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NEW FUNDAMENTAL RELATIONS IN HYPERRINGS AND THE CORRESPONDING QUOTIENT STRUCTURES

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Abstract. In this article, we introduce and analyse the smallest equivalence binary relation χ^* on a hyperring R such that the quotient R/χ^*, the set of all equivalence classes, is a commutative ring with identity and of characteristic m. The characterizations of commutative rings with identity via strongly regular relations is investigated and some properties on the topic are presented. Moreover, we introduce a new strongly regular relation σ_p^* such that the quotient structure is a p-ring.

1. Introduction

The hyperstructure theory was introduced by Marty [12] at the 8th Congress of Scandinavian Mathematicians. Hyperstructures have many applications in several sectors of both pure and applied mathematics, for instance in geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability, rough set theory and so on (see [3]). A hypergroup in the sense of Marty is a non-empty set H endowed by a hyperoperation $* : H \times H \longrightarrow \wp^*(H)$, the set of all non-empty subset of H, which satisfies the associative law and the reproduction axiom. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semihypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all $a, b \in H$. We say that f is a *good*

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homomorphism if for all $a, b \in H$, $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, then for all non-empty subsets A, B of H we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a\rho b, \quad \text{for all} \quad a \in A, \ b \in B.$$

The relation ρ is called strongly regular on the right (on the left) if $x \ \rho \ y \Rightarrow a \cdot x \ \overline{\overline{\rho}} \ a \cdot y (x \ \rho \ y \Rightarrow x \cdot a \ \overline{\overline{\rho}} \ y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called strongly regular if it is strongly regular on the right and on the left. Let H be a hypergroup and ρ an equivalence relation on H. A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a) \otimes \rho(b) = \{\rho(x) | x \in \rho(a) \circ \rho(b)\}$. If ρ is strongly regular, then it readily follows that $\rho(a) \otimes \rho(b) = \{\rho(x) | x \in a \circ b\}$. It is well known $\langle H/\rho, \otimes \rangle$ is a group, that is, $\rho(a) \otimes \rho(b) = \rho(c)$ for all $c \in a \circ b$. Basic definitions and propositions about the hyperstructures are found in [2, 15]. The most general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a *hyperring* if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. We call $(R, +, \cdot)$ a *hyperfield* if $(R, +, \cdot)$ is a hyperring and $(R, \cdot)$ is a hypergroup. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is a Krasner additive hyperring [10] but under some conditions such as $(R, +)$ that is a canonical hypergroup with identity 0 and 0 is a bilaterally absorbing element. If only $\cdot$ is a hyperoperation, we shall say that R is a multiplicative hyperring. Rota [14] introduced the multiplicative hyperrings. In what follows we shall consider one of the most general types of hyperrings.

Definition 1.1. [15] The triple $(R, +, \cdot)$ is a *hyperring* if

(1) $(R, +)$ is a hypergroup;
(2) $(R, \cdot)$ is a semihypergroup;
(3) the hyperoperation " $\cdot$" is distributive over the hyperoperation "$+$", which means that for all x, y, z of R we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

Definition 1.2. An element e of hyperring R is said to be unitary, if for any $x \in R$, we have $x \in x \cdot e \cap e \cdot x$.

The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures. Fundamental equivalence relations link hyperstructure theory to the theory of corresponding classical structures. They also introduce new hyperstructure classes. The fundamental relations β^* and γ^* on hypergroups were defined by Koskas [9] and Freni [7], respectively and studied then by many researchers [1, 4, 11]. The fundamental relations on hyperrings have also been studied by Vougiouklis and Davvaz [6, 16] and others [3, 8, 14]. Let us recall now some important equivalence relations and results of hypergroup and hyperring theory.

Definition 1.3. [9] For all $n > 1$ define the relation β_n on the semi-hypergroup $(H, \cdot)$, as follows:

$$a \ \beta_n \ b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \cup_{i=1}^{n} \beta_i$, where $\beta_1 = \{(x, x) | x \in H\}$ is the diagonal relation on H. Denote by β^* the transitive closure of β. The relation β^* is a strongly regular relation. Also, if H is hypergroup then $\beta = \beta^*$. The relation β^* is the least equivalence relation on a hypergroup H, such that the quotient H/β^* is a group.

Definition 1.4. [16] Let $(R, +, \cdot)$ be a hyperring. We define the relation Γ as follows:

$$x \ \Gamma \ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \ldots, k_n) \in \mathbb{N}^n$$

and

$$\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, (i = 1, \ldots, n)]$$

such that

$$x, y \in \textstyle\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}).$$

Let Γ^* be the transitive closure of Γ. The fundamental relation Γ^* on R can be considered as the smallest equivalence relation such that the quotient R/Γ^* be a ring.

Definition 1.5. [6] Let $(R, +, \cdot)$ be a hyperring. We define the relation α as follows:

$$x \ \alpha \ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \ldots, k_n) \in \mathbb{N}^n, \exists \tau \in \mathbb{S}_n$$

and

$$\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, \exists \tau_i \in \mathbb{S}_{k_i}, (i = 1, \ldots, n)]$$

such that

$$x \in \textstyle\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\tau(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\tau_i(j)}$.

Let α^* be the transitive closure of α. Then α^* is the smallest strongly regular relation on R such that R/α^* is a commutative ring.

Definition 1.6. [11] For any natural number n, we define the relation ρ^n on the hypergroup $(H,\cdot)$, as follows: $\rho^n = \cup_{m\geq 1}\rho^n_m$, where for every integer $m \geq 1$, ρ^n_m is the relation defined as follows:

$$a\ \rho^n_m\ b \Leftrightarrow \exists(x_1,\ldots,x_m)\in H^m :$$
$$x\in\prod_{i=1}^{n}x_i,\ y\in\prod_{i=1}^{n}x_i^{j_i}\quad \text{or}\quad y\in\prod_{i=1}^{n}x_i,\ x\in\prod_{i=1}^{n}x_i^{j_i}$$

where $\forall i \in \{1,2,\ldots,m\}, j_i \in \{1, n+1\}$ and $x_i^{j_i} = x_i\cdot x_i\cdot\ldots\cdot x_i$, ($j_i$ times).

Denote by ρ^{n*} the transitive closure of ρ^n. The relation ρ^{n*} is a strongly regular relation. The relation ρ^{n*} is the smallest equivalence relation on hypergroup H, such that the quotient H/ρ^{n*} is a group. Moreover, for all $x\in H$, $[\rho^{n*}(x)]^{n+1} = \rho^{n*}(x)$ hold, which means that $[\rho^{n*}(x)]^n = e$, the identity of the group H/ρ^{n*}.

