Journal of Algebraic Systems
Vol. 11, No. 2, (2024), pp 13-32

ISOTONIC CLOSURE FUNCTIONS ON A LOCALE

T. HAGHDADI* AND A. A. ESTAJI

Abstract. In this paper, we introduce and study isotonic closure functions on a locale. These are pairs of the form $(L, \underline{\mathrm{cl}}_L)$, where L is a locale and $\underline{\mathrm{cl}}_L : \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ is an isotonic closure function on the sublocales of L. Moreover, we introduce generalized $\underline{\mathrm{cl}}_L$-closed sublocales in isotonic closure locales and discuss some of their properties. Also, we introduce and study the category **ICF** whose objects and morphisms are isotonic closure functions $(L, \underline{\mathrm{cl}}_L)$ and localic maps, respectively.

1. Introduction and Preliminaries

Hausdorff studied closed spaces and isotonic spaces in [8]. Later on, Day [2], Hammer [7, 6] and Habil [4, 5] studied some properties of isotonic spaces. In 1970, Levine [9] initiated the study of the so-called g-closed sets.

Recall that a subset A of a topological space (X, τ) is *g-closed* if the closure of A is included in every open superset of A. Since g-closed sets are natural generalizations of closed sets, they have been widely studied by topologists in recent years.

Let X be a set, $P(X)$ denote its power set, and $\mathrm{cl}\colon P(X) \to P(X)$ be an arbitrary set-valued function, called a *closure function*. Then $\mathrm{cl}(A)$, $A \subseteq X$, is called the *closure* of A, and the pair (X, cl) is called a *generalized closure space*.

DOI: 10.22044/JAS.2022.12101.1627.
MSC(2010): Primary: 06D22; Secondary: 18F99.
Keywords: Isotonic closure function; Locale; Neighborhood function; Category.
Received: 12 July 2022, Accepted: 26 September 2022.
*Corresponding author.

Consider the following axioms of the closure function in which A, B, $A_\lambda \in P(X)$.

(K0) $\mathrm{cl}(\emptyset) = \emptyset$.
(K1) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
(K2) $A \subseteq \mathrm{cl}(A)$.
(K3) $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
(K4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(K5) $\bigcup_{\lambda\in\Lambda} \mathrm{cl}(A_\lambda) = \mathrm{cl}(\bigcup_{\lambda\in\Lambda} A_\lambda)$.

The dual of a given closure function cl is the *interior function* $\mathrm{int}\colon P(X) \to P(X)$ defined by

$$\mathrm{int}(A)\colon = X \setminus \mathrm{cl}(X \setminus A).$$

Given the interior function $\mathrm{int}\colon P(X) \to P(X)$, the closure function can be recovered via

$$\mathrm{cl}(A)\colon = X \setminus (\mathrm{int}(X \setminus A)) \quad \text{for all} \quad A \in P(X).$$

A set $A \in P(X)$ is *closed* in the generalized closure space (X, cl) if $\mathrm{cl}(A) = A$. It is *open* if its complement $X \setminus A$ is closed, or equivalently, $A = \mathrm{int}(A)$ (see [2]).

In the pointfree (localic) approach to topology, topological spaces are replaced by locales, seen as generalized spaces in which points are not explicitly mentioned. Formally, a *locale* L is defined as a special complete lattice (where we denote *top* (respectively, *bottom*) by 1 (respectively, 0)), usually called a *frame*, in which finite meets distribute over arbitrary joins, that is,

$$a \wedge \bigvee S = \bigvee \{a \wedge s\colon s \in S\}$$

for all $a \in L$ and $S \subseteq L$. A *sublocale* of a locale L is a subset $S \subseteq L$, closed under arbitrary meets, such that $\forall x \in L, \forall s \in S(x \to s \in S)$. Among the important examples of sublocales are, for each $a \in L$, the closed sublocales $\mathfrak{c}(a) = \uparrow a = \{b \in L\colon a \le b\}$, the open sublocales $\mathfrak{o}(a) = \{a \to b\colon b \in L\}$. Morover, for every $a \in L$,

$$\mathfrak{b}(a) = \{b \to a\colon b \in L\}$$

is the smallest sublocale containing a. Throughout the paper L and M stand for a locales, unless otherwise noted.

The lattice of all sublocales of L is denoted by $\mathcal{S}\ell(L)$. In this lattice, the meet is the intersection. The join of any collection $\{S_i\colon i \in I\}$ of $\mathcal{S}\ell(L)$ is given by

$$\bigvee_i S_i = \left\{ \bigwedge M\colon M \in \mathcal{S}\ell(L) \text{and } M \subseteq \bigcup_i S_i \right\}.$$

The lattice $\mathcal{S}\ell(L)$, partially ordered by inclusion, is a *coframe*, in the sense that for any $S \in \mathcal{S}\ell(L)$ and any family $\{T_\alpha\}$ of sublocales, the following distributive law holds.

$$S \vee \bigwedge_\alpha T_\alpha = \bigwedge_\alpha (S \vee T_\alpha).$$

The smallest sublocale of L is $\mathsf{O} = \{1\}$, which is known as the *void sublocale*. The largest is, of course, L. We say that sublocales S and T are *disjoint* if $S \cap T = \mathsf{O}$. A sublocale of L is *complemented* if it has a (Boolean) complement in the lattice $\mathcal{S}\ell(L)$. If A is a complemented sublocale, we denote its complement by A^c .

Definition 1.1. The *supplement sublocale* A of L, denoted by $A^\#$ or $L \setminus A$, is

$$A^\# \colon = \bigcap \Big\{ B \in \mathcal{S}\ell(L) \colon B \vee A = L \Big\}.$$

Note that, every supplement sublocale is the dual of pseudocomplementary. It is easy to see that $A^{\#\#} \subseteq A$ and $A \vee A^\# = L$. Also, if A is a complemented sublocale of L, then $A \cap A^\# = \mathsf{O}$ and so, $A^\#$ is the complement of A in the coframe $\mathcal{S}\ell(L)$.

A map $f \colon L \longrightarrow M$ between locales is said to be a *localic map* whenever for every $a \in L$, $b \in M$ and $S \subseteq L$,

(L1) $f(\bigwedge S) = \bigwedge f[S]$ (in particular, $f(1) = 1$),
(L2) $f(f_*(b) \to a) = b \to f(a)$, and
(L3) $f(a) = 1 \Rightarrow a = 1$,

where $f_* \colon M \longrightarrow L$ denotes the left adjoint of f provided by $(L1)$.

A localic map $f \colon L \longrightarrow M$ gives rise to two mappings, namely, $f[-] \colon \mathcal{S}\ell(L) \longrightarrow \mathcal{S}\ell(M)$ and $f_{-1}[-] \colon \mathcal{S}\ell(M) \longrightarrow \mathcal{S}\ell(L)$ defined by

$$f[S] = \big\{ f(x) \colon x \in S \big\}$$

and

$$f_{-1}[T] = \bigvee \big\{ A \in \mathcal{S}\ell(L) | A \subseteq f^{-1}[T] \big\}.$$

Note that $f_{-1}[-]$ is the right adjoint of $f[-]$ (that is, $f[S] \subseteq T$ if and only if $S \subseteq f_{-1}[T]$).

Definition 1.2. Suppose that L is a lattice. A $\wedge$-closed subset $S \subseteq L$ is called almost saturated, whenever if $x, y \in L$, $s \in S$ and $x \wedge y = s$, then there exist $s_1, s_2 \in S$ such that $x \leq s_1$, $y \leq s_2$ and $s = s_1 \wedge s_2$.

Proposition 1.3. *Assume that L is a locale. $B \subseteq L$ is a sublocale if and only if B is closed under arbitrary meets and also almost saturated.*

Proof. $\Rightarrow$) By the hypothesis, there is a nucleus function $j\colon L \to L$ such that $j(L) = B$. It is obvious that B is closed under arbirary meets. Now, Let $x \wedge y = b$ in which $x, y \in L$ and $b \in B$. Take $b_1 = j(x \vee b)$ and $b_2 = j(y \vee b)$. Clearly, $b_1, b_2 \in j(L) = B$. On other hand, we can write:

$$x \leq j(x) \leq j(x \vee b) = b_1 \ , \ y \leq j(y) \leq j(y \vee b) = b_2.$$

In addition,

$$b = b \vee (x \wedge y) = (x \vee b) \wedge (y \vee b) \Rightarrow b = j(b) = j(x \vee b) \wedge j(y \vee b) = b_1 \wedge b_2.$$

$\Leftarrow$) Define $j\colon L \to L$ with $j(x) = \bigwedge \uparrow_B x$ in which

$$\uparrow_B x = \{b \in B : x \leq b\}.$$

It is easily seen that j is a closure operator. Only, it suffices to show that j is a $\wedge$-homomorphism. Since B is almost saturated, it is easy to see that the set $\uparrow_B (x \wedge y) \subseteq \{b_1 \wedge b_2 : x \leq b_1, y \leq b_2\}$. Hence we can write:

$$\begin{aligned} j(x) \wedge j(y) &= (\bigwedge \uparrow_B x) \wedge (\bigwedge \uparrow_B y) \\ &= \bigwedge \{b_1 \wedge b_2 : x \leq b_1, y \leq b_2\} \\ &\leq \bigwedge \uparrow_B (x \wedge y) \\ &= j(x \wedge y). \end{aligned}$$

On the other hand, it is clear that $j(x \wedge y) \leq j(x) \wedge j(y)$. Therefore, the equivality holds. □

This paper is organized as follows. In Section 2, we introduce and study isotonic closure functions. These are pairs of the form $(L, \underline{\mathrm{cl}}_L)$, where L is a locale and $\underline{\mathrm{cl}}_L\colon \mathcal{S\ell}(L) \to \mathcal{S\ell}(L)$ is an isotonic closure function on the sublocales of L. We describe connections between closure functions and interior functions in a locale L. In Section 3, we introduce generalized $\underline{\mathrm{cl}}_L$- closed and generalized $\underline{\mathrm{cl}}_L$-open sublocales in an isotonic closure function and study their fundamental properties. In Section 4, we introduce the category of isotonic closure functions over a locale L and discuss some of its properties.

