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ON TRANSINVERSE OF MATRICES AND ITS APPLICATIONS

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Abstract. Given a matrix A with the elements from a field of characteristic zero, the transinverse $A^{\#}$ of A is defined as the transpose of the matrix obtained by replacing the non-zero elements of A by their inverses and leaving zeros, if any, unchanged. We discuss the properties of this matrix operation in some detail and as an important application, we reinvent the celebrated matrix tree theorem for gain graphs. Characterization of balance in connected gain graphs using its Laplacian matrix becomes an immediate consequence.

1. Introduction

In this article, we introduce a new operation on matrices and discuss its properties. As an important application, the matrix tree theorem for gain graphs is established and characterization of balance in connected gain graphs becomes an immediate consequence. A gain graph is a graph where the edges are given some prescribed orientation and labelled with the elements (called gains) from a group, so that the gains are inverted when we reverse the direction of the edges. Through out this paper, $F^{\times}$, where F is a field of characteristic zero, denotes the multiplicative group of the non-zero elements in F. All the graphs G in this paper are finite and simple. The notation $\Phi = (G, F^{\times}, \varphi)$ denotes a gain graph Φ with the underlying graph G, the underlying

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group $F^\times$ and the gain function φ. For definitions and other details for gain graphs, one may refer to [5]. We use the notation $v \sim u$ or $v \sim e$ according as the vertex v is adjacent to the vertex u or incident with the edge e. The adjacency matrix $A(\Phi) = (a_{ij})$ of a gain graph $\Phi = (G, F^\times, \varphi)$, is defined as

$$a_{ij} = \begin{cases} \varphi(v_i v_j), & \text{if } \overrightarrow{v_i v_j} \in \overrightarrow{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

and $a_{ji} = (\varphi(v_i v_j))^{-1}$. The *Laplacian matrix* of a gain graph Φ is defined as $(\Phi) = D(\Phi) - A(\Phi)$, where $D(\Phi)$ is the degree matrix of Φ. Note that $D(\Phi) = D(G)$ and for a vertex v of Φ, its degree $d(v) = \sum_{e \in E : v \sim e} 1$.

A signed graph [6] can be viewed as a gain graph with the underlying group being the multiplicative subgroup $\{1, -1\}$ of $F^\times$ and a graph as the gain graph with the underlying group as $\{1\}$.

The *gain* $\varphi(C)$, of a cycle $C : v_0 v_1 \dots v_n v_0$, is the product of the gains of its edges. i.e., $\varphi(C) = \varphi(v_0 v_1)\varphi(v_1 v_2) \dots \varphi(v_n v_0)$. A gain graph $\Phi = (G, F^\times, \varphi)$ is said to be *cycle balanced* or simply *balanced*, if $\varphi(C) = 1$ for all cycles C in it. More details regarding the notion of balance in signed and gain graphs and its various applications can be obtained from [5].

The following theorem which deals with the spectral characterization of cycle balance in gain graphs, found in [3], is a significant extension of the well-known theorem of Acharya [1] for signed graphs to gain graphs.

Theorem 1.1 ([3]). *If $\Phi = (G, F^\times, \varphi)$ is a gain graph, then Φ is balanced if and only if Φ and G have the same eigenvalues.*

2. Main Results

2.1. **The transinner product of vectors and the transinverse of matrices.** A few definitions and some new notations are essential to build the theory for the transinverse operation on matrices. First we define a^θ for $a \in F$ by

$$a^\theta = \begin{cases} a^{-1}, & \text{if } a \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

An important observation from the definition is $aa^\theta = 1$ if $a \neq 0$ and 0 otherwise. Also $aa^\theta = bb^\theta$, if and only if, both a and b become simultaneously zeros or non-zeros. For a vector