2. The relation χ^*

In this section, with respect to our previous work [8], we introduce a new relation on a general hyperring denoted by χ^*, which we shall use in order to determine a characterization of a new derived hyperring. We prove that χ^* is the smallest strongly regular equivalence such that the quotient R/χ^* is a commutative ring with identity and of characteristic m.

Let $(R,+,\cdot)$ be a hyperring and 1 be some element of R, in fact 1 is a generic element of R.

Definition 2.1. Let $m\in\mathbb{N}$. We say that a pair

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\right)$$

satisfies the condition $\curlywedge$ if

$$\begin{aligned}&\exists n\in\mathbb{N},\exists(k_1,\ldots,k_n)\in\mathbb{N}^n,\exists(k'_1,\ldots,k'_n)\in\mathbb{N}^n,k'_i\geqslant k_i,\\&\exists(x_{i1},\ldots,x_{ik_i})\in R^{k_i},\exists(y_{i1},\ldots,y_{ik'_i})\in R^{k'_i},l_i\in\{1,m+1\},\\&l_ix_{ij}=x_{ij}+x_{ij}+\ldots+x_{ij}(l_i\ times),\exists\sigma_i\in\mathbb{S}_{k_i},(i=1,\ldots,n)\end{aligned}$$

and there exist d_i and d'_i in $\mathbb{N}$ such that $1\leqslant d_i\leqslant k_i$, $d_i\leqslant d'_i\leqslant k'_i$, $k'_i = k_i + (d'_i - d_i)$ and

$$y_{ij}=\begin{cases}x_{i\sigma_i(j)}, & 1\leqslant j\leqslant d_i;\\ 1, & d_i<j\leqslant d'_i;\\ x_{i\sigma_i(j-d'_i+d_i)}, & d'_i<j\leqslant k'_i.\end{cases}$$

We use

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right)_{\curlywedge}$$

if $\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right)$ satisfies the condition $\curlywedge$. We define

$$\Omega^{\curlywedge,1} := \left\{ \left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right) \mid \left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right)_{\curlywedge} \right\}$$

and $\Omega^1 := \Omega^{\curlywedge,1} \bigcup (\Omega^{\curlywedge,1})^{-1}$.

Example 2.2. Let $m = 1$, $1 := c$ and $R = \{a, b, c, d, e\}$. Consider the hyperring $(R, +, \cdot)$, where the hyperoperations $+$ and $\cdot$ are defined on R as follows:

$+$	a	b	c	d	e
a	$\{a, e\}$	b	$\{c, d\}$	$\{c, d\}$	$\{a, e\}$
b	b	$\{c, d\}$	$\{a, e\}$	$\{a, e\}$	b
c	$\{c, d\}$	$\{a, e\}$	b	b	$\{c, d\}$
d	$\{c, d\}$	$\{a, e\}$	b	b	$\{c, d\}$
e	$\{a, e\}$	b	$\{c, d\}$	$\{c, d\}$	$\{a, e\}$

.	a	b	c	d	e
a	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$
b	$\{a, e\}$	b	$\{c, d\}$	$\{c, d\}$	$\{a, e\}$
c	$\{a, e\}$	$\{c, d\}$	b	b	$\{a, e\}$
d	$\{a, e\}$	$\{c, d\}$	b	b	$\{a, e\}$
e	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$

Then, for example, we have

$$\begin{aligned}&\{(a + b \cdot d, a \cdot c + d \cdot b), (a + b \cdot d, a \cdot c + a \cdot c + d \cdot b),\\ &(a + b \cdot d, a \cdot c + d \cdot b + d \cdot b), (e + b, e + b), (e + b, e + e + b),\\ &(e + b, e + b + b), (a \cdot e + b, a \cdot c \cdot e + d \cdot c)\} \subseteq \Omega^1.\end{aligned}$$

Moreover, if

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{k_i} x_{ij})\right) = a \cdot d + c \cdot d \cdot e \cdot c + b + c \cdot d + c \cdot d$$

then $n = 4$ and $(k_1, k_2, k_3, k_4) = (2, 4, 1, 2)$. Now, for example,

$$(a \cdot e + b, a \cdot c \cdot e + d \cdot c + d \cdot c) \in \Omega^1.$$

In fact, we have $(k'_1, k'_2) = (3, 2)$ and

$$\sum_{i=1}^{n} l_i \left(\prod_{j=1}^{k'_i} y_{ij})\right) = a \cdot c \cdot e + 2(d \cdot c) = a \cdot c \cdot e + d \cdot c + d \cdot c,$$

where $l_1 = 1,\ l_2 = 2,\ y_{11} = x_{11},\ y_{12} = y_{22} = 1 = c,\ y_{13} = x_{12}$, $y_{21} = x_{22}$.

Definition 2.3. We define the relation χ on $(R, +, \cdot)$ as follows:

$$x\ \chi\ y \Longleftrightarrow \exists (A, B) \in \Omega^1; \quad x \in A,\ y \in B.$$

Remark 2.4. The relation χ is reflexive and symmetric and $\beta, \Gamma \subseteq \chi$.

Let χ^* be the transitive closure of χ. In order to analyze the quotient hyperstructure with respect to this equivalence relation, we check that:

Lemma 2.5. *χ^* is a strongly regular equivalence relation both on $(R, +)$ and on $(R, \cdot)$.*

Proof. Clearly χ^* is an equivalence relation. In order to prove that it is strongly regular, it is enough to show that

$$x\ \chi\ y \Longrightarrow \begin{cases} x + a\ \overline{\overline{\chi}}\ y + a, & a + x\ \overline{\overline{\chi}}\ a + y, \\ x \cdot a\ \overline{\overline{\chi}}\ y \cdot a, & a \cdot x\ \overline{\overline{\chi}}\ a \cdot y\ , \end{cases}$$

for all $a \in R$. Since $x\ \chi\ y$, it follows that there exists $(A, B) \in \Omega^1$ such that $x \in A$ and $y \in B$. We distinguish the following situations:

Case 1. If $(A, B) \in \Omega^{\curlywedge,1}$, then there exists a pair

$$\left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i (\prod_{j=1}^{k'_i} y_{ij}) \right)$$

which satisfies the condition $\curlywedge$ and the corresponding permutation of this pair is $\sigma_i \in \mathbb{S}_{k_i}$ and also,

$$A = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad B = \sum_{i=1}^{n} l_i (\prod_{j=1}^{k'_i} y_{ij}).$$

We obtain

$$x + a \subseteq \left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \right) + a \quad \text{and} \quad y + a \subseteq \left(\sum_{i=1}^{n} l_i (\prod_{j=1}^{k'_i} y_{ij}) \right) + a.$$

Now, let $k_{n+1} = 1,\ x_{n+1\ 1} = a,\ k'_{n+1} = 1,\ y_{n+1\ 1} = a,\ l_{n+1} = 1$ and $\sigma_{n+1} = id$. Thus

$$x + a \subseteq \left(\sum_{i=1}^{n+1} (\prod_{j=1}^{k_i} x_{ij}) \right) \quad \text{and} \quad y + a \subseteq \left(\sum_{i=1}^{n+1} l_i (\prod_{j=1}^{k'_i} y_{ij}) \right).$$

It is easy to see that the pair $\left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i (\prod_{j=1}^{k'_i} y_{ij}) \right)$ satisfies the condition $\curlywedge$ and hence, this pair belongs to $\Omega^{\curlywedge,1} \subseteq \Omega^1$.