2. Isotonic closure functions on a locale

Let L be a locale, $\mathcal{S\ell}(L)$ be the set of all sublocales of L, and $\underline{\mathrm{cl}}_L\colon \mathcal{S\ell}(L) \to \mathcal{S\ell}(L)$ be an arbitrary set-valued function, called a *closure function*. We note that this concept is different from the concept of clouser of a sublocale. Moreover, almost all the contents of this section can be generalized for the locales and frames. Consider the following axioms of the closure function for arbitrary sublocales A and B.

(K0) $\underline{\mathrm{cl}}_L(\mathsf{O}) = \mathsf{O}$.
(K1) $A \leqslant B$ implies $\underline{\mathrm{cl}}_L(A) \leqslant \underline{\mathrm{cl}}_L(B)$.
(K2) $A \leqslant \underline{\mathrm{cl}}_L(A)$.
(K3) $\underline{\mathrm{cl}}_L(A \vee B) \leqslant \underline{\mathrm{cl}}_L(A) \vee \underline{\mathrm{cl}}_L(B)$.
(K4) $\underline{\mathrm{cl}}_L(\underline{\mathrm{cl}}_L(A)) = \underline{\mathrm{cl}}_L(A)$.

The following proposition is now an immediate consequence.

Proposition 2.1. *The following conditions are equivalent for an arbitrary closure function* $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$.

(1) $A \leqslant B \leqslant L$ *implies* $\underline{\mathrm{cl}}_L(A) \leqslant \underline{\mathrm{cl}}_L(B)$.
(2) $\underline{\mathrm{cl}}_L(A) \vee \underline{\mathrm{cl}}_L(B) \leqslant \underline{\mathrm{cl}}_L(A \vee B)$.
(3) $\underline{\mathrm{cl}}_L(A \wedge B) \leqslant \underline{\mathrm{cl}}_L(A) \wedge \underline{\mathrm{cl}}_L(B)$.

The dual of a closure function $\underline{\mathrm{cl}}_L$ is the *interior function* $\underline{\mathrm{int}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ defined by

$$\underline{\mathrm{int}}_L(A) = \big(\underline{\mathrm{cl}}_L(A^\#)\big)^\#.$$

Proposition 2.2. *Let* $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ *be a closure function that satisfies the axioms* $(K0)$, $(K1)$, $(K2)$ *and* $(K4)$. *Then, the following statements are true.*

(1) $\underline{\mathrm{int}}_L(L) = L$.
(2) $A \subseteq B$ *implies* $\underline{\mathrm{int}}_L(A) \subseteq \underline{\mathrm{int}}_L(B)$.
(3) $\underline{\mathrm{int}}_L(A) \subseteq A$.
(4) $\underline{\mathrm{int}}_L(\underline{\mathrm{int}}_L(A)) = \underline{\mathrm{int}}_L(A)$.

Proof. Let A and B be sublocales of L.

(1) By $(K0)$,

$$\underline{\mathrm{int}}_L(L) = \big(\underline{\mathrm{cl}}_L(L^\#)\big)^\# = \big(\underline{\mathrm{cl}}_L(\mathsf{O})\big)^\# = (\mathsf{O})^\# = L.$$

(2) If $A \subseteq B$, then $B^\# \subseteq A^\#$. By $(K1)$, $\underline{\mathrm{cl}}_L(B^\#) \subseteq \underline{\mathrm{cl}}_L(A^\#)$ and so, $\big(\underline{\mathrm{cl}}_L(A^\#)\big)^\# \subseteq \big(\underline{\mathrm{cl}}_L(B^\#)\big)^\#$. Therefore, by the definition of interior, $\underline{\mathrm{int}}_L(A) \subseteq \underline{\mathrm{int}}_L(B)$.

(3) By $(K2)$, $A^\# \subseteq \underline{\mathrm{cl}}_L(A^\#)$ and so, $\big(\underline{\mathrm{cl}}_L(A^\#)\big)^\# \subseteq A^{\#\#} \subseteq A$, which means that $\underline{\mathrm{int}}_L(A) \subseteq A$.

(4) By (3), $\underline{\mathrm{int}}_L\big(\underline{\mathrm{int}}_L(A)\big) \subseteq \underline{\mathrm{int}}_L(A)$. To complete the proof, note that by $(k4)$ and $(K1)$ we can write

$$\begin{aligned}\big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#\#} \subseteq \underline{\mathrm{cl}}_L(A^{\#}) &\Longrightarrow \underline{\mathrm{cl}}_L\Big(\big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#\#}\Big) \subseteq \underline{\mathrm{cl}}_L\big(\underline{\mathrm{cl}}_L(A^{\#})\big)\\ &\Longrightarrow \underline{\mathrm{cl}}_L\Big(\big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#\#}\Big) \subseteq \underline{\mathrm{cl}}_L(A^{\#})\\ &\Longrightarrow \big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#} \subseteq \Big(\underline{\mathrm{cl}}_L\Big(\big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#\#}\Big)\Big)^{\#}\\ &\Longrightarrow \underline{\mathrm{int}}_L(A) \subseteq \Big(\underline{\mathrm{cl}}_L\Big(\big(\underline{\mathrm{int}}_L(A)\big)^{\#}\Big)\Big)^{\#}\\ &\Longrightarrow \underline{\mathrm{int}}_L(A) \subseteq \underline{\mathrm{int}}_L\big(\underline{\mathrm{int}}_L(A)\big).\end{aligned}$$

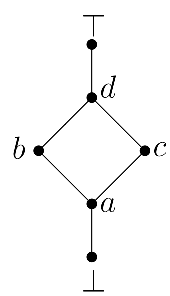
Thus, $\underline{\mathrm{int}}_L(A) = \underline{\mathrm{int}}_L\big(\underline{\mathrm{int}}_L(A)\big)$.

□

A sublocale $A \in \mathcal{S}\ell(L)$ is $\underline{\mathrm{cl}}$-*closed* if $\underline{\mathrm{cl}}_L(A) = A$, and it is $\underline{\mathrm{cl}}$-*open* if $\underline{\mathrm{int}}_L(A) = A$.

Definition 2.3. An *isotonic closure function* is a pair $(L, \underline{\mathrm{cl}}_L)$, where L is a locale and $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ is a closure function that satisfies the axioms $(K0)$ and $(K1)$.

Example 2.4. Let $L = \big\{\bot, a, b, c, d, \top\big\}$ be a locale with the following Hass diagram.



By Proposition 1.3, we have

$$\begin{aligned}\mathcal{S}\ell(L) = \Big\{&\{\top\}, \{b,\top\}, \{c,\top\}, \{d,\top\}, \{\bot,\top\}\{\bot,b,\top\}, \{\bot,c,\top\},\\ &\{\bot,d,\top\}, \{b,d,\top\}, \{c,d,\top\}, \{\bot,b,d,\top\}, \{\bot,c,d,\top\},\\ &\{a,b,c,\top\}, \{a,b,c,d,\top\}, \{\bot,a,b,c,\top\}, L\Big\}.\end{aligned}$$

(1) We define a set-valued function $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ by $\underline{\mathrm{cl}}_L(\{\bot, b, \top\}) = \{a, b, c, \top\}$ and $\underline{\mathrm{cl}}_L(A) = A$ for every sublocale $A \neq \{\bot, b, \top\}$. Then, $\underline{\mathrm{cl}}_L$ is a closure function which satisfies $(K0)$ but not $(K1)$. Hence, $(L, \underline{\mathrm{cl}}_L)$ is not an isotonic closure function.