$$v = (v_1, v_2, \dots, v_n) \in F^n,$$

we define $v^\theta \in F^n$ by $v^\theta = (v_1^\theta, v_2^\theta, \dots, v_n^\theta)$. The *straight inner product* of two vectors $u = (u_1, u_2, \dots, u_n), v = (v_1, v_2, \dots, v_n) \in F^n$ is the function $\langle\ ,\rangle : F^n \times F^n \to F$ given by $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$. Also the function $[\ ,] : F^n \times F^n \to F$ is defined as their *transinner product* given by $[u, v] = \sum\limits_{i=1}^{n} u_i v_i^\theta = \langle u, v^\theta\rangle$. The reader may note that the straight inner product coincides with the usual inner product in the case of $\mathbb{R}^n$ but not for $\mathbb{C}^n$. However, the transinner product, in the case of both $\mathbb{R}^n$ and $\mathbb{C}^n$, coincides with the usual inner product when the co-ordinates of the vectors belong to $\{1, 0, -1\}$. Also, the transinner product coincides with the Hermitian inner product in $\mathbb{C}^n$ when the numbers are restricted to $\mathbb{T} \cup \{0\}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z|^2 = z\overline{z} = 1\}$, a subgroup of $\mathbb{C}^\times$. If we adopt the usual notation e_i to represent the vector in F^n with i^{th} co-ordinate as 1 and others as zeros, then $e_i^\theta = e_i$ and $[e_i, e_j] = \langle e_i, e_j\rangle = \delta_{ij}$.

As usual, we denote by $M_{m\times n}(F)$, the space of all matrices of order $m \times n$ over the field F and $M_n(F)$ denotes the space of all square matrices of order n. Given a matrix $A = (a_{ij}) \in M_{m\times n}(F)$, we denote by A^θ, the matrix of the same order defined by $A^\theta = (a_{ij}^\theta)$ and define its transpose as the *transinverse* of A, denoted by $A^\#$. i.e., $A^\# = (A^\theta)^T = (A^T)^\theta$.

For a $(0, 1)$ or $(0, 1, -1)$-matrix A, $A^\#$ is A^T, the transpose of A. Also if we consider matrices with the entries from $\mathbb{T} \bigcup \{0\}$, then $A^\#$ is A^*, the conjugate transpose of A. The next proposition follows easily from the definition of $A^\#$.

Proposition 2.1. *For $A = (a_{ij}) \in M_n(F)$, $A = A^\#$ if and only if $a_{ij}a_{ji} \in \{0, 1\}$.*

Let us call a matrix $A \in M_n(F)$ satisfying $A = A^\#$ as a *transymmetric matrix*. The adjacency matrix $A(\Phi)$ of a gain graph Φ is a transymmetric matrix. Though, in general, the transinverse operation does not satisfy many of the usual properties enjoyed by the transpose of a matrix or conjugate transpose of a matrix for that matter, it has useful properties like $(A^\#)^\# = A$ and $(\alpha A)^\# = \alpha^\theta A^\#$. Meanwhile it is interesting to note that rank(A) need not be equal to rank$(A^\#)$ in general. In fact, for the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 3 & 3 & 5 \end{bmatrix} \in M_3(\mathbb{R})$, rank$(A) = 2$ whereas rank$(A^\#) = 3$. However, in the case of $2 \times m$ matrices, it is easy to see that rank$(A^\#) = $ rank(A). As such it is natural to pose the following question on this matter.

Problem 2.2. Give necessary and sufficient condition(s) for

$$A \in M_{m\times n}(F)$$

to satisfy $\mathrm{rank}(A^{\#}) = \mathrm{rank}(A)$.

2.2. **Properties of the transinner product and the transinverse.** The proof of the following theorem is omitted as the results follow easily from the definitions.

Theorem 2.3. (i) $\Big[\sum_{i=1}^{n} \alpha_i u_i, v\Big] = \sum_{i=1}^{n} \alpha_i [u_i, v]$.

(ii) $[\alpha u, \beta v] = \alpha\beta^{\theta}[u, v]$

(iii) $[u, u]$ *equals the number of non-zero co-ordinates in* u *and hence* $[u, u] = 0$ *if and only if* $u = 0$. *Also* $[\alpha u, \alpha u] = [u, u]$ *if* $\alpha \neq 0$. *Moreover, if* Y *is a column matrix then* $Y^{\#}Y$ *gives the number of non-zero elements in* Y *and hence* $Y^{\#}Y = O$ *if and only if* $Y = O$.