Therefore, for all $u \in x+a$ and $v \in y+a$, we have $u \in x+a \subseteq A+a$ and $v \in y+a \subseteq B+a$, so $u \; \chi \; v$, because $(A+a, B+a) \in \chi$. Thus $x+a \; \overline{\overline{\chi}} \; y+a$.

Case 2. If there exists a pair $\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k_i'} y_{ij})\right)$ satisfying the condition $\curlywedge$ and

$$x \in \sum_{i=1}^{n} l_i(\prod_{j=1}^{k_i'} y_{ij}) = B \quad \text{and} \quad y \in (\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) = A,$$

then according to *Case*1, $(A+a, B+a) \in (\Omega^{\curlywedge,1})^{-1}$ and so

$$(B+a, A+a) \in \Omega^{\curlywedge,1} \subseteq \Omega^{1}.$$

Thus $x+a \; \overline{\overline{\chi}} \; y+a$.

In the same way, we can show that $a+x \; \overline{\overline{\chi}} \; a+y$. It is easy to see that

$$a+x \; \overline{\overline{\chi^*}} \; a+y \quad \text{and} \quad x+a \; \overline{\overline{\chi^*}} \; y+a.$$

Notice that for $(R, \cdot)$ we have

Case 1. There exists a pair $\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k_i'} y_{ij})\right)$ which satisfies the condition $\curlywedge$ and

$$A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad B = \sum_{i=1}^{n} l_i(\prod_{j=1}^{k_i'} y_{ij}).$$

We obtain

$$x \cdot a \subseteq \left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})\right) \cdot a \quad \text{and} \quad y \cdot a \subseteq \left(\sum_{i=1}^{n} l_i(\prod_{j=1}^{k_i'} y_{ij})\right) \cdot a.$$

Which yields to

$$x \cdot a \subseteq \sum_{i=1}^{n}\left((\prod_{j=1}^{k_i} x_{ij}) \ \cdot a\right) \quad \text{and} \quad y \cdot a \subseteq \sum_{i=1}^{n} l_i\left((\prod_{j=1}^{k_i'} y_{ij}) \ \cdot a\right).$$

We set $f_i = k_i + 1$, $x_{if_i} = a$, $f_i' = k_i' + 1$, $y_{if_i'} = a$, $l_n = 1$ and we define

$$\tau_i(r) = \sigma_i(r) \ (\forall r = 1, \ldots, k_i') \ \textit{and} \ \tau_i(k_i'+1) = k_i'+1.$$

Hence $\tau_i \in \mathbb{S}_{f_i'}$ $(i = 1, \ldots, n)$. Thus

$$x \cdot a \subseteq \left(\sum_{i=1}^{n}(\prod_{j=1}^{f_i} x_{ij})\right) \quad \text{and} \quad y \cdot a \subseteq \left(\sum_{i=1}^{n} l_i(\prod_{j=1}^{f_i'} y_{ij})\right).$$

It is easy to see that the pair $\left(\sum_{i=1}^{n}(\prod_{j=1}^{f_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{f_i'} y_{ij})\right)$ satisfies the condition $\curlywedge$ and hence, this pair belongs to $\Omega^{\curlywedge,1} \subseteq \Omega^{1}$.

Therefore, for all $u \in x \cdot a$ and $v \in y \cdot a$, we have $u \in x \cdot a \subseteq A \cdot a$ and $v \in y \cdot a \subseteq B \cdot a$, so $u\ \chi\ v$, because $(A \cdot a, B \cdot a) \in \chi$. Thus $x \cdot a\ \overline{\overline{\chi}}\ y \cdot a$.

Case 2. If there exists a pair $\big(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\big)$ satisfying the condition $\curlywedge$ and

$$x \in \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}) = B \quad \text{and} \quad y \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) = A,$$

then $(A \cdot a, B \cdot a) \in (\Omega^{\curlywedge,1})^{-1}$ and so $(B \cdot a, A \cdot a) \in \Omega^1$. Thus $x \cdot a\ \overline{\overline{\chi}}\ y \cdot a$. In the same way, we can show that $a \cdot x\ \overline{\overline{\chi}}\ a \cdot y$. It is easy to see that

$$a \cdot x\ \overline{\overline{\chi^*}}\ a \cdot y \quad \text{and} \quad x \cdot a\ \overline{\overline{\chi^*}}\ y \cdot a.$$

Hence χ^* is a strongly regular equivalence relation. □

Theorem 2.6. *Let $(S, +, \cdot)$ be a ring with the multiplication identity element e'. Then $(S, +)$ is commutative.*

Proof. We must show that for all elements $a, b \in S$, $a + b = b + a$. We have

$$(a + b)(e' + e') = (a + b)e' + (a + b)e' = a + b + a + b,$$

$$(a + b)(e' + e') = a(e' + e') + b(e' + e') = a + a + b + b.$$

Therefore, $a + b + a + b = a + a + b + b$ whence $a + b = b + a$. □

Theorem 2.7. *The quotient R/χ^* is singleton or a commutative ring with the identity $\chi^*(1)$ such that for any $x \in R$, $m[\chi^*(x)] = \chi^*(0)$ where $\chi^*(0)$ is the zero element of R/χ^*.*

Proof. By Lemma 2.5, χ^* is a strongly regular equivalence relation; so the quotient structure R/χ^* is a ring with respect to the following operations:

$$\chi^*(x) \oplus \chi^*(y) = \chi^*(z), \quad \forall z \in x + y,$$

$$\chi^*(x) \otimes \chi^*(y) = \chi^*(t), \quad \forall t \in x \cdot y.$$

On the other hand, since $(x \cdot y, y \cdot x) \in \Omega^1$ it follows that

$$\chi^*(x) \otimes \chi^*(y) = \chi^*(y) \otimes \chi^*(x).$$

Moreover, for all $x \in R$ we have $(x, x \cdot 1) \in \Omega^{\curlywedge,1} \subseteq \Omega^1$. Hence for all $t \in x \cdot 1$, we have $x\ \chi\ t$ and so $\chi^*(x) = \chi^*(t)$. Therefore

$$\chi^*(x) \otimes \chi^*(1) = \chi^*(t) = \chi^*(x) = \chi^*(1) \otimes \chi^*(x).$$

