote by $u.dim({}_RM)$ (resp. $u.dim(M_R)$) the uniform dimension of $M$ as left (resp. right) $R$-module. For a ring $R$, if $u.dim({}_RR) < \infty$, then $R$ satisfies $ACC$ on left ring direct summands and $ACC$ on right ring direct summands. So by Corollary 2.22, we have the following.

**Corollary 2.23.** *If $R$ is a left weakly Baer ring with $u.dim({}_RR) < \infty$, then $R$ is a Baer ring.*

## Acknowledgments

The authors thank the referee for his/her constructive comments. This paper is supported by Islamic Azad University Central Tehran Branch (IAUCTB). The authors want to thank the authority of IAUCTB for their support to complete this research.

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**Somaye Mehralinejadian**
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
Email: som.mehralinejadian.sci@iauctb.ac.ir

**Ahmad Moussavi**
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran,
and
Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran.
Email: moussavi.a@azad.ac.ir
Email: moussavi.a@modares.ac.ir

**Shervin Sahebi**
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
Email: sahebi@iauctb.ac.ir

Journal of Algebraic Systems

WEAKLY BAER RINGS

S. MEHRALINEJADIAN, A. MOUSSAVI AND SH. SAHEBI

حلقه‌های بئر ضعیف

سمیه مهرعلی‌نژادیان[۱]، احمد موسوی[۲،۱] و شروین صاحبی[۱]

[۱]گروه ریاضی، واحد تهران مرکز، دانشگاه آزاد اسلامی، تهران، ایران

[۲]گروه ریاضی محض، دانشکده علوم ریاضی، دانشگاه تربیت مدرس، تهران، ایران

در این مقاله، حلقه‌های بئر ضعیف معرفی و مورد مطالعه قرار گرفته است. حلقه $R$ را بئر ضعیف چپ نامیم اگر پوچ‌ساز چپ هر زیرمجموعه ناتهی از $R$، $s$-یکال راست با خودتوان‌های نیم‌مرکزی راست باشد، در این صورت $R$ به پیمانه پوچ‌ساز چپ هر زیرمجموعه ناتهی، تخت است. برخلاف خاصیت‌های بئر یا $PP$ راست، توسیع‌های چندجمله‌ای، ویژگی بئر ضعیف را به ارث می‌برند. در این مقاله همچنین مثال‌هایی برای شرح نتایج ارائه شده است.

کلمات کلیدی: حلقه بئر ضعیف چپ، حلقه $p.q$-بئر ضعیف، حلقه $APP$، ایده‌آل $s$-یکال چپ (راست).