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NEW MAJORIZATION FOR BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. This work aims to introduce and investigate a preordering in $B(\mathcal{H})$, the Banach space of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$. It is called strong majorization and denoted by $S \prec_s T$, for $S, T \in B(\mathcal{H})$. The strong majorization follows the majorization considered by Barnes, but not vice versa. If $S \prec_s T$, then S inherits some properties of T. The strong majorization will be extended for the d-tuples of operators in $B(\mathcal{H})^d$ and is called joint strong majorization denoted by $S \prec_{js} T$, for $S, T \in B(\mathcal{H})^d$. We show that some properties of strong majorization are satisfied for joint strong majorization.

1. Introduction

Let $B(\mathcal{H})$ denote the Banach space of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. The numerical radius and the Crowford number of $T \in B(\mathcal{H})$, respectively are defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : \ x \in \mathcal{H}, \ \|x\| = 1\},$$

and

$$c(T) = \inf\{|\langle Tx, x\rangle| : \ x \in \mathcal{H}, \ \|x\| = 1\}.$$

It is well known that

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|, \tag{1.1}$$

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where $\|T\|$ is the usual operator norm.

In [7], Zamani et al. obtained the following lemma.

Lemma 1.1. [7, Lemma 2.7] *Let $T \in B(\mathcal{H})$. Then for all $x \in \mathcal{H}$ with $\|x\| = 1$, we have*

$$\|T\|^2 + c^2(T) \leq \|Tx\|^2 + |\langle Tx, x\rangle|^2 \leq 4w^2(T). \tag{1.2}$$

For $T \in B(\mathcal{H})$, we denote $R(T)$ for the range of T and $N(T)$ for the null space of T, its adjoint is denoted by T^*.

An operator $T \in B(\mathcal{H})$ is said to be positive if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$.

For Banach spaces X and Y, we denote the Banach space of all bounded linear operators $T : X \to Y$, by $B(X, Y)$.

In [1], Barnes considered the following majorization.

Definition 1.2. [1] Let $T \in B(X, Y)$ and $S \in B(X, Z)$. Then T majorizes S and denoted by $S \prec_B T$ if there exists $M > 0$ such that for all $x \in \mathcal{H}$, we have

$$\|Sx\| \leq M\|Tx\|.$$

In [1], Barnes obtained the following proposition.

Proposition 1.3. [1, Proposition 3] *Let $T \in B(X, Y)$, and $S \in B(X, Z)$. Then the following statements are equivalent.*

1. $S \prec_B T$.
2. *There exists $V \in B(\overline{R(T)}, Z)$ such that $S = VT$.*
3. *Whenever $\{x_n\} \subseteq X$ with $\|Tx_n\| \to 0$, then $\|Sx_n\| \to 0$.*

In [5], Douglas proved the next proposition.

Proposition 1.4. [5] *Let $S, T \in B(\mathcal{H})$. Then the following three conditions are equivalent.*

1. $R(S) \subseteq R(T)$.
2. $S^* \prec_B T^*$.
3. *$S = TU$ for some $U \in B(\mathcal{H})$.*

For more details about numerical radius, norm equalities and majorization, we refer the reader to [2, 3, 4, 6, 7].

We organize this paper as follows. In the next section, we introduce a preorder relation in $B(\mathcal{H})$, which is called strong majorization and denoted by $\prec_s$. Some properties of strong majorization are investigated and we show that strong majorization follows majorization considered by Barnes, but not vice versa. We prove that if $S \prec_s T$, then S inherits some properties of T. In Section 3 we extend the strong majorization for the d-tuples of operators in $B(\mathcal{H})^d$ and is called joint strong majorization denoted by $S \prec_{js} T$, for $S, T \in B(\mathcal{H})^d$. We show that some properties of strong majorization are satisfied for joint strong majorization.

2. Strong majorization

In this section, we introduce a preordering on $B(\mathcal{H})$, we call it, strong majorization and consider some properties of it.