(2) Define a closure function $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ by

$$\begin{aligned}\underline{\mathrm{cl}}_L(\{c,\top\}) &= \underline{\mathrm{cl}}_L(\{d,\top\}) = \underline{\mathrm{cl}}_L(\{\bot,c,\top\})\\ &= \underline{\mathrm{cl}}_L(\{c,d,\top) = \underline{\mathrm{cl}}_L(\{\bot,d,\top\})\\ &= \{\bot,\top\},\end{aligned}$$

$\underline{\mathrm{cl}}_L(\{\bot,c,d,\top\}) = \{\bot,b,d,\top\}$ and for other sublocales A, $\underline{\mathrm{cl}}_L(A) = A$. Then, $(L,\underline{\mathrm{cl}}_L)$ is an isotonic closure function which satisfies $(K4)$. Note that $(L,\underline{\mathrm{cl}}_L)$ does not satisfy $(K3)$. To see this, consider the sublocales $A = \{b,\top\}$ and $B = \{c,\top\}$. Then $A \vee B = \{a,b,c,\top\}$ and so,

$$\underline{\mathrm{cl}}_L(A \vee B) = \underline{\mathrm{cl}}_L\big(\{a,b,c,\top\}\big) = \{a,b,c,\top\}.$$

On the other hand,

$$\underline{\mathrm{cl}}_L(A) \vee \underline{\mathrm{cl}}_L(B) = \{b,\top\} \vee \{\bot,\top\} = \{\bot,b,\top\}.$$

Therefore, $\underline{\mathrm{cl}}_L(A\vee B) \not\subseteq \underline{\mathrm{cl}}_L(A)\vee\underline{\mathrm{cl}}_L(B)$. Also, $A = \{\bot,c,d,\top\}$ implies $A \not\subseteq \underline{\mathrm{cl}}_L(A)$, which means that $\underline{\mathrm{cl}}_L$ does not satisfy $(K2)$.

(3) Define a closure function $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ by

$$\underline{\mathrm{cl}}_L\big(\{\bot,b,\top\}\big) = \{\bot,a,b,c,\top\}$$

and $\underline{\mathrm{cl}}_L(\{\bot,b,d,\top\} = L$. Then, $\underline{\mathrm{cl}}_L$ satisfies the axioms $(K0)$, $(K1)$ and $(K2)$.

(4) Define a closure function $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ by $\underline{\mathrm{cl}}_L(\{\bot,b,d,\top\}) = L$ and $\underline{\mathrm{cl}}_L(A) = A$ for other sublocales A of L. Then, $\underline{\mathrm{cl}}_L$ satisfies the axioms $(K0)$, $(K1)$, $(K2)$ and $(K3)$.

Remark 2.5. Assume that a closure function $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ satisfies $(K2)$. Then, $\underline{\mathrm{cl}}_L(L) = L$ and $\underline{\mathrm{int}}_L(\mathsf{O}) = \mathsf{O}$.

Remark 2.6. Let L be a locale, $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be a closure function, and A be a sublocale of L.

(1) $L \setminus \underline{\mathrm{cl}}_L(A) \subseteq \underline{\mathrm{int}}_L(L \setminus A)$.
(2) If A is complemented, then $L \setminus \underline{\mathrm{cl}}_L(A) = \underline{\mathrm{int}}_L(L \setminus A)$, because

$$\underline{\mathrm{int}}_L(L \setminus A) = \underline{\mathrm{int}}_L(A^{\#}) = \big(\underline{\mathrm{cl}}_L(A^{\#\#})\big)^{\#} = \big(\underline{\mathrm{cl}}_L(A)\big)^{\#} = L \setminus \underline{\mathrm{cl}}_L(A).$$

(3) $L \setminus \underline{\mathrm{int}}_L(L \setminus A) \subseteq \underline{\mathrm{cl}}_L(A)$.
(4) If A and $\underline{\mathrm{cl}}_L(A)$ are complemented, then $L\setminus\underline{\mathrm{int}}_L(L\setminus A) = \underline{\mathrm{cl}}_L(A)$, because

$$L \setminus \underline{\mathrm{int}}_L(L \setminus A) = \big(\underline{\mathrm{int}}_L(A^{\#})\big)^{\#} = \big(\underline{\mathrm{cl}}_L(A^{\#\#})\big)^{\#\#} = \underline{\mathrm{cl}}_L(A).$$

The condition that A and $\underline{\mathrm{cl}}_L(A)$ are complementary is necessary. This is the content of the following example.

Example 2.7. (1) Let the locale L and $\underline{\mathrm{cl}}\colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be as in Example 2.4(2). Consider the sublocale $A = \{c, \top\}$. Then, it is clear that no complement exists for A. Also,

$$\underline{\mathrm{int}}_L\Big(\big(\{c, \top\}\big)^{\#}\Big) = \underline{\mathrm{int}}_L(L) = L.$$

On the other hand, $\Big(\underline{\mathrm{cl}}_L\big(\{c, \top\}\big)\Big)^{\#} = \{a, b, c, d, \top\}$. Therefore, $\underline{\mathrm{int}}_L\Big(\big(\{c, \top\}\big)^{\#}\Big) \neq \Big(\underline{\mathrm{cl}}_L\big(\{c, \top\}\big)\Big)^{\#}$.

(2) Let the locale L and $\underline{\mathrm{cl}}\colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be as in Example 2.4(1). Consider the sublocale $A = \{b, \top\}$. It is clear that no complements exist for the sublocales A and $\underline{\mathrm{cl}}_L(A)$. Moreover,

$$\underline{\mathrm{int}}_L\big(\{b, \top\}^{\#}\big) = \underline{\mathrm{int}}_L(L) = L$$

and so, $\Big(\underline{\mathrm{int}}_L\big(\{b, \top\}^{\#}\big)\Big)^{\#} = (L)^{\#} = \{\top\}$. On the other hand, $\underline{\mathrm{cl}}_L\big(\{b, \top\}\big) = \{b, \top\}$. Then, $\underline{\mathrm{cl}}_L\big(\{b, \top\}\big) \neq \Big(\underline{\mathrm{int}}_L\big(\{b, \top\}^{\#}\big)\Big)^{\#}$.

The proof of the following lemma is straightforward.

Lemma 2.8. *Let $(L, \underline{\mathrm{cl}}_L)$ be an isotonic closure function which satisfies $(K2)$. Then, for every sublocale A of L, the following statements are true.*

(1) $\big(\underline{\mathrm{cl}}_L(A)\big)^{\#} \subseteq \underline{\mathrm{cl}}_L(A^{\#})$.
(2) $\underline{\mathrm{int}}_L(A) \subseteq \underline{\mathrm{cl}}_L(A)$.

Lemma 2.9. *Let L be a locale whose all sublocales are complemented. Then, $(L, \underline{\mathrm{cl}}_L)$ is an isotonic closure function if and only if $\underline{\mathrm{int}}_L\colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ satisfies the following conditions.*

(1) $\underline{\mathrm{int}}_L(L) = L$.
(2) *For arbitrary sublocales A and B of L with $A \leqslant B$, $\underline{\mathrm{int}}_L(A) \leqslant \underline{\mathrm{int}}_L(B)$.*

Proof. $\Rightarrow$) Let $A \leqslant B$. Then, by $(K1)$, $\underline{\mathrm{cl}}_L(B^{\#}) \leqslant \underline{\mathrm{cl}}_L(A^{\#})$. Therefore $\big(\underline{\mathrm{cl}}_L(A^{\#})\big)^{\#} \leqslant \big(\underline{\mathrm{cl}}_L(B^{\#})\big)^{\#}$, which means that $\underline{\mathrm{int}}_L(A) \leqslant \underline{\mathrm{int}}_L(B)$. Also,

$$\underline{\mathrm{int}}_L(L) = \big(\underline{\mathrm{cl}}_L(L^{\#})\big)^{\#} = \big(\underline{\mathrm{cl}}_L(\mathsf{O})\big)^{\#} = (\mathsf{O})^{\#} = L.$$

$\Leftarrow$) If $\underline{\mathrm{int}}_L(L) = L$, then

$$\underline{\mathrm{cl}}_L(\mathsf{O}) = \big(\underline{\mathrm{int}}_L(\mathsf{O}^{\#})\big)^{\#} = \big(\underline{\mathrm{int}}_L(L)\big)^{\#} = \big(L\big)^{\#} = \mathsf{O}.$$

Now, let $A \leqslant B$. Then, by (2), $\big(\underline{\mathrm{int}}_L(A^{\#})\big)^{\#} \leqslant \big(\underline{\mathrm{int}}_L(B^{\#})\big)^{\#}$. Then by Remark 2.6, $\underline{\mathrm{cl}}_L(A) \leqslant \underline{\mathrm{cl}}_L(B)$. Therefore, $(L, \underline{\mathrm{cl}}_L)$ is an isotonic closure function. □

Definition 2.10. Let $(L, \underline{\mathrm{cl}}_L)$ be a closure function and M be a sublocale of L. Then $\underline{\mathrm{cl}}_M \colon \mathcal{S}\ell(M) \to \mathcal{S}\ell(M)$, defined by

$$A \mapsto M \cap \underline{\mathrm{cl}}_L(A),$$

is the *relativization* of $\underline{\mathrm{cl}}_L$ to M. The pair $(M, \underline{\mathrm{cl}}_M)$ is called a *sub-closure function* of $(L, \underline{\mathrm{cl}}_L)$.

It is easy to see that, if $(L, \underline{\mathrm{cl}}_L)$ is an isotonic closure function, then $(M, \underline{\mathrm{cl}}_M)$ is an isotonic closure function.

Definition 2.11. A property $\mathbb{B}$ of a closure function $(L, \underline{\mathrm{cl}}_L)$ is *hereditary* if every sub-closure function $(M, \underline{\mathrm{cl}}_M)$ of $(L, \underline{\mathrm{cl}}_L)$ also has the property $\mathbb{B}$.