Hence R/χ^* is a commutative ring with identity. Also, for all $x \in R$, $(x, (m+1)x) \in \Omega^1$ and thus for any $t \in (m+1)x$, we have $x\ \chi^*\ t$. Thus

$$\chi^*(x) = \chi^*(t) = (m + 1)[\chi^*(x)].$$

Hence, for every $x \in R$; $m[\chi^*(x)] = \chi^*(0)$ where $\chi^*(0)$ is the zero element of the ring R/χ^* and the proof is complete. □

Remark 2.8. By Theorem 2.7, we have $\gamma, \alpha \subseteq \chi$.

Corollary 2.9. *If the quotient R/α^* is a commutative ring with identity 1 and for all $x \in R$, $(m+1)x = x$ holds in the hyperring R, thus $\alpha^* = \chi^*$.*

Theorem 2.10. *The relation χ^* is the smallest equivalent relation such that the quotient R/χ^* is a commutative ring with the identity $\chi^*(1)$ and for any $x \in R$; $m[\chi^*(x)] = \chi^*(0)$.*

Proof. Let θ be an equivalence relation such that R/θ is a commutative ring with the identity $\theta(1)$ and for any $x \in R$, $m[\theta(x)] = \theta(0)$. Let $\phi_{1,m} : R \longrightarrow R/\theta$ be the canonical projection; so $\phi_{1,m}$ is a good homomorphism. Assume that $x \ \chi \ y$. Let there exist a pair $\big(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\big)$ which satisfies the condition $\curlywedge$ and $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$, $y \in B = \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})$. Therefore

$$\phi_{1,m}(x) = \oplus\sum_{i=1}^{n}(\otimes\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad \phi_{1,m}(y) = \oplus\sum_{i=1}^{n} l_i(\otimes\prod_{j=1}^{k'_i} y_{ij}).$$

Since $\phi_{1,m}(1) = \theta(1)$ is the identity element of commutative ring R/θ by Definition 2.1, and for any $x \in R$; $(m+1)[\theta(x)] = \theta(0)$, we get $\phi_{1,m}(x) = \phi_{1,m}(y)$. Similarly, if $(A,\ B)$ satisfy the condition $\curlywedge$ and $x \in B$, $y \in A$, we obtain $x \ \theta \ y$. Thus, $x \ \chi \ y$ implies that $x \ \theta \ y$. Finally, let $x \ \chi^* \ y$. Since θ is transitively closed, we obtain

$$x \in \chi^*(y) \Longrightarrow x \in \theta(y).$$

Therefore $\chi^* \subseteq \theta$. □

Example 2.11. Let $R = \{a, b, c, d, e, f\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperations $+$ and $\cdot$ are defined on R as follows:

$+$	a	b	c	d	e	f
a	a	$\{b,f\}$	c	d	e	$\{b,f\}$
b	$\{b,f\}$	c	d	e	a	c
c	c	d	e	a	$\{b,f\}$	d
d	d	e	a	$\{b,f\}$	c	e
e	e	a	$\{b,f\}$	c	d	a
f	$\{b,f\}$	c	d	e	a	c

$.$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	$\{b,f\}$	c	d	e	$\{b,f\}$
c	a	c	e	$\{b,f\}$	d	c
d	a	d	$\{b,f\}$	e	c	d
e	a	e	d	c	$\{b,f\}$	e
f	a	$\{b,f\}$	c	d	e	$\{b,f\}$

Set $m = 5$ and $1 = b$. It is easy to see that $R/\chi^* \cong \mathbb{Z}_5$. Hence, R/χ^* is a commutative ring with identity such that for any $\chi^*(x) \in R/\chi^*$ we have $5 \otimes \chi^*(x) = \chi^*(x)$. In this example $\chi^* = \Gamma^*$.

Example 2.12. Let

$$R = M_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\},\ 1 = h = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix},\ m = 1.$$

It is easy to see that R/χ^* is a commutative ring with identity $\chi^*(h)$ such that $R/\chi^* \cong \mathbb{Z}_2$ and $R/\alpha^* \cong \mathbb{Z}_3$. So $\chi^* \neq \alpha^*$. If $1 = h = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ and $m = 3$ then $R/\chi^* = R/\alpha^* \cong \mathbb{Z}_3$.

Example 2.13. Let

$$R_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\},\ R_2 = (\mathbb{Z}, +, \cdot),$$

$R = R_1 \times R_2$ and $1 = h' = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 2 \right) \in R$ and $m = 2$.

It is easy to see that R/χ^* is a commutative ring with identity $\chi^*(h)$ such that $R/\chi^* \cong \mathbb{Z}_3$ and R/α^* is an infinite commutative ring. So $\chi^* \neq \alpha^*$.

Definition 2.14. [13] Let S be a ring. If there exists a positive integer n such that for all $a \in S$, $na = 0$, then the smallest such that positive integer is called the characteristic of S. If no such positive integer exists, then S is said to be of characteristic zero.

Corollary 2.15. *Following Definition 2.1, we conclude the ring R/χ^* is of characteristic n such that $n \leq m$.*

Example 2.16. Let R be a non-empty set. We define $x+y = x \cdot y = R$ for every $x, y \in R$. If $m = 3$, then $\Gamma^* = \chi^* = R \times R$ and so $R/\chi^* = \{0\}$.

We recall that a K_R-semihypergroup is semihypergroup constructed from a semihypergroup $(R, +)$ and a family $\{A(x)\}_{x \in R}$ of non-empty and mutually disjoint subsets of R. Set $K_R = \bigcup_{x \in R} A(x)$ and define the hyperoperation $\oplus$ on K_R as follows:

$$\forall (a,b) \in K_R^2;\ a \in A(x),\ b \in A(y),\ a \oplus b = \bigcup_{z \in x+y} A(z).$$

Then $(R, +)$ is a hypergroup if and only if $(K_R, \oplus)$ is a hypergroup.

Theorem 2.17. *Let $(R, +, \cdot)$ be a hyperring. Then the following assertions hold:*

(i) $(K_R, \oplus, \circ)$ is a hyperring.
(ii) If $r \in R$ and $r' \in A_r$, then $R/\chi_r^* \cong R/\chi_{r'}^*$.