Definition 2.1. Let $S, T \in B(\mathcal{H})$. We say that T strong majorizes S and denoted by $S \prec_s T$ if there exists $M > 0$ such that for all $x \in \mathcal{H}$,

$$|\langle Sx, x\rangle| \leq M|\langle Tx, x\rangle|. \tag{2.1}$$

Clearly, strong majorization is a preordering relation on $B(\mathcal{H})$, i.e., it is reflexive and transitive. Obviously, $S \prec_s T$ if and only if $S^* \prec_s T^*$. By taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in (2.1), we get

$$w(S) \leq Mw(T). \tag{2.2}$$

Proposition 2.2. *Let $S, T \in B(\mathcal{H})$. If $S \prec_s T$, then $S \prec_B T$.*

Proof. By assumption, there exists $M > 0$ such that for all $x \in \mathcal{H}$, we have (2.1). The inequalities (1.1) and (2.2) follow that

$$0 \leq w(S) \leq Mw(T) \leq M\|T\|,$$

so

$$4w^2(S) \leq 4M^2\|T\|^2. \tag{2.3}$$

On the other hand, (1.2) concludes the following inequalities for $x \in \mathcal{H}$ with $\|x\| = 1$,

$$\|Sx\|^2 \leq \|Sx\|^2 + |\langle Sx, x\rangle|^2 \leq 4w^2(S),$$

and

$$\begin{aligned} 4M^2\|T\|^2 &\leq 4M^2(\|T\|^2 + c^2(T)) \\ &\leq 4M^2(\|Tx\|^2 + |\langle Tx, x\rangle|^2) \\ &\leq 4M^2(\|Tx\|^2 + \|Tx\|^2\|x\|^2) \\ &\leq 8M^2\|Tx\|^2. \end{aligned}$$

The above inequalities and (2.3) follow that

$$\|Sx\|^2 \leq 8M^2\|Tx\|^2,$$

so for all $x \in \mathcal{H}$

$$\|Sx\| \leq \sqrt{8}M\|Tx\|. \tag{2.4}$$

Therefore $S \prec_B T$. □

The inequality (2.4) concludes that $N(T) \subseteq N(S)$. Also, by taking the supremum in (2.4) over $x \in \mathcal{H}$ with $\|x\| = 1$, it follows that

$$\|S\| \leq \sqrt{8}M\|T\|.$$

Remark 2.3. Let $S, T \in B(\mathcal{H})$ be such that $S = aT$, for some $a \in \mathbb{C} \setminus \{0\}$. Clearly, $T \prec_s S$ and $S \prec_s T$, but for $a \neq 1$, we have $S \neq T$, i.e., in general the strong majorization is not a partial ordering.

Now we obtain nontrivial example of (2.1).

Example 2.4. Let

$$\mathcal{H} = \ell_2 = \{x = (x_n) : \ \textstyle\sum_{n=1}^{\infty} |x_n|^2 < \infty\}$$

be the Hilbert space with the inner product $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \bar{y_n}$, where $x = (x_n)$ and $y = (y_n)$ are in ℓ_2. Suppose that $S, T \in B(\mathcal{H})$ are defined by

$$Sx = (0, 0, x_3, x_4, \ldots), \text{ and } Tx = (0, x_2, x_3, \ldots), \text{ for } x = (x_n) \in \mathcal{H}.$$

Hence for $x = (x_n) \in \ell_2$, we have

$$\langle Sx, x\rangle = |x_3|^2 + |x_4|^2 + \cdots ,$$
$$\langle Tx, x\rangle = |x_2|^2 + |x_3|^2 + |x_4|^2 + \cdots .$$

Clearly

$$|\langle Sx, x\rangle| \leq |\langle Tx, x\rangle|,$$

and so $S \prec_s T$.

The next example obtains in general the inverse of Proposition 2.2 is not correct.

Example 2.5. Let $n \in \mathbb{N} \setminus \{2\}$ be even and $\mathcal{H} = \mathbb{C}^n$ be a Hilbert space with inner product $\langle x, y\rangle = \sum\limits_{i=1}^{n} x_i \bar{y_i}$, for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$. Let S be the right shift operator on $\mathcal{H}$ defined by $Sx = (0, x_1, x_2, \ldots, x_{n-1})$. Thus

$$\|Sx\|^2 = \langle Sx, Sx\rangle = |x_1|^2 + |x_2|^2 + \cdots + |x_{n-1}|^2, \tag{2.5}$$

$$\langle Sx, x\rangle = \bar{x_2}x_1 + \bar{x_3}x_2 + \cdots + \bar{x_n}\, x_{n-1}. \tag{2.6}$$