(iv) $u = \sum_{i=1}^{n} [u, e_i] e_i$ *and* $u^{\theta} = \sum_{i=1}^{n} [e_i, u] e_i$

(v) *If* $A \in M_{m\times n}(F)$ *then* $AA^{\#} = O$ *(and similarly* $A^{\#}A = O$*) if and only if* $A = O$.

Note that a significant difference of the transinner product from the straight inner product is that the linearity is lost in the second component as $[u, \alpha v + \beta w]$ need not be equal to $\alpha^{\theta}[u, v] + \beta^{\theta}[u, v]$ in general and hence the reversal law fails in the case of matrix products. i.e., $(AB)^{\#} \neq B^{\#}A^{\#}$ in general (See Theorem 2.6 for a special case where it is true).

As a passing reference before we move on to the next section, let us make some remarks for further research. Denoting the i-th row of a matrix A by R_i, note that the matrix $AA^{\#} = ([R_i, R_j])$. This one resembles in definition with the Grammian matrix $(\langle R_i, R_j\rangle)$ with the straight inner product in action for the latter in place of the transinner product for the former. Also, it would be worthwhile to explore the properties of those matrices A satisfying $AA^{\#} = A^{\#}A$ (we call them *transnormal* matrices) and $AA^{\#} = A^{\#}A = I$ (we call them *transunitary* matrices). Also looking on the spectral properties of $A^{\#}$ would also give rise to many new ideas. Though we wish to explore these ideas somewhere else in great detail, the following simple results are some interesting observations.

Theorem 2.4. *If* $D = diag(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *where each* $\alpha_i \neq 0$*, then* $DD^{\#} = I_n = D^{\#}D$.

Theorem 2.5. *If* P *is a permutation matrix, then* $P^{\#} = P^{-1} = P^T$.

The following result shows a particular case where the reversal law holds good.

Theorem 2.6. *If* P *is a permutation matrix and* D *is a diagonal matrix, then* $(PD)^{\#} = D^{\#}P^{\#}$.

Theorem 2.7. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(F)$ *is transnormal if and only if one of the following conditions is satisfied.*

(i) $b = c = 0$

(ii) $b, c \neq 0$ *and either* $a = d$ *or* $ad + bc = 0$.

Proof. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} if and only if A^{\#} = \begin{bmatrix} a^\theta & c^\theta \\ b^\theta & d^\theta \end{bmatrix}$. Then, a simple calculation provides that $AA^{\#} = \begin{bmatrix} aa^\theta + bb^\theta & ac^\theta + bd^\theta \\ ca^\theta + db^\theta & cc^\theta + dd^\theta \end{bmatrix}$ and similarly $A^{\#}A = \begin{bmatrix} aa^\theta + cc^\theta & a^\theta b + c^\theta d \\ ab^\theta + d^\theta c & bb^\theta + dd^\theta \end{bmatrix}$. Equating $AA^{\#}$ and $A^{\#}A$, first we observe that $bb^\theta = cc^\theta$. Now this would imply that either b and c both zeros or b and c are both non-zeros. In case $b = c = 0$, we may have a and d as arbitrary. In the second case when b and c are both non-zero, the equality $AA^{\#} = A^{\#}A$ gives on simple computation that either $a = d$ or $ad + bc = 0$. The converse follows easily on actual evaluations of the products based on the two conditions along with the fact that $bb^\theta = cc^\theta$. □

Now we characterize transunitary matrices in the following theorem.

Theorem 2.8. $A \in M_n(F)$ *is transunitary if and only if there exists a permutation matrix* P *such that* $A = PD$ *where* D *is a diagonal matrix with none of its diagonal elements are zeros.*

Proof. If A is transunitary, then $AA^{\#} = I$ implies that each row should have exactly one non-zero scalar and column in which that scalar exists should have other elements as zeros. This proves that A is a permutation of diagonal matrix, diagonal of which contains no zeros. That is, $A = PD$ where D is a diagonal matrix with none of its diagonal elements are zeros and P is a a permutation matrix. Conversely suppose that $A = PD$. This gives $A^{\#} = D^{\#}P^{\#}$. Then an easy computation of $AA^{\#}$ and $A^{\#}A$ completes the proof. □

Corollary 2.9. *$A \in M_2(F)$ is transunitary if and only if either $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ or $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ for non-zero a, b, c and d.*

An important class of matrices, known as circulant matrices belong to the set of transnormal matrices.

