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ON THE MINIMAXNESS AND ARTINIANNESS DIMENSIONS

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ABSTRACT. Let R be a commutative Noetherian ring, I, J be ideals of R such that $J \subseteq I$, and M a finitely generated R-module. In this paper, we prove that the invariants $A_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M)$ is not Artinian for all $t \in \mathbb{N}_0\}$ and $\inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M)$ is not minimax for all $t \in \mathbb{N}_0\}$ are equal. In particular, we show that the invariants $A_I^I(M)$ and $\inf\{i \in \mathbb{N}_0 \mid H_I^i(M)$ is not minimax$\}$ are equal. We also establish the local-global principle, $A_I^J(M) = \inf\{A_{IR_{\mathfrak{p}}}^{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Spec}(R)\}$, in some cases.

1. Introduction

Let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H_I^i(M) = \varinjlim_{n \geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

Local cohomology was first defined and studied by Grothendieck. Let M be a finitely generated R-module. The *J-finiteness dimension $f_I^J(M)$ of M relative to I* is defined by

$$f_I^J(M) = \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \neq 0 \text{ for all } t \in \mathbb{N}_0\},$$

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with the usual convention that the infimum of the empty set of integers is interpreted as ∞. It is immediate from [5, Proposition 9.1.2] that

$$f_I(M) := f_I^I(M) = \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not finitely generated}\}.$$

As a generalization of $f_I^J(M)$, we define the *J-Artinianness dimension* $A_I^J(M)$ *of* M *relative to* I by

$$A_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \text{ is not Artinian for all } t \in \mathbb{N}_0\}.$$

Let $A_I(M) := A_I^I(M)$. It is easy to see that

$$A_I(M) \geq \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not Artinian}\}.$$

There are several examples show that the above inequality is strict. Set

$$\mu_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \text{ is not minimax for all } t \in \mathbb{N}_0\}.$$

By the definitions, it is clear that $f_I^J(M) \leqslant A_I^J(M) \leqslant \mu_I^J(M)$. In this paper, we show that if $J \subseteq I$, then $A_I^J(M) = \mu_I^J(M)$.

In [2], Asadollahi and Naghipour proved that

$$f_I^J(M) = \inf\{f_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{Spec}(R)\},$$

whenever Ass $H_I^{f_I^J(M)}(M)$ is finite or

$$f_I(M) \neq \{i \in \mathbb{N}_0 | H_I^i(M) \text{ is not } J\text{-cofinite}\}.$$

We also generalize this local-global principle for the invariant $A_I^J(M)$. More precisely, we prove the following.

Theorem 1.1. *Let M be a finitely generated R-module, I, J be ideals of R and one of the following conditions* (1) *or* (2) *holds:*

(1) Ass $H_I^{A_I^J(M)}(M)$ *is finite;*
(2) $J \subseteq I$ *and* $A_I(M) \neq \{i \in \mathbb{N}_0 | H_I^i(M)$ *is not J-cominimax*$\}$. *(Recall that an R-module K is said to be J-cominimax if* Supp $K \subseteq V(J)$ *and* $\operatorname{Ext}_R^i(R/J, K)$ *is minimax for all* $i \geq 0$*)*

Then $A_I^J(M) = \inf\{A_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{Spec}(R)\}$.

The following theorem plays an important role in this paper.

Theorem 1.2. *Let M be an R-module, I, J ideals of R such that* $\operatorname{Supp} M \subseteq V(I)$ *and* $\operatorname{Hom}_R(R/I, M)$ *is minimax. If for each* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$ *there is a non-negative integer $t_\mathfrak{p}$ such that* $(J^{t_\mathfrak{p}} M)_\mathfrak{p} = 0$*, then $J^t M$ is Artinian for some positive integer t.*

As a consequence of Theorem 1.2, we prove the following criterion for the minimaxness.