Let T be the operator on $\mathcal{H}$ defined by the block diagonal $n \times n$ matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, it follows that

$$Tx = (x_2, x_1, x_4, x_3, \ldots, x_n, x_{n-1}),$$

and

$$\|Tx\|^2 = \langle Tx, Tx\rangle = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2, \tag{2.7}$$

$$\langle Tx, x\rangle = \bar{x}_1 x_2 + \bar{x}_2 x_1 + \bar{x}_3 x_4 + \bar{x}_4 x_3 + \cdots + \overline{x_{n-1}}\, x_n + \overline{x_n}\, x_{n-1}. \tag{2.8}$$

The relations (2.5) and (2.7) follow that for all $x \in \mathbb{C}^n$, we have

$$\|Sx\| \leq \|Tx\|,$$

that is $S \prec_B T$. But for $x = (0, 1, 1, 0, \ldots, 0) \in \mathbb{C}^n$, the relations (2.6) and (2.8) follow that

$$\langle Sx, x\rangle = 1, \qquad \langle Tx, x\rangle = 0,$$

and so $S \not\prec_s T$.

Proposition 2.6. *Let $S, T \in B(\mathcal{H})$. If $S \prec_s T$, then the following statements hold.*

(i) *There exists $V \in B(\overline{R(T)}, \mathcal{H})$ such that $S = VT$.*
(ii) *Whenever $\{x_n\} \subseteq \mathcal{H}$ with $\|Tx_n\| \to 0$, then $\|Sx_n\| \to 0$.*

Proof. Propositions 2.2 and 1.3 follow the assertions. □

Theorem 2.7. *Let $S, T \in B(\mathcal{H})$. If $S \prec_s T$, then the following statements are true.*

(i) *If $S_1, S_2 \in B(\mathcal{H})$ and $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ such that $S_1 \prec_s T$, $S_2 \prec_s T$, then $\alpha_1 S_1 + \alpha_2 S_2 \prec_s T$.*
(ii) *If T is self-adjoint, then $Re(S) \prec_s T$, $Im(S) \prec_s T$, where $Re(S) = \frac{S+S^*}{2}$ and $Im(S) = \frac{S-S^*}{2i}$.*
(iii) *$R(S^*) \subseteq R(T^*)$ and $R(S) \subseteq R(T)$.*

Proof. (i) By assumption, there exist two positive numbers M_1, M_2 such that for all $x \in \mathcal{H}$

$$\begin{aligned}|\langle(\alpha_1 S_1 + \alpha_2 S_2)x, x\rangle| &\leq |\alpha_1||\langle S_1 x, x\rangle| + |\alpha_2||\langle S_2 x, x\rangle| \\ &\leq (|\alpha_1| M_1 + |\alpha_2| M_2)|\langle Tx, x\rangle|,\end{aligned}$$

that is $\alpha_1 S_1 + \alpha_2 S_2 \prec_s T$.

(ii) Since $S \prec_s T$ follows $S^* \prec_s T^*$, and by hypothesis $T = T^*$, we have $S^* \prec_s T$. Now the assertions follow by part (i).

(iii) According to Propositions 1.4, 1.3 and 2.2 and since $S \prec_s T$ if and only if $S^* \prec_s T^*$, the assumption concludes $R(S^*) \subseteq R(T^*)$ and $R(S) \subseteq R(T)$. □

Theorem 2.8. *Let $S, R, T \in B(\mathcal{H})$. If $S \prec_s T$, then*

(i) $T^*S \prec_s T^*T$ *and* $S^*T \prec_s T^*T$,
(ii) $S^*S \prec_s T^*T$,
(iii) $R^*SR \prec_s R^*TR$,
(iv) $T^*S \pm S^*T \prec_s T^*T$.

Proof. (i) Since $S \prec_s T$ implies $S \prec_B T$, so there exists $M > 0$ such that for all $x \in \mathcal{H}$,

$$\begin{aligned}|\langle T^*Sx, x\rangle| &= |\langle Sx, Tx\rangle| \\ &\leq \|Sx\|\|Tx\| \\ &\leq M\|Tx\|^2 \\ &= M|\langle Tx, Tx\rangle| \\ &= M|\langle T^*Tx, x\rangle|.\end{aligned}$$

That is $T^*S \prec_s T^*T$.
As $S \prec_s T$ implies $S^* \prec_s T^*$, so $T^*S \prec_s T^*T$ follows that $S^*T \prec_s T^*T$.