Proof. (i) We define the multiplicative hyperoperation $\circ$ as follows:

$$\forall (a,b) \in K_R^2;\ a \in A(x),\ b \in A(y),\ a \circ b = \bigcup_{z \in x.y} A(z).$$

So, by this definition, it is not difficult to see that $(K_R, \oplus, \circ)$ is a hyperring.

(ii) Let $r \in R$ and $r' \in A_r$. In this case we have

$$x\ \chi_r\ y \Longleftrightarrow A(x)\ \chi_{r'}\ A(y)$$

for every $(x,y) \in R^2$, and hence

$$R/\chi_r^* \cong K_R/\chi_{r'}^*.$$

□

Let $\phi : R \longrightarrow R/\chi^*$ be the canonical projection and let $D(R)$ be the *kernel* of ϕ, so if we denote the zero element of R/χ^* by $\overline{0}$, then $D(R) = \phi^{-1}(\overline{0})$.

Lemma 2.18. *Let R be a hyperring. Then*

$$R \cdot D(R) \subseteq D(R) \quad \textit{and} \quad D(R) \cdot R \subseteq D(R).$$

Proof. For all $a \in R \cdot D(R)$ there exists $r \in R$ and $x \in D(R)$ such that $a \in r \cdot x$. So

$$\chi^*(a) = \chi^*(r \cdot x) = \chi^*(r) \otimes \chi^*(x) = \chi^*(r) \otimes \overline{0} = \overline{0}.$$

Similarly, $D(R) \cdot R \subseteq D(R)$. □

Lemma 2.19. *If R is a Krasner hyperring, then*

$$\chi^*(0)=\overline{0} \quad \text{and} \quad \chi^*(-x)=-\chi^*(x) \quad \text{for all} \quad x\in R.$$

Theorem 2.20. *If R is a Krasner hyperring, then $D(R)$ is a hyperideal of R.*

Proof. We have $0\in D(R)$. Let $x,y\in D(R)$, then for every $z\in x+y$, we have $\chi^*(z)=\chi^*(x)\oplus\chi^*(y)=\overline{0}\oplus\overline{0}=\overline{0}$ which yields that $z\in D(R)$, and so $x+y\subseteq D(R)$. Since $x\in D(R)$, then there exists $-x\in R$ such that $0\in x-x$. So

$$D(R)=\chi^*(0)=\chi^*(x-x)=\chi^*(x)\oplus\chi^*(-x)=\overline{0}\oplus\chi^*(-x)=\chi^*(-x)$$

and hence $-x\in D(R)$. Therefore, $D(R)$ is a hyperideal of R. Also, it is clear that for any $r\in R$ and $x\in D(R)$ we have $r\otimes x\in D(R)$ and $x\otimes r\in D(R)$. □

3. The transitivity of the relation χ

In this section we determine some necessary and sufficient conditions for the relation χ to be transitive.

In what follows, we let M to be a non-empty subset of a hyperring $(R,+,\cdot)$.

Definition 3.1. We say that M is a χ^*-part of R if

$$\textstyle\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\bigcap M\neq\emptyset$$

implies that $\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\subseteq M$ for all $\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})$ such that

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\right) \quad \text{or} \quad \left(\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij}),\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\right)$$

satisfies the condition $\curlywedge$.

Using this notion we obtain the following characterization:

Proposition 3.2. *The following conditions are equivalent:*

(*i*) *M is a χ^*-part;*
(*ii*) *$x\in M$, $x\ \chi\ y\Longrightarrow y\in M$;*
(*iii*) *$x\in M$, $x\ \chi^*\ y\Longrightarrow y\in M$.*

Proof. $(i)\Rightarrow(ii)$ Let $(x,y)\in R^2$ be such that $x\in M$ and $x\ \chi\ y$. Suppose that there exists a pair $\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\right)$ which satisfies the condition $\curlywedge$ and is such that $x\in A=\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})$, $y\in B=\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})$. Since $x\in M$; $\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\cap M\neq\emptyset$ and so, we get $\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\subseteq M$ by Definition 3.1. Thus $y\in M$.

Similarly, if $\big(A=\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\ B=\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\big)$ satisfies the condition $\curlywedge$ and $x\in B$, $y\in A$, then $\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\cap M\neq\emptyset$ and so, we obtain $\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\subseteq M$, then $y\in M$ by Definition 3.1.

$(ii)\Rightarrow(iii)$ Let $(x,y)\in R^2$ be such that $x\in M$ and $x\ \chi^*\ y$. So, there exist $t\in\mathbb{N}$ and $(w_0=x,w_1,\dots,w_t=y)\in R^{t+1}$ such that

$$x=w_0\ \chi\ w_1\ \chi\ w_2\dots\chi\ w_{t-1}\ \chi\ w_t=y.$$

Since $x\in M$, applying (ii) t times, we obtain $y\in M$.

$(iii)\Rightarrow(i)$ let $\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\cap M\neq\emptyset$ and $\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})$ be such that

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\right)\quad\text{or}\quad\left(\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij}),\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\right)$$

satisfies the condition $\curlywedge$. Suppose that

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\right)$$

satisfies the condition $\curlywedge$. Since $\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})\cap M\neq\emptyset$; it follows that there exists $x\in\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})$. Let $y\in\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})$. From

$$(\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij}),\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij}))\in\Omega^{\curlywedge,1}\subseteq\Omega^1$$

it follows that $x\ \chi^*\ y$. Thus by (iii) we have $y\in M$ and so

$$\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})\subseteq M.$$

The proof of the other case is similar. □

Before proving the next theorem, we introduce the following notion.

Definition 3.3. Let $A=\sum_{i=1}^{n}(\prod_{j=1}^{k_i}x_{ij})$ and $B=\sum_{i=1}^{n}l_i(\prod_{j=1}^{k'_i}y_{ij})$. For all $n\geqslant 1$, set:

(N_1) $P_{\xi_n}(x)=\bigcup\{B\mid x\in A,(A,B)\ \textit{or}\ (B,A)\ \text{satisfies the condition}\ \curlywedge\}$;
(N_2) $P(x)=\bigcup_{n\geq 1}P_{\xi_n}(x)$.