Corollary 1.3. *Let M be an R-module and I an ideal of R such that* $\operatorname{Supp} M \subseteq \mathrm{V}(I)$. *Then* $\operatorname{Hom}_R(R/I, M)$ *is minimax and for each* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$ *there is a non-negative integer* $t_{\mathfrak{p}}$ *such that* $(I^{t_{\mathfrak{p}}} M)_{\mathfrak{p}} = 0$ *if and only if M is minimax.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. An R-module L is said to be *minimax*, if there exists a finitely generated submodule N of L such that L/N is Artinian. The class of minimax modules was introduced by H. Zöschinger [11] and he has given in [11, 12] many equivalent conditions for a module to be minimax. We shall use $\operatorname{Max}(R)$ to denote the set of all maximal ideals of R. Also, for any ideal I of R, we denote $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I\}$ by $V(I)$. For any unexplained notation and terminology we refer the reader to [5] and [9].

2. Artinianness dimension

In this section, we define and study the Artinianness dimension of a finitely generated R-module M with respect to the ideal I. We also prove local-global principle for this invariant in some case. Our main results are Theorems 2.7, 2.9 and 2.11.

Definition 2.1. Let M be a finitely generated R-module and I, J ideals of R. We define the *J-Artinianness dimension $A_I^J(M)$ of M with respect to I* by

$$A_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \text{ is not Artinian for all } t \in \mathbb{N}_0\}.$$

Note that $A_I^J(M)$ is either a positive integer or ∞. We also denote $A_I^I(M)$ by $A_I(M)$ and call it the *Artinianness dimension of M with respect to I*.

The following example shows that the invariant $A_I^J(M)$ is different from the invariant $\inf\{i \mid H_I^i(M) \text{ is not Artinian}\}$, in general.

Example 2.2. *Let M be a finitely generated R-module. It is easy to see that*

$$A_I(M) \geqslant \inf\{i \mid H_I^i(M) \textit{ is not Artinian}\}.$$

Now let $\mathfrak{p}$ be a non-maximal prime ideal of R and I be an ideal of R such that $I \subseteq \mathfrak{p}$. Set $M := R/\mathfrak{p}$. Since $\Gamma_I(M) = M$, $\Gamma_I(M)$ is not Artinian. Also $I^t\Gamma_I(M) = 0$, for some $t \in \mathbb{N}$. Hence $I^t\Gamma_I(M)$ is Artinian. Thus, the above inequality may be strict. □

Now we prove the following useful lemmas.

Lemma 2.3. *Let M be an R-module. Then M is Artinian if and only if M is minimax and* $\operatorname{Supp} M \subseteq \operatorname{Max}(R)$.

Proof. Let M be minimax. Then there is a finitely generated submodule N of M such that M/N is Artinian. Since N is finitely generated and $\operatorname{Supp} N \subseteq \operatorname{Max}(R)$, N is Artinian. Therefore, it is immediate from the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

that M is Artinian. □

Lemma 2.4. *Let M be an R-module and I an ideal of R such that* $\operatorname{Hom}_R(R/I, M)$ *is minimax. Then,* $\operatorname{Hom}_R(R/I^n, M)$ *is minimax for all $n \in \mathbb{N}$.*

Proof. We proceed by induction on n. If $n = 1$, there is nothing to prove. Suppose that $n > 1$ and the case $n-1$ is setteled. Since I^{n-1}/I^n is finitely generated R/I-module, I^{n-1}/I^n is a homomorphic image of $(R/I)^r$ for some $r \in \mathbb{N}$. It is immediate from the monomorphism $\operatorname{Hom}_R(I^{n-1}/I^n, M) \longrightarrow \operatorname{Hom}_R(R/I, M)^r$ that $\operatorname{Hom}_R(I^{n-1}/I^n, M)$ is minimax. By applying the functor $\operatorname{Hom}_R(-, M)$ to the short exact sequence

$$0 \longrightarrow I^{n-1}/I^n \longrightarrow R/I^n \longrightarrow R/I^{n-1} \longrightarrow 0,$$

we get the exact sequence

$$\operatorname{Hom}_R(R/I^{n-1}, M) \longrightarrow \operatorname{Hom}_R(R/I^n, M) \longrightarrow \operatorname{Hom}_R(I^{n-1}/I^n, M). \quad (\dagger)$$

We conclude from inductive assumption and the exact sequence (†) that $\operatorname{Hom}_R(R/I^n, M)$ is minimax. This completes the inductive steps. □

The following theorem plays a key role in this paper.