(ii) As $S \prec_s T$ follows $S \prec_B T$, so there is $M > 0$ such that for all $x \in \mathcal{H}$,

$$\begin{aligned}|\langle S^*Sx, x\rangle| &= |\langle Sx, Sx\rangle| \\ &= \|Sx\|^2 \\ &\leq M\|Tx\|^2 \\ &= M|\langle Tx, Tx\rangle| \\ &= M|\langle T^*Tx, x\rangle|.\end{aligned}$$

That is $S^*S \prec_s T^*T$.

(iii) Since $S \prec_s T$, there exists $M > 0$ such that for all $x \in \mathcal{H}$,

$$\begin{aligned}|\langle R^*SRx, x\rangle| &= |\langle SRx, Rx\rangle| \\ &\leq M|\langle TRx, Rx\rangle| \\ &= M|\langle R^*TRx, x\rangle|.\end{aligned}$$

(iv) Part (i) and Theorem 2.7 imply the assertion. □

Theorem 2.9. *Let* $S, T \in B(\mathcal{H})$ *such that* $S \prec_s T$ *and* M *be a subspace of* $\mathcal{H}$. *If* $TM \subseteq M^\perp$, *then* $SM \subseteq M^\perp$.

Proof. By assumption, there exists $N > 0$ such that for all $x \in \mathcal{H}$,

$$|\langle Sx, x\rangle| \leq N|\langle Tx, x\rangle|. \tag{2.9}$$

By hypothesis $x \in M$ implies that $Tx \in M^\perp$. Assume that $x \in M$, so $\langle Tx, x\rangle = 0$, thus (2.9) implies that $\langle Sx, x\rangle = 0$, that is $Sx \in M^\perp$. Therefore $SM \subseteq M^\perp$. □

Theorem 2.10. *Let $S, T \in B(\mathcal{H})$ and $S \prec_s T$. If S and T are both self-adjoint, then $S^n \prec_s T^n$, where $n = 2^m$, for all $m \in \mathbb{N}$.*

Proof. We proceed by induction. For $m = 1$, according to part (ii) of Theorem 2.8, we have

$$S^*S \prec_s T^*T.$$

Since by hypothesis $S^* = S$ and $T^* = T$, it follows that $S^2 \prec_s T^2$. Now suppose that for $n = 2^m$ and $m \in \mathbb{N}$, we have $S^n \prec_s T^n$. Again we use part (ii) of Theorem 2.8 to conclude that

$$(S^n)^*S^n \prec_s (T^n)^*T^n,$$

since $S^* = S$ and $T^* = T$, we get $S^{2n} \prec_s T^{2n}$. Thus the result holds for $2n = 2^{m+1}$. This completes the induction. □

For $T \in B(\mathcal{H})$, the Davis-Wielandt radius of T is defined by

$$dw(T) = \sup\left\{\sqrt{|\langle Tx, x\rangle|^2 + \|Tx\|^4} : \ x \in \mathcal{H}, \ \|x\| = 1\right\}.$$

The total cosine of T is defined by

$$|\cos|T = \inf\left\{\frac{|\langle Tx, x\rangle|}{\|Tx\| \ \|x\|} : \ x \in \mathcal{H}, \ Tx \neq 0, \ x \neq 0\right\}.$$

These concepts will be used in the next theorem.

Theorem 2.11. *Suppose that $S, T \in B(\mathcal{H})$ and $S \prec_s T$, i.e., there exists $M > 0$ such that for all $x \in \mathcal{H}$,*

$$|\langle Sx, x\rangle| \leq M|\langle Tx, x\rangle|. \tag{2.10}$$

Then the following statements hold.

(i) *For all $x \in \mathcal{H}$, $\sqrt{|\langle Sx, x\rangle|^2 + \|Sx\|^4} \leq N\sqrt{|\langle Tx, x\rangle|^2 + \|Tx\|^4}$, and so $dw(S) \leq N\ dw(T)$, for some $N > 0$.*

(ii) *$|\cos|S \leq \sqrt{8}M^2|\cos|T$.*

(iii) *$c(S) \leq M\ c(T)$.*

Proof. (i) By Proposition 2.2, for all $x \in \mathcal{H}$, we have $\|Sx\| \leq \sqrt{8}M\|Tx\|$ and so

$$\|Sx\|^4 \leq 64M^4\|Tx\|^4. \tag{2.11}$$

Also, (2.10) follows that

$$|\langle Sx, x\rangle|^2 \leq M^2|\langle Tx, x\rangle|^2. \tag{2.12}$$

The relations (2.11) and (2.12) conclude that for some $N > 0$, we have

$$\sqrt{|\langle Sx, x\rangle|^2 + \|Sx\|^4} \leq N\sqrt{|\langle Tx, x\rangle|^2 + \|Tx\|^4}.$$