Proposition 3.4. *For all $x\in R$, $P(x)=\{y\in R\mid x\ \chi\ y\}$.*

Proof. Let $x \in R$ and $y \in P(x)$. So, there exists $n \geq 1$ such that $y \in P_{\xi_n}(x)$. Thus, there exists $\sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})$ such that

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right) \quad \text{or} \quad \left(\sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}), \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})\right)$$

satisfies the condition $\curlywedge$ where

$$x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y \in \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}).$$

Therefore $\big(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\big) \in \Omega^{\curlywedge,1} \subseteq \Omega^1$ and hence, $x\ \chi\ y$. Thus $P(x) \subseteq \{y \in R \mid x\ \chi\ y\}$. The proof of the reverse of the inclusion is obvious. □

Lemma 3.5. *Let $(R, +, \cdot)$ be a hyperring and let M be a χ^*-part of R. If $x \in M$, then $P(x) \subseteq M$.*

Proof. If $y \in P(x)$, then $x\ \chi\ y$. Suppose that there exists a pair

$$\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})\right), \left(\sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}), \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})\right)$$

satisfies the condition $\curlywedge$ and is such that $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})$. Since $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \cap M$ and M is a χ^*-part, it follows by Definition 3.1 that

$$y \in \textstyle\sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}) \subseteq M$$

and so $y \in M$. Similarly, if $\big(\sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij}), \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})\big)$, satisfies the condition $\curlywedge$ and is such that $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in \sum_{i=1}^{n} l_i(\prod_{j=1}^{k'_i} y_{ij})$, then by Definition 3.1, we obtain $y \in M$. □

Theorem 3.6. *Let $(R, +, .)$ be a hyperring. The following conditions are equivalent:*

(*i*) *χ is transitive;*
(*ii*) *For any $x \in R$, $\chi^*(x) = P(x)$;*
(*iii*) *For any $x \in R$, $P(x)$ is a χ^*-part of R.*

Proof. $(i) \Rightarrow (ii)$ By Proposition 3.4, for all pairs $(x, y) \in R^2$ we have

$$y \in \chi^*(x) \Longleftrightarrow x\ \chi\ y \Longleftrightarrow y \in P(x).$$

$(ii) \Rightarrow (i)$ By Proposition 3.2, if M is a non-empty subset of R, then M is a χ^*-part of R if and only if it is a union of equivalence classes modulo χ^*. In particular, every equivalence class modulo χ^* is a χ^*-part of R.

$(ii) \Leftrightarrow (iii)$ Let $x\ \chi\ y$ and $y\ \chi\ z$, so $x \in P(y)$ and $y \in P(z)$ by Proposition 3.4. Since $P(z)$ is a χ^*-part, by Lemma 3.5, we have $P(y) \subseteq P(z)$ and hence $x \in P(z)$. Therefore, $x\ \chi\ z$ by Proposition 3.4 and the proof is complete. □

4. New strongly regular relation

In this section, we introduce and analyse a new smallest strongly regular relation on a hyperring R, denoted by σ_p^* such that R/σ_p^* is a p-ring.

Definition 4.1. For prime number p we define

$$\Re^{\Gamma,p} := \left\{ \left(\sum_{i=1}^{n} l'_i (\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}^{l_{ij}}) \right) \mid l'_i \in \{1, p+1\}, l_{ij} \in \{1, p\} \right\},$$

$$\Re_\Gamma^p := \Re^{\Gamma,p} \bigcup (\Re^{\Gamma,p})^{-1}.$$

Example 4.2. Let $p = 2$ and $R = \{a, b, c, d, e, f\}$. Let the hyperoperations $+$ and $\cdot$ are defined on R as follows:

$+$	a	b	c	d	e	f
a	$\{a, f\}$	b	c	$\{d, e\}$	$\{d, e\}$	$\{a, f\}$
b	b	$\{a, f\}$	$\{d, e\}$	c	c	b
c	c	$\{d, e\}$	$\{a, f\}$	b	b	c
d	$\{d, e\}$	c	b	$\{a, f\}$	$\{a, f\}$	$\{d, e\}$
e	$\{d, e\}$	c	b	$\{a, f\}$	$\{a, f\}$	$\{d, e\}$
f	$\{a, f\}$	b	c	$\{d, e\}$	$\{d, e\}$	$\{a, f\}$

$\cdot$	a	b	c	d	e	f
a	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$
b	$\{a, f\}$	b	$\{a, f\}$	b	b	$\{a, f\}$
c	$\{a, f\}$	c	$\{a, f\}$	c	c	$\{a, f\}$
d	$\{a, f\}$	$\{d, e\}$	$\{a, f\}$	$\{d, e\}$	$\{d, e\}$	$\{a, f\}$
e	$\{a, f\}$	$\{d, e\}$	$\{a, f\}$	$\{d, e\}$	$\{d, e\}$	$\{a, f\}$
f	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$	$\{a, f\}$

Then $(R, +, \cdot)$ is a non-commutative hyperring which is not a ring. For example, we have

$$\begin{aligned}&\{(a \cdot b + d \cdot c \cdot e, a \cdot b + d \cdot c \cdot e),\\ &(a \cdot b + a \cdot b + a \cdot b + d \cdot c \cdot e, a \cdot b + d \cdot c \cdot e),\\ &(a \cdot b + d \cdot c \cdot e, a \cdot a \cdot b + d \cdot c \cdot e \cdot e)\} \subseteq \Re_\Gamma^p.\end{aligned}$$

Moreover, if $\sum_{i=1}^{n} l'_i\left(\prod_{j=1}^{k_i} x_{ij}\right) = a\cdot b+d+d+d+c\cdot e\cdot f$ then $n=3$ and $(k_1,k_2,k_3)=(2,1,3)$ and $l'_1=1,\ l'_2=3,\ l'_3=1$. Now, for example,

$$(a\cdot b+d+d+d+c\cdot e\cdot f, a\cdot b^2+d^2+c\cdot e^2\cdot f^2)\in \Re^p_\Gamma.$$

In fact, we have $\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}}) = a\cdot b^2+d^2+c\cdot e^2\cdot f^2$, where $l_{11}=1,\ l_{12}=2,\ l_{21}=2,\ l_{31}=1,\ l_{32}=2$ and $l_{33}=2$.

Definition 4.3. We define the relation σ_p on $(R,+,\cdot)$ as follows:

$$x\ \sigma_p\ y \Longleftrightarrow\ \exists (A,B)\in \Re^\Gamma_p, \quad x\in A,\ y\in B.$$

Remark 4.4. The relation σ_p is reflexive and symmetric and $\beta,\Gamma\subseteq\sigma_p$ and also $\rho^n\subseteq\sigma_p$. Moreover, if for any $x\in R$, $(p+1)x=x$ and $x^p=x$, then $\Gamma=\sigma_p$.

Let σ_p^* be the transitive closure of σ_p. In order to analyse the quotient hyperstructure with respect to this equivalence relation, we state the following theorem.