Theorem 2.5. *Let M be an R-module, I, J be ideals of R such that* $\operatorname{Supp} M \subseteq V(I)$ *and* $\operatorname{Hom}_R(R/I, M)$ *is minimax. If for each* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$ *there is a non-negative integer $t_{\mathfrak{p}}$ such that* $(J^{t_{\mathfrak{p}}} M)_{\mathfrak{p}} = 0$*, then $J^t M$ is Artinian for some positive integer t.*

Proof. For each $t \in \mathbb{N}_0$, the set $\operatorname{Ass} J^t M$ is finite. Thus $\operatorname{Supp} J^t M$ is a closed subset of $\operatorname{Spec}(R)$ (in the Zariski topology), and so the descending chain

$$\cdots \supseteq \operatorname{Supp} J^t M \supseteq \operatorname{Supp} J^{t+1} M \supseteq \cdots$$

is eventually stationary. Therefore there is a non-negative integer t_0 such that, for each $t \geqslant t_0$, $\operatorname{Supp} J^t M = \operatorname{Supp} J^{t_0} M$. Let

$$\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max} R.$$

By assumption, $(J^tM)_{\mathfrak{p}} = 0$ for some integer $t \geqslant t_0$. So $\mathfrak{p} \notin \operatorname{Supp} J^{t_0}M$. Hence, $\operatorname{Supp} J^{t_0}M \subseteq \operatorname{Max}(R)$. As $\operatorname{Hom}_R(R/I, J^{t_0}M)$ is minimax and

$$\operatorname{Supp}\operatorname{Hom}_R(R/I, J^{t_0}M) \subseteq \operatorname{Supp} J^{t_0}M \subseteq \operatorname{Max}(R),$$

it follows from Lemma 2.3 that $\operatorname{Hom}_R(R/I, J^{t_0}M)$ is Artinian. Since $\operatorname{Supp} J^{t_0}M \subseteq V(I)$, it yields from Melkersson's theorem [10, Theorem 1.3] that $J^{t_0}M$ is Artinian. □

As a consequence of Theorem 2.5, we prove the following corollary.

Corollary 2.6. *Let M be an R-module and I an ideal of R such that* $\operatorname{Supp} M \subseteq \mathrm{V}(I)$. *Then* $\operatorname{Hom}_R(R/I, M)$ *is minimax and there is a non-negative integer* $t_{\mathfrak{p}}$ *such that* $(I^{t_{\mathfrak{p}}}M)_{\mathfrak{p}} = 0$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$ *if and only if M is minimax.*

Proof. It follows from Theorem 2.5 that there is a non-negative integer t_0 such that $I^{t_0}M$ is Artinian. Hence,by [6, Lemma 2.1], $M/(0:_M I^{t_0})$ is Artinian. According to Lemma 2.4, $(0:_M I^{t_0})$ is minimax. Therefore, by the exact sequence

$$0 \longrightarrow (0:_M I^{t_0}) \longrightarrow M \longrightarrow M/(0:_M I^{t_0}) \longrightarrow 0$$

of R-modules and R-homomorphisms, M is minimax, as required. □

Now, we are ready to state and prove the first main result of this paper which concludes that

$$A_I(M) = \inf\{i \in \mathbb{N}_0 \mid H^i_I(M) \text{ is not minimax}\}.$$

Theorem 2.7. *Let M be a finitely generated R-module and I, J be ideals of R such that $J \subseteq I$. Then*

$$A^J_I(M) = \inf\{i \in \mathbb{N}_0 \mid J^tH^i_I(M) \textit{ is not minimax}\}.$$

In particular,

$$\begin{aligned} A_I(M) &= \inf\{i \in \mathbb{N}_0 \mid I^tH^i_I(M) \textit{ is not minimax}\} \\ &= \inf\{i \in \mathbb{N}_0 \mid H^i_I(M) \textit{ is not minimax}\}. \end{aligned}$$

Proof. Let

$$\mu^J_I(M) := \inf\{i \in \mathbb{N}_0 \mid J^tH^i_I(M) \text{ is not minimax}\}.$$

Obviously, $A^J_I(M) \leq \mu^J_I(M)$.