By taking the supremum over $x \in \mathcal{H}$ such that $\|x\| = 1$, we get

$$dw(S) \leq N\ dw(T).$$

(ii) The hypothesis implies that for all $x \in \mathcal{H}$ with $Sx \neq 0,\ x \neq 0$,

$$\frac{|\langle Sx, x\rangle|}{\|Sx\|\ \|x\|} \leq M \frac{|\langle Tx, x\rangle|}{\|Sx\|\ \|x\|},$$

and so by taking the infimum over $x \in \mathcal{H}$ with $Sx \neq 0,\ x \neq 0$, we have

$$\begin{aligned} |\cos|S = \inf \frac{|\langle Sx, x\rangle|}{\|Sx\|\ \|x\|} &\leq M \inf \frac{|\langle Tx, x\rangle|}{\|Sx\|\ \|x\|} \\ &= M \sup \frac{\|Sx\|\ \|x\|}{|\langle Tx, x\rangle|} \\ &\leq \sqrt{8} M^2 \sup \frac{\|Tx\|\ \|x\|}{|\langle Tx, x\rangle|} \\ &= \sqrt{8} M^2 \inf \frac{|\langle Tx, x\rangle|}{\|Tx\|\ \|x\|} \\ &= \sqrt{8} M^2 |\cos|T. \end{aligned}$$

In (2.10), if for some $x \in \mathcal{H}$, we have $\langle Tx, x\rangle = 0$, then $\langle Sx, x\rangle = 0$, and so $|\cos|S = 0 = |\cos|T$. Therefore in the above inequalities, we assume that for all $x \in \mathcal{H}$, we have $\langle Sx, x\rangle \neq 0$.

(iii) The relation (2.10) follows part (iii). □

Let M be a closed subspace of $\mathcal{H}$. If there exists a closed subspace N of $\mathcal{H}$ with $\mathcal{H} = M \oplus N$, then M is called complemented.
By Proposition 2.2, $S \prec_s T$ follows $S \prec_B T$, and so the following three proposition hold by [1, Theorem 13, Proposition 6, Proposition 5].

Proposition 2.12. *Let $S, T \in B(\mathcal{H})$ and $S \prec_s T$. If $\overline{R(T)}$ is complemented, then there exists $V \in B(\mathcal{H})$ such that $S = VT$.*

Proof. By Proposition 2.2 and [1, Theorem 13], the assertion follows. □

If $S, T \in B(\mathcal{H})$ and $S \prec_s T$, then S inherits some properties of T. Proposition 2.2 and [1, Proposition 6] follow the next proposition.

Proposition 2.13. *Let $S, T \in B(\mathcal{H})$ and $S \prec_s T$. Then the following statements are true.*

(i) *If T is a compact operator, then S is so.*
(ii) *If T is a weakly compact operator, then S is so.*
(iii) *If T is a strictly singular operator, then S is so.*

For $T \in B(\mathcal{H})$, $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ is the spectral radius of T.

Proposition 2.14. *Let $S, T \in B(\mathcal{H})$, and $S \prec_s T$, i.e., there exists $M > 0$ such that for all $x \in \mathcal{H}$,*

$$|\langle Sx, x\rangle| \leq M|\langle Tx, x\rangle|.$$

Then the following statements hold.

(i) *If $N(T) = N(S)$ and $R(S)$ is closed, then $R(T)$ is closed.*
(ii) *If $TS = ST$, then for all $n \in \mathbb{N}$, $S^n \prec_B T^n$ and $r(S) \leq \sqrt{8}M\ r(T)$ and so if T is quasinilpotent, then S is so.*

Proof. By Proposition 2.2, there exists $M > 0$ such that for all $x \in \mathcal{H}$,

$$\|Sx\| \leq \sqrt{8}M\|Tx\|.$$

Now [1, Proposition 5] implies (i) and (ii). □

3. Joint strong majorization

This section deals with the extend of strong majorization for the d-tuples of operators in $B(\mathcal{H})^d$, as follows.