Theorem 4.5. *σ_p^* is a strongly regular equivalence relation and the quotient R/σ_p^* is a ring such that for any $x\in R$ we have $[\sigma_p^*(x)]^p=\sigma_p^*(x)$ and $p[\sigma_p^*(x)]=\sigma_p^*(0)$, where $\sigma_p^*(0)$ the zero element of R/σ_p^*.*

Proof. Clearly σ_p^* is an equivalence relation. First, we will check that

$$x\sigma_p y\Longrightarrow \begin{cases} x+a\ \overline{\overline{\sigma_p}}\ y+a, & a+x\ \overline{\overline{\sigma_p}}\ a+y,\\ x\cdot a\ \overline{\overline{\sigma_p}}\ y\cdot a, & a\cdot x\ \overline{\overline{\sigma_p}}\ a\cdot y\ , \end{cases}$$

for all $a\in R$. From $x\sigma_p y$, it follows that there exists $(A,B)\in\Re^\Gamma_p$ such that $x\in A$ and $y\in B$. If $(A,B)\in\Re^{\Gamma,p}$, then $A=\sum_{i=1}^{n} l'_i(\prod_{j=1}^{k_i} x_{ij})$ and $B=\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}})$. Then

$$x+a\subseteq\left(\sum_{i=1}^{n} l'_i(\prod_{j=1}^{k_i} x_{ij})\right)+a \quad\text{ and }\quad y+a\subseteq\left(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}})\right)+a.$$

Now, let $k_{n+1}=1$, $x_{n+1\ 1}=a$, $l'_{n+1}=1$ and $l_{n+1\ 1}=1$. Thus

$$x+a\subseteq\left(\sum_{i=1}^{n+1} l'_i(\prod_{j=1}^{k_i} x_{ij})\right) \quad\text{ and }\quad y+a\subseteq\left(\sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}})\right).$$

Therefore, for all $u\in x+a\subseteq A+a$ and $v\in y+a\subseteq B+a$ we have $u\sigma_p v$, because $(A+a,B+a)\in\Re^\Gamma_p$. Thus $x+a\ \overline{\overline{\sigma_p}}\ y+a$. In the same way we can show that $a+x\ \overline{\overline{\sigma_p}}\ a+y$. It is easy to see that

$$a+x\ \overline{\overline{\sigma_p^*}}\ a+y \quad\text{ and }\quad x+a\ \overline{\overline{\sigma_p^*}}\ y+a.$$

Now, notice that

$$x \cdot a \subseteq \left(\sum_{i=1}^{n} l'_i (\prod_{j=1}^{k_i} x_{ij}) \right) \cdot a \quad \text{and} \quad y \cdot a \subseteq \left(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}^{l_{ij}}) \right) \cdot a.$$

We set $k'_i = k_i + 1$, $x_{ik'_i} = a$, $l'_i = 1$ for any $1 \leq i \leq n$ and $l_{ik'_i} = 1$. Thus

$$x \cdot a \subseteq \left(\sum_{i=1}^{n} l'_i (\prod_{j=1}^{k'_i} x_{ij}) \right) \quad \text{and} \quad y \cdot a \subseteq \left(\sum_{i=1}^{n} (\prod_{j=1}^{k'_i} x_{ij}^{l_{ij}}) \right).$$

Therefore, for all $u \in x \cdot a \subseteq A \cdot a$ and $v \in y \cdot a \subseteq B \cdot a$ we have $u\sigma_p v$, because $(A \cdot a, B \cdot a) \in \Re_p^{\Gamma}$. Thus $x \cdot a \ \overline{\overline{\sigma_p}}\ y \cdot a$ and Then $a \cdot x \ \overline{\overline{\sigma_p}}\ a \cdot y$. If $x \in B$ and $y \in A$ in the same way the proof is hold. Therefore, the quotient R/σ_p^* is a ring under the following operations:

$$\sigma_p^*(x) \oplus \sigma_p^*(y) = \sigma_p^*(z), \quad \forall z \in x + y,$$

$$\sigma_p^*(x) \otimes \sigma_p^*(y) = \sigma_p^*(t), \quad \forall t \in x \cdot y.$$

On the other hand, for all $x \in R$ since $((p+1)x, x) \in \Re_p^{\Gamma}$, it follows that $(p+1)[\sigma_p^*(x)] = \sigma_p^*(x)$. Moreover, for any $x \in R$, $(x, x^p) \in \Re_p^{\Gamma}$, so $\sigma_p^*(x) = [\sigma_p^*(x)]^p$. □

Corollary 4.6. *If $(R, +, \cdot)$ is a hyperfield, then R/σ_p^* is a field with $[\sigma_p^*(x)]^{p-1} = \sigma_p^*(1)$. Moreover, if $p = 2$ then $R/\sigma_p^* = \{\sigma_p^*(0), \sigma_p^*(1)\}$.*

Definition 4.7. [13] A ring in which every element has characteristic power of the prime p and $x^p = x$ is called a p-ring.

Corollary 4.8. *Let p be the smallest prime number such that R/σ_p^* is a ring with $[\sigma_p^*(x)]^p = \sigma_p^*(x)$ and $p[\sigma_p^*(x)] = \sigma_p^*(0)$ for any $x \in R$. Then R/σ_p^* is a p-ring.*

Theorem 4.9. *The quotient R/σ_p^* with identity is a commutative ring.*

Proof. See [13]. □

Corollary 4.10. *Let R be a hyperring which has at least an unitary element. Then the quotient R/σ_p^* is a commutative ring.*

Definition 4.11. Let $(R, +, \cdot)$ be a hyperring. We say that R is n-idempotent if there exists a constant n, where $2 \leq n \in \mathbb{N}$ such that $x \in x^n$, for all $x \in R$.