Lemma 2.12. *The properties* $(K0)$*,* $(K1)$ *and* $(K2)$ *are hereditary in any closure function* $(L, \underline{\mathrm{cl}}_L)$*.*

Proof. This is straightforward. □

In the following example, we show that the axiom $(K4)$ is not hereditary.

Example 2.13. Let the locale L and $\underline{\mathrm{cl}} \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be as in Example 2.4(2). Consider the sublocale $M = \{\bot, c, d, \top\}$ of L. By Definition 2.10,

$$\begin{aligned}\underline{\mathrm{cl}}_M(\{c, \top\}) = \underline{\mathrm{cl}}_M(\{d, \top\}) &= \underline{\mathrm{cl}}_M(\{\bot, c, \top\}) \\ &= \underline{\mathrm{cl}}_M(\{c, d, \top\}) = \underline{\mathrm{cl}}_M(\{\bot, d, \top\}) \\ &= \{\bot, \top\},\end{aligned}$$

$\underline{\mathrm{cl}}_M(\{\top\}) = \{\top\}$ and $\underline{\mathrm{cl}}_M(\{\bot, c, d, \top\}) = \{\bot, d, \top\}$. Hence

$$\underline{\mathrm{cl}}_M\big(\underline{\mathrm{cl}}_M(\{\bot, c, d, \top\})\big) = \{\bot, \top\},$$

which implies that $\underline{\mathrm{cl}}_M\big(\underline{\mathrm{cl}}_M(\{\bot, c, d, \top\})\big) \neq \underline{\mathrm{cl}}_M(\{\bot, c, d, \top\})$.

Definition 2.14. Let L be a locale and $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be a closure function on L. Then, the *neighborhood function* $\mathcal{N} \colon L \to \mathcal{P}(\mathcal{S}\ell(L))$ and the *convergent function* $\mathcal{N}^* \colon L \to \mathcal{P}(\mathcal{S}\ell(L))$ are respectively defined as follows:

$$\mathcal{N}(x) = \big\{N \in \mathcal{S}\ell(L)\ ;\ x \in \underline{\mathrm{int}}_L(N)\big\}$$

and

$$\mathcal{N}^*(x) = \big\{N \in \mathcal{S}\ell(L)\ ;\ x \in \underline{\mathrm{cl}}_L(N)\big\}.$$

A sublocale B is a *neighborhood* of sublocale A, if $B \in \mathcal{N}(x)$ for all $x \in A$.

By the above definition, the following lemma is obvious.

Lemma 2.15. *For any isotonic closure function* $(L, \underline{\mathrm{cl}}_L)$, $B \in \mathcal{N}(A)$ *if and only if* $A \subseteq \underline{\mathrm{int}}_L B$.

Proposition 2.16. *Let* $(L, \underline{\mathrm{cl}}_L)$ *be an isotonic closure function. Then,* $\mathcal{N}(a) = \mathcal{N}(\mathfrak{b}(a))$ *for every* $a \in L$.

Proof. Let $N \in \mathcal{N}(a)$. Then $a \in \underline{\mathrm{int}}_L(N)$ and so $\mathcal{N}(\mathfrak{b}(a)) \subseteq \underline{\mathrm{int}}_L(N)$. Hence by Lemma 2.15, $N \in \mathcal{N}(\mathfrak{b}(a))$. Now, let $N \in \mathcal{N}(\mathfrak{b}(a))$. Then, $N \in \mathcal{N}(x \to a)$ for every $x \in L$. Put $x = 1$, so $N \in \mathcal{N}(1 \to a) = \mathcal{N}(a)$. Therefore, $\mathcal{N}(\mathfrak{b}(a)) = \mathcal{N}(a)$. □

Proposition 2.17. *Let* $(L, \underline{\mathrm{cl}})$ *be a closure function. If* A *and* $\underline{\mathrm{cl}}_L(A)$ *are complemented sublocales of* L *and* $A \in \mathcal{N}^*(x)$, *then* $A^{\#} \notin \mathcal{N}(x)$ *for every* $\top \neq x \in L$.

Proof. Let $\top \neq x \in L$ and A be a complemented sublocale of L. Then by Remark 2.6,

$$A \in \mathcal{N}^*(x) \Rightarrow x \notin \big(\underline{\mathrm{cl}}_L(A)\big)^{\#} \Rightarrow x \notin \mathrm{int}(A^{\#}) \Rightarrow A^{\#} \notin \mathcal{N}(x).$$

□

The condition that A is complemented is necessary. This is the content of the following example.

Example 2.18. Let the locale L and $\underline{\mathrm{cl}}_L \colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ be as in Example 2.4(2). Then,

$$\mathcal{N}(b) = \Big\{\mathfrak{o}(c), \mathfrak{c}(b), \mathfrak{o}(d), \mathfrak{c}(c), \mathfrak{b}(a), \mathfrak{c}(a), < \bot, d >, < \bot, b, d >, \\ < \bot, c, d >, L\Big\}.$$

and

$$\mathcal{N}^*(b) = \Big\{\mathfrak{b}(b), \mathfrak{o}(c), \mathfrak{c}(b), \mathfrak{b}(a), \mathfrak{c}(a), \mathfrak{o}(d), < \bot, b, d >, L\Big\}.$$

Now, let $A =< \bot, b, d >$. Therefore, A is not a complemented sublocale. It is easy to see that $A \in \mathcal{N}^*(b)$ and $A^{\#} \in \mathcal{N}(b)$.

In the following example we show that the converse of the above proposition is not necessarily true.

Example 2.19. Let $L = \big\{\bot, a \wedge b = x, a, b, \top = a \vee b\big\}$. It is obvious that by Proposition 1.3,

$$\mathcal{S}\ell(L) = \Big\{\{\top\}, \{\bot, \top\}, \{a, \top\}, \{b, \top\}, \{\bot, a, \top\}, \{\bot, b, \top\}, \\ \{x, a, b, \top\}, L\Big\}.$$

It is easily seen that $\mathcal{S}\ell(L)$ is a Boolean algebra. Supposing the function $\underline{\mathrm{cl}}_L$ is identity, clearly, the function $\underline{\mathrm{int}}_L$ is also identity. Now, if we take $A = \{a, \top\}$, then $x \notin A = \underline{\mathrm{cl}}_L(A)$ and so $A \notin \mathcal{N}^*(x)$. In addition, $x \notin \{\bot, b, \top\} = A^\# = \underline{\mathrm{int}}_L(A^\#)$ and consequently $A^\# \in \mathcal{N}(x)$.

Lemma 2.20. *Let L be a locale whose all sublocales are complemented. Then, $(L, \underline{\mathrm{cl}}_L)$ is an isotonic closure function if and only if the neighborhood function $\mathcal{N}\colon L \to \mathcal{P}(\mathcal{S}\ell(L))$ satisfies the following conditions.*

1. *For every $a \in A$, $L \in \mathcal{N}(a)$.*
2. *$A \in \mathcal{N}(a)$ and $A \leqslant B$ imply $B \in \mathcal{N}(a)$, for every $a \in L$.*

Proof. $\Rightarrow$) Let $a \in A$. By Lemma 2.9, $\underline{\mathrm{int}}_L(L) = L$ and so, $a \in \underline{\mathrm{int}}_L(L)$. This means that $L \in \mathcal{N}(a)$. Let $A \in \mathcal{N}(a)$ and $A \leqslant B$. Since $A \leqslant B$, by Lemma 2.9, $\underline{\mathrm{int}}_L(A) \leqslant \underline{\mathrm{int}}_L(B)$. So, $a \in \underline{\mathrm{int}}_L(B)$.

$\Leftarrow$) It is clear that $\underline{\mathrm{int}}_L(L) \leqslant L$. Let $a \in L$. By (1), $L \in \mathcal{N}(a)$, which means that $a \in \underline{\mathrm{int}}_L(L)$. Hence, $\underline{\mathrm{int}}_L(L) = L$. Now, let $A \leqslant B$ and $a \in \underline{\mathrm{int}}_L(A)$. Then, $A \in \mathcal{N}(a)$ and so by (2), $B \in \mathcal{N}(a)$. Hence, $a \in \underline{\mathrm{int}}_L(B)$ and consequently, $\underline{\mathrm{int}}_L(A) \subseteq \underline{\mathrm{int}}_L(B)$. Therefore, by Lemma 2.9, $(L, \underline{\mathrm{cl}}_L)$ is isotonic. □

Proposition 2.21. *Let L be a locale whose all sublocales are complemented. Let $\underline{\mathrm{cl}_1}_L$ and $\underline{\mathrm{cl}_2}_L$ be closure functions on L. Then, the following conditions are equivalent.*

1. *$\underline{\mathrm{cl}_1}_L(A) \subseteq \underline{\mathrm{cl}_2}_L(A)$ for all $A \in \mathcal{S}\ell(L)$.*
2. *$\underline{\mathrm{int}_2}_L(A) \subseteq \underline{\mathrm{int}_1}_L(A)$ for all $A \in \mathcal{S}\ell(L)$.*
3. *$\mathcal{N}_2(x) \subseteq \mathcal{N}_1(x)$ for all $x \in L$.*
4. *$\mathcal{N}_1^*(x) \subseteq \mathcal{N}_2^*(x)$ for all $x \in L$.*

Proof. The proof is straightforward. □

Definition 2.22. Let $(L, \underline{\mathrm{cl}}_L)$ and $(M, \underline{\mathrm{cl}}_M)$ be isotonic closure functions. A localic map $f\colon L \longrightarrow M$ is

1. *continuous* if $\underline{\mathrm{cl}}_L(f_{-1}[B]) \leqslant f_{-1}[\underline{\mathrm{cl}}_M(B)]$ for every $B \in \mathcal{S}\ell(L)$;
2. *closure-preserving* if $f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M\big(f(A)\big)$ for any $A \in \mathcal{S}\ell(L)$.