Definition 3.1. Let $S = (S_1, \ldots, S_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ be two d-tuples of operators. We say that T joint strong majorizes S and denoted by $S \prec_{js} T$, if there exists $M > 0$ such that for all $1 \leq i \leq d$ and all $x \in \mathcal{H}$,

$$|\langle S_i x, x\rangle| \leq M|\langle T_i x, x\rangle|.$$

That is $S \prec_{js} T$ if and only if $S_i \prec_s T_i$ for all $1 \leq i \leq d$.

Clearly, the above inequality follows that

$$\left(\sum_{i=1}^{d} |\langle S_i x, x\rangle|^2\right)^{\frac{1}{2}} \leq M \left(\sum_{i=1}^{d} |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}}. \tag{3.1}$$

Let $S^* = (S_1^*, \ldots, S_d^*) \in B(\mathcal{H})^d$ be the adjoint operator of a d-tuples $S = (S_1, \ldots, S_d)$ in $B(\mathcal{H})^d$. Clearly, $S \prec_{js} T$ if and only if $S^* \prec_{js} T^*$. We say that S is self-adjoint if $S^* = S$.
In 1981, M. Chō et al. introduced the joint operator norm and the joint numerical radius for a d-tuples $T = (T_1, \ldots, T_d)$ of operators defined on $\mathcal{H}$, respectively as follows [4],

$$\|T\| := \sup\left\{\left(\sum_{i=1}^{d} \|T_i x\|^2\right)^{\frac{1}{2}} : \ x \in \mathcal{H}, \ \|x\| = 1\right\},$$

and

$$w(T) = \sup \left\{ \left(\sum_{i=1}^{d} |\langle T_i x, x \rangle|^2 \right)^{\frac{1}{2}} : x \in \mathcal{H}, \ \|x\| = 1 \right\}.$$

For a d-tuples $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$, the Davis-Wielandt radius and the Crawford number of T, respectively defined by

$$dw(T) = \sup \left\{ \sqrt{\sum_{i=1}^{d} |\langle T_i x, x \rangle|^2 + \left(\sum_{i=1}^{d} \|T_i x\|^2 \right)^2} : x \in \mathcal{H}, \ \|x\| = 1 \right\},$$

and

$$c(T) = \inf \left\{ \left(\sum_{i=1}^{d} |\langle T_i x, x \rangle|^2 \right)^{\frac{1}{2}} : x \in \mathcal{H}, \ \|x\| = 1 \right\}.$$

The total cosine of T is defined by

$$|\cos|T = \inf \left\{ \frac{\left(\sum_{i=1}^{d} |\langle T_i x, x \rangle|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{d} \|T_i x\|^2 \right)^{\frac{1}{2}} \|x\|} : x \in \mathcal{H}, \ x \neq 0, \ \sum_{i=1}^{d} \|T_i x\|^2 \neq 0 \right\}.$$

Proposition 2.2, Theorem 2.11 and (3.1) follow the next theorem for two d-tuples of operators.

Theorem 3.2. *Let $S = (S_1, \ldots, S_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ be two d-tuples of operators and $S \prec_{js} T$, i.e., there exists $M > 0$ such that for all $x \in \mathcal{H}$, and $1 \leq i \leq d$ we have*

$$|\langle S_i x, x \rangle| \leq M |\langle T_i x, x \rangle|.$$

Then the following statements hold.

(i) *For all $x \in \mathcal{H}$,*

$$\sqrt{\sum_{i=1}^{d} |\langle S_i x, x \rangle|^2 + \left(\sum_{i=1}^{d} \|S_i x\|^2 \right)^2} \leq N \sqrt{\sum_{i=1}^{d} |\langle T_i x, x \rangle|^2 + \left(\sum_{i=1}^{d} \|T_i x\|^2 \right)^2}$$

and so $dw(S) \leq N\, dw(T)$, for some $N > 0$.

(ii) $|\cos|S \leq \sqrt{8} M^2 |\cos|T$.

(iii) $c(S) \leq M\, c(T)$.

(iv) $w(S) \leq M w(T)$.

(v) $\|S\| \leq \sqrt{8} M \|T\|$.

Let $S = (S_1, \ldots, S_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ be two d-tuples of operators. We consider ST by $ST = (S_1 T_1, \ldots, S_d T_d)$.

Theorem 3.3. *Let* $S = (S_1, \ldots, S_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ *be two d-tuples of operators such that* $S \prec_{js} T$ *and* $\bigcap_{i=1}^{d} \overline{R(T_i)} \neq \{0\}$. *Then there exists a d-tuples* V *in* $B\left(\bigcap_{i=1}^{d} \overline{R(T_i)}, \mathcal{H}\right)^d$ *such that* $S = VT$.