For the converse, let $i < \mu^J_I(M)$. Then, there exists $t' \in \mathbb{N}_0$ such that $J^{t'}H^i_I(M)$ is minimax for all $i < \mu^J_I(M)$. So there is a finitely generated submodule N_i of $J^{t'}H^i_I(M)$ such that $J^{t'}H^i_I(M)/N_i$ is Artinian. Hence,

$$(JR_{\mathfrak{p}})^{t'}H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) \simeq (J^{t'}H^i_I(M))_{\mathfrak{p}} \simeq (N_i)_{\mathfrak{p}}$$

is a finitely generated $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$. Since $JR_{\mathfrak{p}} \subseteq IR_{\mathfrak{p}}$ and $(JR_{\mathfrak{p}})^{t'} H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is $IR_{\mathfrak{p}}$-torsion, there exists $t_{\mathfrak{p}} \in \mathbb{N}_0$ such that

$$(J^{t_{\mathfrak{p}}+t'} H^i_I(M))_{\mathfrak{p}} \simeq (JR_{\mathfrak{p}})^{t_{\mathfrak{p}}+t'} H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0.$$

Now, it follows from Theorem 2.5 that there is a positive integer t such that $J^t J^{t'} H^i_I(M)$ is Artinian.

Now, let $J = I$. It follows from the above argument and [7, Lemma 2.2] that

$$\begin{aligned} A_I(M) &= \inf\{i \in \mathbb{N}_0 \mid I^t H^i_I(M) \text{ is not minimax}\} \\ &= \inf\{i \in \mathbb{N}_0 \mid H^i_I(M) \text{ is not minimax}\}. \end{aligned}$$

□

Now we restate Theorem 2.7 in another terms, which is sometime useful.

Corollary 2.8. *Let M be finitely generated R-module, I, J be ideals of R such that $J \subseteq I$ and $s \in \mathbb{N}$. Then the following statements are equivalent.*

(1) *There exists $r \in \mathbb{N}_0$ such that $J^r H^i_I(M)$ is Artinian for all $i < s$;*
(2) *There exists $r \in \mathbb{N}_0$ such that $J^r H^i_I(M)$ is minimax for all $i < s$.*

In particular, if $J = I$, then the following statements are equivalent.

(1) *There exists $r \in \mathbb{N}_0$ such that $I^r H^i_I(M)$ is Artinian for all $i < s$;*
(2) *There exists $r \in \mathbb{N}_0$ such that $I^r H^i_I(M)$ is minimax for all $i < s$;*
(3) *$H^i_I(M)$ is minimax for all $i < s$.*

Proof. The proof is concluded from Theorem 2.7. □

The following theorem is a generalization of [2, Theorem 2.3] which shows that local-global principle for the invariant $A^J_I(M)$ holds whenever $\operatorname{Ass} H^{A^J_I(M)}_I(M)$ is a finite set.

Theorem 2.9. *Let M be a finitely generated R-module and I, J be ideals of R such that $\operatorname{Ass} H^{A^J_I(M)}_I(M)$ is a finite set. Then,*

$$\begin{aligned} A^J_I(M) &= \inf\{A^{JR_{\mathfrak{p}}}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \inf\{A^{JR_{\mathfrak{m}}}_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}}) | \mathfrak{m} \in \operatorname{Max}(R)\}. \end{aligned}$$

Proof. Let $r := A^J_I(M)$. Clearly,

$$r \leq \inf\{A^{JR_{\mathfrak{p}}}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) | \mathfrak{p} \in \operatorname{Spec}(R)\} \leq \inf\{A^{JR_{\mathfrak{m}}}_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}}) | \mathfrak{m} \in \operatorname{Max}(R)\}.$$