Proposition 2.23. *Let $(L, \underline{\mathrm{cl}}_L)$ and $(M, \underline{\mathrm{cl}}_M)$ be isotonic closure functions, and $f\colon L \to M$ a localic map. Then, the following statements are equivalent.*

1. *$f\colon L \to M$ is continuous.*
2. *$f\colon L \to M$ is closure-preserving.*
3. *If $f(A) \leqslant B$, then $f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M(B)$ for all $A \in \mathcal{S}\ell(L)$ and $B \in \mathcal{S}\ell(M)$.*

Proof. 1 ⇒ 3) Suppose that $f\colon L \to M$ is a continuous localic map, $A \in \mathcal{S}\ell(L)$, $B \in \mathcal{S}\ell(M)$ and $f(A) \leqslant B$. Then $A \leqslant f_{-1}[B]$ and so, $\underline{\mathrm{cl}}_L(A) \leqslant \underline{\mathrm{cl}}_L(f_{-1}[B])$. Hence, by (1),

$$\underline{\mathrm{cl}}_L(A) \leqslant \underline{\mathrm{cl}}_L(f_{-1}[B]) \subseteq f_{-1}(\underline{\mathrm{cl}}_M(B)).$$

Now, since $f_{-1}[.]$ is the right adjoint of $f[.]$,

$$f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant f f_{-1}\big(\underline{\mathrm{cl}}_M([B])\big) \leqslant \underline{\mathrm{cl}}_M(B).$$

3 ⇒ 1) Let B be a sublocale of M and set $A = f_{-1}[B]$. Then $f(A) \leqslant B$ and so by (3), $f(\underline{\mathrm{cl}}_L(A)) \subseteq \underline{\mathrm{cl}}_M(B)$. Thus,

$$f\big(\underline{\mathrm{cl}}_L(f_{-1}[B])\big) = f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M(B)$$

and so, $\underline{\mathrm{cl}}_L(f_{-1}[B]) \leqslant f_{-1}\big(\underline{\mathrm{cl}}_M(B)\big)$. This means that f is continuous.

2 ⇒ 3) Let f be closure-preserving and $f(A) \leqslant B$. Since $f\colon L \to M$ is closure-preserving, $f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M\big(f(A)\big)$ and by $(K1)$,

$$\underline{\mathrm{cl}}_M\big(f(A)\big) \leqslant \underline{\mathrm{cl}}_M(B).$$

Hence, $f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M(B)$.

3 ⇒ 2) Let A be a sublocale of L and set $B = f(A)$. By (3),

$$f\big(\underline{\mathrm{cl}}_L(A)\big) \leqslant \underline{\mathrm{cl}}_M(B) = \underline{\mathrm{cl}}_M\big(f(A)\big),$$

which means that f is closure-preserving. □

Proposition 2.24. *Let $(L, \underline{\mathrm{cl}}_L)$ and $(M, \underline{\mathrm{cl}}_M)$ be two isotonic closure functions that satisfies in axiom $(K2)$ and $(k4)$. Then localic map $f\colon L \to M$ is continuous if and only if for every $\underline{\mathrm{cl}}$-closed sublocale B of M, $f_{-1}[B]$ is a $\underline{\mathrm{cl}}$-closed sublocale of L.*

Proof. ⇒) Let B be a $\underline{\mathrm{cl}}$-closed sublocale M. Since, $f\colon L \to M$ is continuous and B is $\underline{\mathrm{cl}}$-closed, we infer that

$$\underline{\mathrm{cl}}_L\big(f_{-1}(B)\big) \leqslant f_{-1}\big(\underline{\mathrm{cl}}_M(B)\big) = f_{-1}(B).$$

Now, by $(K2)$, $f_{-1}(B) \leqslant \underline{\mathrm{cl}}_L\big(f_{-1}(B)\big)$. Hence $f_{-1}[B]$ is a $\underline{\mathrm{cl}}$-closed sublocale of L.

⇐) Let B be a sublocale of M. Then by $(K4)$, $\underline{\mathrm{cl}}_M(B)$ is a $\underline{\mathrm{cl}}$-closed sublocale of M and so $f_{-1}\big(\underline{\mathrm{cl}}_M(B)\big)$ is a $\underline{\mathrm{cl}}$-closed sublocale of L. Now, by $(K2)$, $B \leqslant \underline{\mathrm{cl}}_M(B)$. Since $f_{-1}\big(\underline{\mathrm{cl}}_M(B)\big)$ is $\underline{\mathrm{cl}}$-closed, we have

$$\underline{\mathrm{cl}}_L\big(f_{-1}(B)\big) \leqslant \underline{\mathrm{cl}}_L\big(f_{-1}(\underline{\mathrm{cl}}_M(B))\big) = f_{-1}(\underline{\mathrm{cl}}_M(B)).$$

Therefore, localic map $f\colon L \to M$ is continuous. □

Proposition 2.25. *Let L be a locale such that sublocales A, B, $\underline{\mathrm{cl}}_L(A)$ and $\underline{\mathrm{cl}}_L(B)$ be complemented. Then, the following conditions are equivalent for any isotonic closure function $(L, \underline{\mathrm{cl}}_L)$.*

(1) $\underline{\mathrm{cl}}_L(A) \wedge B = A \wedge \underline{\mathrm{cl}}_L(B) = \mathsf{O}$.
(2) *There exist* $U \in \mathcal{N}(A)$ *and* $V \in \mathcal{N}(B)$ *such that*
$$A \wedge V = U \wedge B = \mathsf{O}.$$

Proof. 1 ⇒ 2) Let A and B be sublocales of L and
$$\underline{\mathrm{cl}}_L(A) \wedge B = A \wedge \underline{\mathrm{cl}}_L(B) = \mathsf{O}.$$
Since $\underline{\mathrm{cl}}_L(A) \wedge B = \mathsf{O}$,
$$B \leqslant \big(\underline{\mathrm{cl}}_L(A)\big)^{\#} = \big(\underline{\mathrm{int}}_L(A^{\#})\big)^{\#\#} \leqslant \underline{\mathrm{int}}_L(A^{\#}),$$
that is, $A^{\#} \in \mathcal{N}(B)$. Thus, there exists $V = A^{\#} \in \mathcal{N}(B)$ such that $A \wedge V = A \wedge A^{\#} = \mathsf{O}$. Similarly, we obtain $A \leqslant \underline{\mathrm{int}}_L(B^{\#})$, that is, there exists $B^{\#} \in \mathcal{N}(A)$ with $B^{\#} \wedge B = \mathsf{O}$.

2 ⇒ 1) Let A and B be sublocales of L. By (2), there exist sublocales U and V of L such that $A \leqslant \underline{\mathrm{int}}_L(U)$, $B \leqslant \underline{\mathrm{int}}_L(V)$, $A \wedge V = \mathsf{O}$, and $U \wedge B = \mathsf{O}$. Since $A \wedge V = \mathsf{O}$ implies $V \leqslant A^{\#}$, Propositition 2.2 shows that $B \leqslant \underline{\mathrm{int}}_L(V) \leqslant \underline{\mathrm{int}}_L(A^{\#})$. Hence, $\underline{\mathrm{cl}}_L(A) = \big(\underline{\mathrm{int}}_L(A^{\#})\big)^{\#} \leqslant B^{\#}$. Since B is a complemented sublocale, we conclude that $\underline{\mathrm{cl}}_L(A) \wedge B = \mathsf{O}$. The same argument yields $A \wedge \underline{\mathrm{cl}}_L(B) = \mathsf{O}$. □

3. G-closed sublocales

In this section, we introduce generalized closed sublocales in isotonic closure function and discuss some of their properties.

Definition 3.1. Let $(L, \underline{\mathrm{cl}}_L)$ be an isotonic closure function.

(1) A sublocale A of L is called a *generalized* $\underline{\mathrm{cl}}$*-closed sublocale* (briefly, g-$\underline{\mathrm{cl}}$-closed sublocale) if $\underline{\mathrm{cl}}_L(A) \subseteq G$ whenever G is a $\underline{\mathrm{cl}}$-open sublocale of $(L, \underline{\mathrm{cl}}_L)$ with $A \subseteq G$.
(2) A sublocale A of L is called a *generalized* $\underline{\mathrm{cl}}$*-open sublocale* (briefly, *g*-$\underline{\mathrm{cl}}$-open sublocale) if $F \subseteq \underline{\mathrm{int}}_L(A)$ whenever F is a $\underline{\mathrm{cl}}$-close sublocale of $(L, \underline{\mathrm{cl}}_L)$ with $F \subseteq A$.