Corollary 4.12. *Let $(R, +, \cdot)$ be an p-idempotent hyperring for a prime number p. Then in Definition 2.1, if $m = p$ and $\sigma_i = id$, then R/χ^* is a p-ring with identity $\chi^*(1)$.*

Lemma 4.13. *Every idempotent element of the quotient ring R/σ_p^* is in the center of R/σ_p^*.*

Proof. Let $\bar{0} = \sigma_p^*(0)$ the zero element of R/σ_p^* and $\bar{x} = \sigma_p^*(x) \in R/\sigma_p^*$ for all $x \in R$. Let $\bar{y}$ be an idempotent in the ring R/σ_p^* and $\bar{x}$ an arbitrary element of R/σ_p^*. Then

$$[(\bar{y} \otimes \bar{x} \otimes \bar{y}) \oplus (-\bar{y} \otimes \bar{x})]^2 = \bar{0}.$$

Thus $\bar{y} \otimes \bar{x} \otimes \bar{y} = \bar{y} \otimes \bar{x}$, and a similar calculation shows that

$$\bar{y} \otimes \bar{x} \otimes \bar{y} = \bar{x} \otimes \bar{y}.$$

Hence $\bar{y} \otimes \bar{x} = \bar{x} \otimes \bar{y}$ and $\bar{y}$ is in the center. □

Theorem 4.14. *σ_p^* is the smallest equivalence relation such that the quotient R/σ_p^* is a p-ring.*

Proof. Let θ be a strongly regular equivalence such that the quotient R/σ_p^* is a p-ring. Let $\phi_p : R \longrightarrow R/\theta$ be the canonical projection, which is good homomorphism. We check that $x\sigma_p y$ implies that $x\theta y$. If there exists a pair $\left(A = \sum_{i=1}^{n} l_i'(\prod_{j=1}^{k_i} x_{ij}), B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}})\right)$ and $x \in A = \sum_{i=1}^{n} l_i'(\prod_{j=1}^{k_i} x_{ij})$, $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}^{l_{ij}})$, then we obtain

$$\phi_p(x) = \oplus \sum_{i=1}^{n+1} l_i' \left(\otimes \prod_{j=1}^{k_i} x_{ij}) \right) \quad \text{and} \quad \phi_p(y) = \oplus \sum_{i=1}^{n+1} \left(\otimes \prod_{j=1}^{k_i} x_{ij}^{l_{ij}} \right).$$

By the p-ring of R/θ, it follow that $\phi_p(x) = \phi_p(y)$. Hence $x\theta y$. Similarly, if $x \in B$ and $y \in A$, we obtain $x\theta y$. Finally, let $x\sigma_p^* y$. Since θ is transitive, we obtain

$$x \in \sigma_p^*(y) \Longrightarrow x \in \theta(y).$$

Therefore $\sigma_p^* \subseteq \theta$. □

Example 4.15. Let $p = 5$ and $R = \{a, b, c, d, e, f\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperations $+$ and $\cdot$ are defined on R as follows:

$+$	a	b	c	d	e	f
a	a	$\{b,f\}$	c	d	e	$\{b,f\}$
b	$\{b,f\}$	c	d	e	a	c
c	c	d	e	a	$\{b,f\}$	d
d	d	e	a	$\{b,f\}$	c	e
e	e	a	$\{b,f\}$	c	d	a
f	$\{b,f\}$	c	d	e	a	c

$.$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	$\{b,f\}$	c	d	e	$\{b,f\}$
c	a	c	e	$\{b,f\}$	d	c
d	a	d	$\{b,f\}$	e	c	d
e	a	e	d	c	$\{b,f\}$	e
f	a	$\{b,f\}$	c	d	e	$\{b,f\}$

It is easy to see that $R/\sigma_5^* \cong \mathbb{Z}_5$. Hence, R/σ_5^* is a commutative ring with identity such that for any $\sigma_5^*(x) \in R/\sigma_5^*$ we have $5 \otimes \sigma_5^*(x) = \sigma_5^*(0)$ and $[\sigma_5^*(x)]^5 = \sigma_5^*(x)$.

Example 4.16. Let $p = 2$ and $R = \{a, b, c, d, e\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperations $+$ and $\cdot$ are defined on R as follows:

$+$	a	b	c	d	e
a	a	$\{b,e\}$	c	d	$\{b,e\}$
b	$\{b,e\}$	a	d	c	a
c	c	d	a	$\{b,e\}$	d
d	d	c	$\{b,e\}$	a	c
e	$\{b,e\}$	a	d	c	a

$.$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{b,e\}$	a	$\{b,a\}$	$\{b,e\}$
c	a	a	c	c	a
d	a	$\{b,e\}$	c	d	$\{b,e\}$
e	a	$\{b,e\}$	a	$\{b,e\}$	$\{b,e\}$

It is easy to see that $R/\sigma_2^* \cong (P(X), \triangle, \cap)$ such that $X = \{1, 2\}$ and $P(X)$ power set of X. It is easy to show that $(P(X), \triangle, \cap)$ is p-ring. Hence R/σ_2^* is a p-ring where for all $\sigma_2^*(x) \in R/\sigma_2^*$ we have $2 \otimes \sigma_2^*(x) = \sigma_2^*(0)$ and $[\sigma_2^*(x)]^2 = \sigma_2^*(x)$.

Example 4.17. Let R be the hyperring in Example 2.16. Set $T = \mathbb{Z}_3 \times \mathbb{Z}_2 \times R$ and $p = 3$. Then we have

$$T/\alpha^* \cong \mathbb{Z}_3 \times \mathbb{Z}_2, \qquad T/\sigma_3^* \cong \mathbb{Z}_3.$$

So $\{(x, x)|x \in T\} \subset \alpha^* \subset \sigma_3^* \subset T \times T$.

5. Conclusions

The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures. In the theory of hyperrings, fundamental relations make a connection between hyperrings and ordinary rings. In this paper, we introduced and studied the smallest equivalence binary relation on a general hyperring R which was denoted by χ^* such that the quotient R/χ^* is a commutative ring with identity and of characteristic m. The characterizations of commutative rings with identity via strongly regular relations is investigated and some properties on the topic are presented. Finally, we introduced a new strongly regular relation which was denoted by σ^* such that the quotient structure is a p-ring. Several properties, examples and characterizations of the strongly regular relation σ^*, especially in quotient structure on general hyperrings, have been investigated.

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NEW FUNDAMENTAL RELATIONS IN HYPERRINGS AND THE CORRESPONDING QUOTIENT STRUCTURES

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رابطه‌های اساسی جدید در ابرحلقه‌ها و ساختارهای خارج‌قسمتی متناظر با آن‌ها

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هدف اصلی این مقاله معرفی و تجزیه و تحلیل کوچکترین رابطه دوتایی هم‌ارزی به نام χ^* روی ابرحلقه R است به‌طوری‌که حلقه خارج‌قسمتی R/χ^*، مجموعه تمام کلاس‌های هم‌ارزی، یک حلقه جابجایی با عضو همانی و از مشخصه m است. برخی خصوصیات حلقه‌های جابجایی با عنصر همانی را از طریق این رابطه‌ی منظم قوی مورد بررسی قرار داده و ویژگی‌های آن‌ها را در این زمینه ارائه می‌کنیم. به‌علاوه، یک رابطه منظم قوی جدید به نام σ_p^* را معرفی کرده به‌طوری‌که ساختار خارج قسمتی آن یک p–حلقه است.

کلمات کلیدی: ابرحلقه، p–حلقه، رابطه منظم قوی.