Assume on the contrary that

$$r < \inf\{A_{IR_{\mathfrak{m}}}^{JR_{\mathfrak{m}}}(M_{\mathfrak{m}}) | \mathfrak{m} \in \operatorname{Max}(R)\},$$

and look for a contradiction. Similar to the proof of Theorem 2.5, there is a non-negative integer t_0 such that, for each $t \geqslant t_0$,

$$\operatorname{Supp} J^t H_I^r(M) = \operatorname{Supp} J^{t_0} H_I^r(M).$$

Let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$. There is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$. Since $r < A_{IR_{\mathfrak{m}}}^{JR_{\mathfrak{m}}}(M_{\mathfrak{m}})$, there exists an integer $s \geq t_0$ such that $(JR_{\mathfrak{m}})^s H_{IR_{\mathfrak{m}}}^r(M_{\mathfrak{m}})$ is Artinian. Hence,

$$(J^s H_I^r(M))_{\mathfrak{p}} \simeq ((JR_{\mathfrak{m}})^s H_{IR_{\mathfrak{m}}}^r(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} = 0.$$

So, Ass $J^{t_0} H_I^r(M) = \operatorname{Supp} J^{t_0} H_I^r(M) \subseteq \operatorname{Max}(R)$ is a finite set. Let

$$\operatorname{Supp} J^{t_0} H_I^r(M) := \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}.$$

For each $1 \leq i \leq n$, there exists $t_i \geq t_0$ such that

$$(J^{t_i} H_I^r(M))_{\mathfrak{m}_i} \simeq (JR_{\mathfrak{m}_i})^{t_i} H_{IR_{\mathfrak{m}_i}}^r(M_{\mathfrak{m}_i})$$

is Artinian. Set $t := \max\{t_1 \ldots, t_n\}$. Then, $(J^t H_I^r(M))_{\mathfrak{m}_i}$ is Artinian for all $1 \leq i \leq n$. Now, it follows from [1, Lemma 2.1] that $J^t H_I^r(M)$ is Artinian, which is a contradiction. □

We prepare the grand by some definitions and a proposition to prove local-global principle for the invariant $A_I^J(M)$ in another case. Let J be an ideal of R. The $J-$cominimaxness of an $R-$module was introduced in [3]. An R-module K is said to be J-cominimax if Supp $K \subseteq V(J)$ and $\operatorname{Ext}_R^i(R/J, K)$ is minimax for all $i \geq 0$. Let M be a finitely generated R-module and I, J be ideals of R such that $J \subseteq I$. We define *the J-cominimaxness dimension $c_I^J(M)$ of M relative to I* by

$$c_I^J(M) = \inf\{i \in \mathbb{N}_0 | H_I^i(M) \text{ is not } J\text{-cominimax }\}.$$

Proposition 2.10. *Let M be a finitely generated R-module and I, J be ideals of R such that $J \subseteq I$. Then,*

$$A_I(M) = \inf\{A_I^J(M), c_I^J(M)\}.$$

Proof. Clearly $A_I(M) \leq A_I^J(M)$. If $r := c_I^J(M) < A_I(M)$. Then, there exists a non-negative integer t such that $I^t H_I^i(M)$ is Artinian for all $i \leq r$. Hence, by Corollary 2.8, $H_I^i(M)$ is minimax for all $i \leq r$. Since

$$\operatorname{Supp} H_I^r(M) \subseteq V(I) \subseteq V(J),$$

$H_I^r(M)$ is J-cominimax, which is imposable. Thus, $A_I(M) \leq c_I^J(M)$. So,

$$A_I(M) \leq \inf\{A_I^J(M), c_I^J(M)\}.$$

Finally, assume on the contrary that

$$s := A_I(M) < \inf\{A_I^J(M), c_I^J(M)\},$$

and look for a contradiction. Since $s < A_I^J(M)$, there exists a non-negative integer t such that $J^tH_I^s(M)$ is Artinian. Hence, by [6, Lemma 2.1], $H_I^s(M)/(0 :_{H_I^s(M)} J^t)$ is Artinian. As $s < c_I^J(M)$, it follows from Lemma 2.4 that $(0 :_{H_I^s(M)} J^t)$ is minimax. Now, we deduce from the short exact sequence