Example 3.2. Let $(L, \underline{\mathrm{cl}}_L)$ be an isotonic closure function. Then, the sublocales O and L are g-$\underline{\mathrm{cl}}$-closed and *g*-$\underline{\mathrm{cl}}$-open.

Remark 3.3. Every $\underline{\mathrm{cl}}$-closed sublocale is g-$\underline{\mathrm{cl}}$-closed. The converse is not true, as can be seen from the following example.

Example 3.4. Let $(L, \underline{\mathrm{cl}}_L)$ be the isotonic closure function given in Example 2.4 and $A = \mathfrak{o}(c)$. It is easy to see that A is a g-$\underline{\mathrm{cl}}$-closed sublocale. But $\underline{\mathrm{cl}}_L\big(\mathfrak{o}(c)\big) = \mathfrak{o}(d)$ and so A is not $\underline{\mathrm{cl}}$-closed.

Proposition 3.5. *Let* $(L, \underline{\mathrm{cl}}_L)$ *be an isotonic closure function such that* $\underline{\mathrm{cl}}_L$ *satisfies* $(K2)$*. Then, the set* $Gc(L)$ *of all g-*$\underline{\mathrm{cl}}$*-closed sublocales forms a* $\bigvee$*-semilattice.*

Proof. Let A and B be g-$\underline{\mathrm{cl}}$-closed sublocales of L, and G be an open sublocale of L such that $A \vee B \subseteq G$. Then, $A \subseteq G$ and $B \subseteq G$. Since A and B are g-$\underline{\mathrm{cl}}$-closed, $\underline{\mathrm{cl}}_L(A) \subseteq G$ and $\underline{\mathrm{cl}}_L(B) \subseteq G$. Then, by $(K2)$,

$$\underline{\mathrm{cl}}_L(A \vee B) \subseteq \underline{\mathrm{cl}}_L(A) \vee \underline{\mathrm{cl}}_L(B) \subseteq G.$$

Therefore, $A \vee B$ is g-$\underline{\mathrm{cl}}$-closed. □

In the following example, we show that the intersection of two g-$\underline{\mathrm{cl}}$-closed sublocales need not be a g-$\underline{\mathrm{cl}}$-closed sublocale.

Example 3.6. Let $(L, \underline{\mathrm{cl}}_L)$ be the isotonic closure function given in Example 2.4. Consider the g-$\underline{\mathrm{cl}}$-closed sublocales $A =< \perp, c, d >$ and $B = \mathfrak{b}(a)$. Then, $A \cap B = \mathfrak{b}(c)$ is not g-$\underline{\mathrm{cl}}$-closed. To see this, consider the open sublocale $G = \mathfrak{c}(c)$. Then, $A \cap B \subseteq G$ but $\underline{\mathrm{cl}}_L(A \cap B) \nsubseteq G$. Therefore, $A \cap B$ is not g-$\underline{\mathrm{cl}}$-closed.

Lemma 3.7. *Let F be a complemented sublocale of L such that $\underline{\mathrm{cl}}_L(F) = F$. Then, $F^{\#}$ is a $\underline{\mathrm{cl}}_L$-open sublocale of L.*

Proof. By the definition of interior,

$$\underline{\mathrm{int}}_L(F^{\#}) = \big(\underline{\mathrm{cl}}_L(F^{\#\#})\big)^{\#} = \big(\underline{\mathrm{cl}}_L(F)\big)^{\#} = F^{\#},$$

which means that $F^{\#}$ is a $\underline{\mathrm{cl}}_L$-open sublocale of L. □

Lemma 3.8. *Let $(L, \underline{\mathrm{cl}}_L)$ be an isotonic closure function such that $\underline{\mathrm{cl}}_L$ satisfies $(K2)$. If A and B are $\underline{\mathrm{cl}}_L$-open sublocales, then $A \vee B$ is $\underline{\mathrm{cl}}_L$-open.*

Proof. Let A and B be $\underline{\mathrm{cl}}_L$-open sublocales. Then by Proposition 2.2,

$$\underline{\mathrm{int}}_L(A) \vee \underline{\mathrm{int}}_L(B) \subseteq \underline{\mathrm{int}}_L(A \vee B)$$

and so, $A \vee B \subseteq \underline{\mathrm{int}}_L(A \vee B)$. Since $\underline{\mathrm{cl}}_L$ satisfies $(K2)$,

$$\underline{\mathrm{int}}_L(A \vee B) \subseteq A \vee B.$$

Then, $\underline{\mathrm{int}}_L(A \vee B) = A \vee B$. This means that $A \vee B$ is a $\underline{\mathrm{cl}}_L$-open sublocale. □

Proposition 3.9. *Let $(L, \underline{\mathrm{cl}}_L)$ be an isotonic closure function such that $\underline{\mathrm{cl}}_L$ satisfies $(K2)$. If A is a g-$\underline{\mathrm{cl}}$-closed sublocale and, F is complemented and $\underline{\mathrm{cl}}$-closed in $(L, \underline{\mathrm{cl}}_L)$, then $A \cap F$ is g-$\underline{\mathrm{cl}}$-closed.*

Proof. Let G be a $\underline{\mathrm{cl}}$-open sublocale of $(L, \underline{\mathrm{cl}}_L)$ such that $A \cap F \subseteq G$. Then, $A \subseteq G \cup F^{\#}$. By Lemmas 3.7 and 3.8, $G \cup F^{\#}$ is a $\underline{\mathrm{cl}}$-open sublocale and so, $\underline{\mathrm{cl}}_L(A) \subseteq G \cup F^{\#}$. Then, $\underline{\mathrm{cl}}_L(A) \cap F \subseteq G$. Since F is $\underline{\mathrm{cl}}$-closed,

$$\underline{\mathrm{cl}}_L(A \cap F) \subseteq \underline{\mathrm{cl}}_L(A) \cap \underline{\mathrm{cl}}_L(F) = \underline{\mathrm{cl}}_L(A) \cap F \subseteq G.$$

Hence, $A \cap F$ is g-$\underline{\mathrm{cl}}$-closed. □

Proposition 3.10. *Let* $(L, \underline{\mathrm{cl}}_L)$ *be an isotonic closure function such that* $\underline{\mathrm{cl}}_L$ *satisfies* $(K2)$*. Also, let* A *be a sublocale of* L *which is both* $\underline{\mathrm{cl}}$*-open and g-*$\underline{\mathrm{cl}}$*-closed. Then,* A *is* $\underline{\mathrm{cl}}$*-closed.*

Proof. By $(K2)$, $A \subseteq \underline{\mathrm{cl}}_L(A)$. On the other hand, $A \subseteq A$ and, A is both $\underline{\mathrm{cl}}$-open and g-$\underline{\mathrm{cl}}$-closed. Thus, $\underline{\mathrm{cl}}_L(A) \subseteq A$. Therefore, $\underline{\mathrm{cl}}_L(A) = A$ and so, A is $\underline{\mathrm{cl}}$-closed. □

Proposition 3.11. *Let* $(L, \underline{\mathrm{cl}}_L)$ *be an isotonic closure function such that* $\underline{\mathrm{cl}}_L$ *satisfies* $(K4)$*. If* A *is a g-*$\underline{\mathrm{cl}}$*-closed sublocale of* $(L, \underline{\mathrm{cl}}_L)$ *such that* $A \subseteq B \subseteq \underline{\mathrm{cl}}_L(A)$*, then* B *is a g-*$\underline{\mathrm{cl}}$*-closed subset of* $(L, \underline{\mathrm{cl}}_L)$*.*

Proof. Let G be a $\underline{\mathrm{cl}}$-open sublocale of $(L, \underline{\mathrm{cl}}_L)$ such that $B \subseteq G$. Then, $A \subseteq G$. Since A is g-$\underline{\mathrm{cl}}$-closed, $\underline{\mathrm{cl}}_L(A) \subseteq G$. Now, by $(K4)$,

$$\underline{\mathrm{cl}}_L(B) \subseteq \underline{\mathrm{cl}}_L(\underline{\mathrm{cl}}_L(A)) = \underline{\mathrm{cl}}_L(A) \subseteq G.$$

Hence, B is a g-$\underline{\mathrm{cl}}$-closed sublocale. □

4. The category of isotonic closure functions

In this section, we introduce the category of isotonic closure functions over a locale L and discuss some of its properties.

Definition 4.1. Let $(L, \underline{\mathrm{cl}}_L)$ and $(M, \underline{\mathrm{cl}}_M)$ be isotonic closure functions. A function $\varphi : L(L, \underline{\mathrm{cl}}_L) \to (M, \underline{\mathrm{cl}}_M)$ is called a *morphism* if φ as a function from L to M is a localic map and also $\varphi\big(\underline{\mathrm{cl}}_L(A)\big) \subseteq \underline{\mathrm{cl}}_M\big(\varphi(A)\big)$ for every $A \in \mathcal{S}\ell(L)$.