Proof. Since $S \prec_{js} T$ implies that $S_i \prec_s T_i$, for all $1 \leq i \leq d$, so by Proposition 2.6, there are $V_i \in B(\overline{R(T_i)}, \mathcal{H})$ such that $S_i = V_i T_i$. Thus $S = VT$ and $V = (V_1, \ldots, V_d) \in B\left(\bigcap_{i=1}^{d} \overline{R(T_i)}, \mathcal{H}\right)^d$. □

Theorems 2.7 and 2.8 are satisfied for the d-tuples of operators as follows.

Proposition 3.4. *Let* $\alpha, \beta \in \mathbb{C} \backslash \{0\}$ *and* $S,\ R,\ T \in B(\mathcal{H})^d$. *If* $S \prec_{js} T$, $R \prec_{js} T$, *then* $\alpha S + \beta R \prec_{js} T$.

Proof. Let

$$S = (S_1, \ldots, S_d), R = (R_1, \ldots, R_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d.$$

As $S \prec_{js} T$ and $R \prec_{js} T$ conclude that $S_i \prec_s T_i$ and $R_i \prec_s T_i$, for all $1 \leq i \leq d$, so by Theorem 2.7, we have $\alpha S_i + \beta R_i \prec_s T_i$, for all $1 \leq i \leq d$. Therefore $\alpha S + \beta R \prec_{js} T$. □

Theorem 3.5. *Let* $S, R, T \in B(\mathcal{H})^d$, *and* $S \prec_{js} T$. *Then*

(i) $T^*S \prec_{js} T^*T$ *and* $S^*T \prec_{js} T^*T$,
(ii) $S^*S \prec_{js} T^*T$,
(iii) $R^*SR \prec_{js} R^*TR$,
(iv) $T^*S \pm S^*T \prec_{js} T^*T$.

Proof. Since $S \prec_{js} T$ implies that $S_i \prec_s T_i$, for all $1 \leq i \leq d$, the proof follows by Theorem 2.8 and Proposition 3.4. □

Theorem 3.6. *Let* $S = (S_1, \ldots, S_d), T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ *be two self-adjoint d-tuples of operators and* $S \prec_{js} T$. *Then* $S^n \prec_{js} T^n$, *where* $n = 2^m$, *for all* $m \in \mathbb{N}$.

Proof. It follows by Theorem 2.10. □

4. Conclusion

We define a preordering in $B(\mathcal{H})$ and call it strong majorization which is stronger than majorization considered by Barnes. Thus all results that Barnes proved, are satisfied for strong majorization. In Example 2.5, we show that $S \prec_B T$ but $S \not\prec_s T$. One can find some conditions on S, T that Barnes's majorization implies strong majorization, also find the properties of strong majorization that aren't inherited from Barnes's majorization.

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NEW MAJORIZATION FOR BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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احاطه‌سازی جدید برای عملگرهای خطی کراندار در فضاهای هیلبرت

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در این مقاله می‌خواهیم پیش ترتیبی در $B(\mathcal{H})$ فضای باناخ تمام عملگرهای خطی و کراندار تعریف شده بر فضای هیلبرت مختلط $\mathcal{H}$ معرفی و بررسی کنیم. آن‌را احاطه‌سازی قوی می‌نامیم و برای $S,T\in B(\mathcal{H})$ به‌صورت $S\prec_s T$ نمایش می‌دهیم. احاطه‌سازی قوی، احاطه‌سازی بارنز را نتیجه می‌دهد، ولی عکس آن برقرار نیست. هرگاه $S\prec_s T$ آن‌گاه برخی از ویژگی‌های T به عملگر S به ارث می‌رسد. احاطه‌سازی قوی را برای d تایی از عملگرها در فضای $B(\mathcal{H})^d$ توسیع می‌دهیم و آن‌را احاطه‌سازی قوی توام می‌نامیم و برای $S,T\in B(\mathcal{H})^d$ به‌صورت $S\prec_{js} T$ نمایش می‌دهیم. نشان می‌دهیم که برخی از ویژگی‌های احاطه‌سازی قوی برای احاطه‌سازی قوی توام برقرار است.

کلمات کلیدی: احاطه‌سازی قوی، فضای هیلبرت، عملگر مثبت.