$$0 \longrightarrow (0 :_{H_I^s(M)} J^t) \longrightarrow H_I^s(M) \longrightarrow H_I^s(M)/(0 :_{H_I^s(M)} J^t) \longrightarrow 0$$

that $H_I^s(M)$ is minimax. So, it follows from Theorem 2.7 that $H_I^i(M)$ is minimax for all $i \leq s$. Again, by Theorem 2.7, there exists a non-negative integer t such that $I^tH_I^i(M)$ is Artinian for all $i \leq s$, which is a contradiction. This contradiction completes the proof of the proposition. □

Now, we are going to prove local-global principle for $A_I^J(M)$ in another case.

Theorem 2.11. *Let M be a finitely generated R-module and let I, J be ideals of R such that $J \subseteq I$ and $A_I(M) \neq c_I^J(M)$. Then,*

$$\begin{aligned} A_I^J(M) &= \inf\{A_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \inf\{A_{IR_\mathfrak{m}}^{JR_\mathfrak{m}}(M_\mathfrak{m}) | \mathfrak{m} \in \operatorname{Max}(R)\}. \end{aligned}$$

Proof. As $A_I(M) \neq c_I^J(M)$, it follows from Proposition 2.10 that $A_I^J(M) \leq c_I^J(M)$. Hence, $A_I(M) = A_I^J(M)$. It follows from Corollary 2.8 that $H_I^i(M)$ is minimax for all $i < A_I^J(M)$. Thus Ass $H_I^{A_I^J(M)}(M)$ is a finite set, by [4, Theorem 2.3]. Now, the result follows from Theorem 2.9. □

Let M be a finitely generated R-module and let I, J be ideals of R such that $J \subseteq I$ and $IM \neq M$. As $A_I^J(M) = \mu_I^J(M)$, it follows from [7, Corollaries 2.9, 2.10, 2.11 and Theorem 2.12] that

$$A_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) \geq l \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R) \Longleftrightarrow A_I^J(M) \geq l,$$

where $l \in \{1, 2, \operatorname{grade}_M I, f_I(M)\}$.

In [8], Khashyarmanesh and Salarian show that if R is a homomorphic image of a Gorenstein ring, then

$$f_I^J(M) = \inf\{f_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) | \mathfrak{p} \in \operatorname{Spec}(R)\},$$

for every choice of ideals I, J of R and every choice of the finitely generated R-module M. It is an open problem to us whether the similar statement holds for the invariant $A_I^J(M)$ when R is a homomorphic image of a Gorenstein ring.

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مطالبی در مورد بعدهای آرتینی و مینیماکس

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فرض کنید R یک حلقه جابجایی و نوتری، I و J ایده‌ال‌هایی از R به‌طوری‌که $I \subseteq J$ و M یک R-مدول با تولید متناهی باشد. در این مقاله نشان می‌دهیم که

$$A_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid \text{برای هر } t \in \mathbb{N}_0 \text{ آرتینی نیست } J^t H_I^i(M)\}$$

و

$$\inf\{i \in \mathbb{N}_0 \mid \text{برای هر } t \in \mathbb{N}_0,\ J^t H_I^i(M) \text{ مینیماکس نیست}\}$$

برابرند. به‌ویژه نشان می‌دهیم که

$$\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ مینیماکس نیست}\},\quad A_I^I(M)$$

با هم برابرند. همچنین در برخی حالات ثابت می‌کنیم که

$$A_I^J(M) = \inf\{A_{IR_p}^{JR_p}(M_p) \mid p \in \mathrm{Spec}(R)\}$$

کلمات کلیدی: مدول آرتینی، مدول کوهومولوژی موضعی، مدول مینیماکس.