Proposition 4.2. *Isotonic closure functions and morphisms of isotonics form a category denote by* ***ICF****.*

Proposition 4.3. *The category* ***ICF*** *has an initial object.*

Proof. We show that $(\mathsf{O}, \underline{\mathrm{cl}}_{\mathsf{O}})$ is an initial object, where $\mathsf{O} = \{\top\}$ and $\underline{\mathrm{cl}}_{\mathsf{O}} : \mathcal{S}\ell(\mathsf{O}) \longrightarrow \mathcal{S}\ell(\mathsf{O})$, defined by $\underline{\mathrm{cl}}_{\mathsf{O}}(\mathsf{O}) = \mathsf{O}$, is an isotonic closure function. Let $(L, \underline{\mathrm{cl}}_L)$ be an arbitrary isotonic closure function. Then $f : (\mathsf{O}, \underline{\mathrm{cl}}_{\mathsf{O}}) \longrightarrow (L, \underline{\mathrm{cl}}_L)$, defined by $f(1) = 1_L$, is a localic map. Moreover, for the sublocale O, $f\big(\underline{\mathrm{cl}}_{\mathsf{O}}(\mathsf{O})\big) = f(\mathsf{O}) = \mathsf{O}_L$ and $\underline{\mathrm{cl}}_L\big(f(\mathsf{O})\big) = \underline{\mathrm{cl}}_L(\mathsf{O}) = \mathsf{O}_L$. Hence, $f : (\mathsf{O}, \underline{\mathrm{cl}}_{\mathsf{O}}) \longrightarrow (L, \underline{\mathrm{cl}}_L)$ is a morphism. It is clear that f is unique. □

Theorem 4.4. *The category* ***ICF*** *has a terminal object.*

Proof. We show that $(\mathbf{2}, \underline{\mathrm{cl}}_{\mathbf{2}})$ is a terminal object, where $\mathbf{2}$ is the locale $\{0, 1\}$ and $\underline{\mathrm{cl}}_{\mathbf{2}} : \mathcal{S}\ell(\mathbf{2}) \longrightarrow \mathcal{S}\ell(\mathbf{2})$ is defined by $\underline{\mathrm{cl}}_{\mathbf{2}}(\mathsf{O}) = \mathsf{O}$ and $\underline{\mathrm{cl}}_{\mathbf{2}}(\mathbf{2}) = \mathbf{2}$. It is clear that $(\mathbf{2}, \underline{\mathrm{cl}}_{\mathbf{2}})$ is an isotonic closure function. Let $(L, \underline{\mathrm{cl}}_L)$ be an arbitrary isotonic closure function. Then $f : L \longrightarrow \mathbf{2}$, defined by $f(1) = 1$ and $f(a) = 0$ for every $1 \neq a \in L$, is a localic map. Now, let A be a sublocale of L. If $f(A) = \mathsf{O}$, then $A = \mathsf{O}$ and so,

$$f\big(\underline{\mathrm{cl}}_L(A)\big) = \mathsf{O} = \underline{\mathrm{cl}}_{\mathbf{2}}\big(f(A)\big).$$

If $f(A) \neq \mathsf{O}$, then $\underline{\mathrm{cl}}_{\mathbf{2}}\big(f(A)\big) = \mathbf{2}$ and so, $f\big(\underline{\mathrm{cl}}_L(A)\big) \subseteq \underline{\mathrm{cl}}_{\mathbf{2}}\big(f(A)\big)$. Hence, $f : (L, \underline{\mathrm{cl}}_L) \longrightarrow (\mathbf{2}, \underline{\mathrm{cl}}_{\mathbf{2}})$ is a morphism. It is clear that f is unique. □

We consider **LOC** as the category with locales for objects and the localic maps for morphisms.

Remark 4.5. [11] The epimorphisms in **LOC** are precisely the onto localic maps.

Lemma 4.6. *Let $f : (L, \underline{\mathrm{cl}}_L) \longrightarrow (M, \underline{\mathrm{cl}}_M)$ be a morphism in* **ICF**. *Then, f is an epimorphism in* **ICF** *if and only if f is an epimorphism in* **Loc**.

Proof. Necessity. Suppose that f is an epimorphism in **ICF** and $f_1, f_2 : M \longrightarrow K$ are localic maps such that $f_1 \circ f = f_2 \circ f$. We define $\underline{\mathrm{cl}}_K : \mathcal{S}\ell(K) \longrightarrow \mathcal{S}\ell(K)$ by $\underline{\mathrm{cl}}_K(\mathsf{O}) = \mathsf{O}$ and $\underline{\mathrm{cl}}_K(A) = K$ for every $\mathsf{O} \neq A \in \mathcal{S}\ell(K)$. Then, $(K, \underline{\mathrm{cl}}_K)$ is an isotonic closure function. For any sublocale A of M,

$$f_1\big(\underline{\mathrm{cl}}_M(A)\big) \subseteq \underline{\mathrm{cl}}_K\big(f_1(A)\big).$$

$$\begin{array}{ccc} \mathcal{S}\ell(M) & \xrightarrow{\underline{\mathrm{cl}}_M} & \mathcal{S}\ell(M) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathcal{S}\ell(K) & \xrightarrow[\underline{\mathrm{cl}}_K]{} & \mathcal{S}\ell(K) \end{array}$$

Hence, $f_1 : (M, \underline{\mathrm{cl}}_M) \longrightarrow (K, \underline{\mathrm{cl}}_K)$ is a morphism. Similarly, $f_2 : (M, \underline{\mathrm{cl}}_M) \longrightarrow (K, \underline{\mathrm{cl}}_K)$ is a morphism and $f_1 \circ f = f_2 \circ f$. Since f is right-cancellable in **ICF**, we obtain $f_1 = f_2$. Then, $f : L \to M$ is an epimorphism in **LOC**.

Sufficiency. This is clear. □

Proposition 4.7. *Let $f : (L, \underline{\mathrm{cl}}_L) \longrightarrow (M, \underline{\mathrm{cl}}_M)$ be a morphism in* **ICF**. *Then, f is an epimorphism in* **ICF** *if and only if f is a surjective localic map.*

Proof. By Remark 4.5 and Lemma 4.6, the proof is straightforward. □

Proposition 4.8. *Let $f : (L, \underline{\mathrm{cl}}_L) \to (M, \underline{\mathrm{cl}}_M)$ be a morphism in* **ICF**. *Then, f is a monomorphism in* **ICF** *if and only if f is a monomorphism in* **LOC**

Proof. Necessity. Let $K \overset{g}{\underset{h}{\rightrightarrows}} L$ be localic maps such that $f \circ g = f \circ h$. Consider the isotonic closure function $\underline{\mathrm{cl}}_K : \mathcal{S}\ell(K) \to \mathcal{S}\ell(K)$ defined by $\underline{\mathrm{cl}}_K(A) = \mathsf{O}$ for all sublocales A of K. For every sublocale A of K,

$$h\big(\underline{\mathrm{cl}}_K(A)\big) = h(\mathsf{O}) = \mathsf{O}$$

and so,

$$h\big(\underline{\mathrm{cl}}_K(A)\big) \subseteq \underline{\mathrm{cl}}_L\big(h(T)\big).$$

Therefore, $h : (K, \underline{\mathrm{cl}}_K) \to (L, \underline{\mathrm{cl}}_L)$ is a morphism in **ICF**. Similarly, $g : (K, \underline{\mathrm{cl}}_K) \to (L, \underline{\mathrm{cl}}_L)$ is a morphism and $f \circ g = f \circ h$. Since f is left-cancellable in **ICF**, we conclude that $g = h$.
Sufficiency. This is clear. ☐

Lemma 4.9. [11] *Let $f : L \to M$ be a localic map, and S a sublocale of M. Then,*

$$\begin{array}{ccc} f_{-1}[S] & \xrightarrow{k=\subseteq} & L \\ \big\downarrow g & & \big\downarrow f \\ S & \xrightarrow[j=\subseteq]{} & M \end{array}$$

is a pullback in **LOC**.

Lemma 4.10. *Let $f : (L, \underline{\mathrm{cl}}_L) \longrightarrow (M, \underline{\mathrm{cl}}_M)$ be a morphism, and S a sublocale of M. Then $g : (f_{-1}[S], \underline{\mathrm{cl}}_{f_{-1}[S]}) \to (S, \underline{\mathrm{cl}}_S)$, defined by $g(x) = f(x)$, is a morphism in* **ICF**, *where $(f_{-1}[S], \underline{\mathrm{cl}}_{f_{-1}[S]})$ and $(S, \underline{\mathrm{cl}}_S)$ are sub-closure functions of $(L, \underline{\mathrm{cl}}_L)$ and $(M, \underline{\mathrm{cl}}_M)$, respectively.*

Proof. It is clear that $g : f_{-1}[S] \to S$ is a localic map. Consider a sublocale B of $f_{-1}[S]$. Then,

$$\begin{aligned} g\big(\underline{\mathrm{cl}}_{f_{-1}[S]}(B)\big) &= f\big(\underline{\mathrm{cl}}_L(B) \cap f_{-1}[S]\big) \\ &= f\big(\underline{\mathrm{cl}}_L(B)\big) \cap f\big(f_{-1}[S]\big) \\ &\subseteq \underline{\mathrm{cl}}_M\big(f(B)\big) \cap S \\ &= \underline{\mathrm{cl}}_S\big(f(B)\big) \\ &= \underline{\mathrm{cl}}_S\big(g(B)\big). \end{aligned}$$

This means that $g : (f_{-1}[S], \underline{\mathrm{cl}}_{f_{-1}[S]}) \to (S, \underline{\mathrm{cl}}_S)$ is a morphism in **ICF**. ☐

Proposition 4.11. *Let $f : (L, \underline{\mathrm{cl}}_L) \longrightarrow (M, \underline{\mathrm{cl}}_M)$ be a morphism and let S be a sublocale of M. Then, the following square is a pullback in*

ICF.

$$\begin{array}{ccc} (f_{-1}[S], \underline{\mathrm{cl}}_{f_{-1}[S]}) & \xrightarrow{k=\subseteq} & (L, \underline{\mathrm{cl}}_L) \\ \Big\downarrow g & & \Big\downarrow f \\ (S, \underline{\mathrm{cl}}_S) & \xrightarrow[j=\subseteq]{} & (M, \underline{\mathrm{cl}}_M) \end{array}$$

Proof. Let $(K, \underline{\mathrm{cl}}_K)$ be an isotonic closure function, and let $\alpha : (K, \underline{\mathrm{cl}}_K) \to (L, \underline{\mathrm{cl}}_L)$ and $\beta : (K, \underline{\mathrm{cl}}_K) \to (S, \underline{\mathrm{cl}}_S)$ be right morphisms such that $f \circ \alpha = j \circ \beta$. By Lemma 4.9, there exists a unique localic map $\gamma : K \to f_{-1}[S]$, defined by $\gamma(x) = \alpha(x)$, such that $k \circ \gamma = \alpha$ and $g \circ \gamma = \beta$. Now, we show that γ is a right morphism in **ICF**. Let B be a sublocale of K. Then, $\gamma(B)$ is a sublocale of $f_{-1}[S]$. Therefore,

$$\begin{aligned} \gamma\big(\underline{\mathrm{cl}}_K(B)\big) &= k\Big(\gamma\big(\underline{\mathrm{cl}}_K(B)\big)\Big) \\ &= k\Big(\gamma\big(\underline{\mathrm{cl}}_K(B)\big) \cap f_{-1}[S]\Big) \\ &\subseteq k\Big(\gamma\big(\underline{\mathrm{cl}}_K(B)\big)\Big) \cap k\big(f_{-1}[S]\big) \\ &= \alpha\big(\underline{\mathrm{cl}}_K(B)\big) \cap f_{-1}[S] \\ &\subseteq \underline{\mathrm{cl}}_L\big(\alpha(B)\big) \cap f_{-1}[S] \\ &= \underline{\mathrm{cl}}_{f_{-1}[S]}\big(\alpha(B)\big) \\ &= \underline{\mathrm{cl}}_{f_{-1}[S]}\big(\gamma(B)\big). \end{aligned}$$

This means that $\gamma : (K, \underline{\mathrm{cl}}_K) \to (f_{-1}[S], \underline{\mathrm{cl}}_{f_{-1}[S]})$ is a morphism in **ICF**, $k \circ \gamma = \alpha$ and $g \circ \gamma = \beta$. ☐

Lemma 4.12. [11] *Let $f_1, f_2 : L \to M$ be a pair of localic maps. Then, (E, ι_E) is the equalizer of (f_1, f_2) in* **LOC**, *where*

$$E = \big\{s | \forall x, f_1(x \to s) = f_2(x \to s)\big\},$$

and $\iota_E : E \longrightarrow L$ is the inclusion map.

Proposition 4.13. *Let $f_1, f_2 : (L, \underline{\mathrm{cl}}_L) \to (M, \underline{\mathrm{cl}}_M)$ be morphisms. Then $(E, \underline{\mathrm{cl}}_E)$ is the equalizer of (f_1, f_2) in* **ICF**, *where*

$$E = \big\{s | \forall x, f_1(x \to s) = f_2(x \to s)\big\},$$

and $\underline{\mathrm{cl}}_E$ is the relativization of $\underline{\mathrm{cl}}_L$ to E.

Proof. Let $g : (K, \underline{\mathrm{cl}}_K) \to (L, \underline{\mathrm{cl}}_L)$ be a right morphism such that $f_1 \circ g = f_2 \circ g$. Since E is the equalizer of localic maps $L \overset{f_1}{\underset{f_2}{\rightrightarrows}} M$, we conclude the existence of a unique localic map $h : K \longrightarrow E$, defined by $h(x) = g(x)$, such that $\iota_E \circ h = g$. We show that

$$h : (K, \underline{\mathrm{cl}}_K) \longrightarrow (E, \underline{\mathrm{cl}}_E)$$

is a morphism in **ICF**. To see this, let T be a sublocale of K. Then, $h\big(\underline{\mathrm{cl}}_K(T)\big)$ is a sublocale of E and

$$h\big(\underline{\mathrm{cl}}_K(T)\big) = g\big(\underline{\mathrm{cl}}_K(T)\big) \subseteq \underline{\mathrm{cl}}_L\big(g(T)\big).$$

$$\begin{array}{ccc} \mathcal{S}\ell(K) & \xrightarrow{\underline{\mathrm{cl}}_K} & \mathcal{S}\ell(K) \\ \big\downarrow h & & \big\downarrow h \\ \mathcal{S}\ell(E) & \xrightarrow[\underline{\mathrm{cl}}_E]{} & \mathcal{S}\ell(E \end{array}$$

Then

$$h\big(\underline{\mathrm{cl}}_K(T)\big) \subseteq \underline{\mathrm{cl}}_L\big(g(T)\big) \cap E = \underline{\mathrm{cl}}_E\big(g(T)\big) = \underline{\mathrm{cl}}_E\big(h(T)\big)$$

and so, h is a unique morphism in **ICF**. □

Acknowledgement

We thank the referee most heartily for comments that have improved the paper. At the same time, we emphasize that Definition 1.2, Proposition 1.3 and Example 2.19, were stated by the referee.

References

1. Ch. Boonpok, Generalized Closed Sets in Isotonic Spaces, *Int. Journal of Math. Analysis*, **5**(5) (2011), 241–256.
2. M. M. Day, Convergence, closure and neighborhood, *Duke Math. J.*, **11** (1944), 181–199.
3. M. J. Ferreira, J. Picado and S. M. Pinto, Remainders in pointfree topology, *Topology Appl.*, **245** (2018), 21–45.
4. E. D. Habil and Kh. A. Elzenati, Connectedness in isotonic space, *Turk J Math.*, **30** (2006), 247–262.
5. E. D. Habil and Kh. A. Elzenati, Topological properties in isotonic spaces, *IUG Journal of Natural Studies*, **16**(2) (2008), 1–14.
6. P. C. Hammer, Extended topology: Continuity I, *Portug. Math.*, **25** (1964), 77–93.
7. P. C. Hammer, Extended topology: Set-valued set functions, *Nieuw Arch. Wisk. III*, **10** (1962), 55–77.
8. F. Hausdorff, Gestufte Raume, *Fund. Math.*, **25** (1935), 486–502.
9. N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo.*, **19** (1970), 89–96.
10. J. Picado and A. Pultr, *Frames and locales topology without points*, Birkhäuser, New York, 2012.
11. J. Picado and A. Pultr, On equalizers in the Category of locales, *Appl. Categ. Structures*, **29** (2011), 267–283.
12. J. Slapal, Closure operations for digital topology, *Theoret. Comput. Sci.*, **305** (2003), 457–471.

13. B. M. R. Stadler and P. F. Stadler, Basic properties of closure spaces, *J. Chem. Inf. Comput. Sci.*, **42** (2002), 577–585.
14. B. M. R. Stadler and P. F. Stadler, Higher separation axioms in generalized closure spaces, *Commentationes Mathematicae Warszawa*, **43** (2003), 257–273.

Toktam Haghdadi
Department of Basic Sciences, Birjand University of Technology, P.O. Box 226, Birjand, Iran.
Email: haghdadi@birjandut.ac.ir

Ali Akbar Estaji
Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran.
Email: aaestaji@hsu.ac.ir

Journal of Algebraic Systems

ISOTONIC CLOSURE FUNCTIONS ON A LOCALE

T. HAGHDADI AND A. A. ESTAJI

توابع بسته هم کشش روی یک لکل

تکتم حقدادی[1] و علی اکبر استاجی[2]

[1]گروه علوم پایه، دانشگاه صنعتی بیرجند، بیرجند، ایران

[2]دانشکده ریاضی و علوم کامپیوتر، دانشگاه حکیم سبزواری، سبزوار، ایران

در این مقاله، ما مفهوم توابع بسته هم‌کشش روی یک لکل را معرفی کرده و مورد مطالعه قرار داده‌ایم. این رده از توابع، زوج‌هایی به صورت $(L, \underline{\mathrm{cl}}_L)$ هستند که در آن L یک لکل و $\underline{\mathrm{cl}}_L\colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(L)$ یک تابع بسته هم‌کششی روی زیرلکل‌های L است. به‌علاوه زیرلکل‌های $\underline{\mathrm{cl}}_L$- بسته تعمیم یافته را معرفی کرده و برخی از خواص آن‌ها را مورد بحث قرار داده‌ایم. هم‌چنین ما رسته **ICF** را که اشیاء و ریخت‌های آن به ترتیب توابع بسته هم‌کشش و نگاشت‌های لکلیک هستند، معرفی کرده‌ایم.

کلمات کلیدی: توابع بسته هم‌کشش، لکل، توابع همسایگی، رسته.