Theorem 2.10. *The circulant matrices are transnormal.*

Proof. Note the fact that $C^{\#}$ is also a circulant matrix whenever C is. So they commute and the proof is complete. □

Theorem 2.11. *Any tridiagonal Toeplitz matrix is transnormal.*

Proof. A tridiagonal Toeplitz matrix A can be expressed as

$$A = aI + bU + cL,$$

so that $A^{\#} = a^{\theta}I + c^{\theta}U + b^{\theta}L$ where I is the identity matrix with the order as that of A and U and L are respectively the upper and lower triangular matrices with entry one at proper places. An easy verification gives $AA^{\#} = A^{\#}A$. □

2.3. **Geometric significance of the transinner product.** Inner product is generally used for dealing with the orthogonality of vectors and the resultant ideas like projection and other allied concepts. Here we make an attempt to study the geometrical significance of the transinner product. As $[u, v] \neq [v, u]$ in general, we define for $S \subseteq F^n$, two associated sets $S^L = \{v \in F^n : [v, u] = 0 \quad \forall u \in S\}$ and $S^R = \{v \in F^n : [u, v] = 0 \quad \forall u \in S\}$. When $S = \{0\}$, then $S^L = S^R = F^n$ and when $S = F^n$, then $S^L = S^R = \{0\}$.

Proposition 2.12. *S^L is a subspace of F^n.*

Proof. As $[\alpha v_1 + \beta v_2, u] = \alpha[v_1, u] + \beta[v_2, u]$, S^L is a subspace of F^n. □

Though S^R in general need not be a subspace, we have the following.

Proposition 2.13. *$(S^R)^{\theta}$ is a subspace of F^n.*

Proof. Take $v^{\theta}, w^{\theta} \in (S^R)^{\theta}$. This implies that $v, w \in S^R$ or $[u, v] = 0$ and $[u, w] = 0$ for all $u \in S$. Now

$$[u, (\alpha v^{\theta} + \beta w^{\theta})^{\theta}] = \langle u, \alpha v^{\theta} + \beta w^{\theta} \rangle = \alpha[u, v] + \beta[u, w] = 0.$$

Thus $(\alpha v^{\theta} + \beta w^{\theta})^{\theta} \in S^R$ or $\alpha v^{\theta} + \beta w^{\theta} \in (S^R)^{\theta}$ which completes the proof. □

3. Applications: Matrix tree theorem for gain graphs and its consequences

In [2], Seth Chaiken deals with the matrix tree theorem for signed and gain graphs in a much more general setting. The matrix tree theorem for signed graphs is discussed in detail by Zaslavsky [6] also. Recently, Yi Wang et.al., [4] discussed the determinant of the Laplacian matrix of a complex unit gain graph. In this section, we discuss another way of dealing with the matrix tree theorem for gain graphs (of course with gains from the multiplicative group of a field of characteristic zero) using the transiverse operation applied to the incidence matrices.

For an oriented edge $\vec{e} = \overrightarrow{v_i v_k}$, we take v_i as the tail of edge $\vec{e}$, denoted by $t(\vec{e})$ and v_k as the head of edge $\vec{e}$, denoted by $h(\vec{e})$.

The (oriented) *incidence matrix* of a gain graph $\Phi = (V, \vec{E}, \varphi)$ defined as the $|V| \times |E|$ matrix $\mathrm{H} = \mathrm{H}(\Phi) = (\eta_{v\vec{e}})$ given by

$$\eta_{ve} = \begin{cases} 1 & \text{if } t(\vec{e}) = v, \\ -(\varphi(\vec{e}))^{-1} & \text{if } h(\vec{e}) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\eta_{v_i\vec{e}}\eta^{\theta}_{v_k\vec{e}} = -\varphi(\vec{e})$ whenever there is an edge $\vec{e} = \overrightarrow{v_i v_k} \in \vec{E}$ and $\eta_{v_i\vec{e}}\eta^{\theta}_{v_k\vec{e}} = 0$ if v_i or v_k is not incident with the edge $\vec{e}$.

Theorem 3.1. *If $\Phi = (G, F^{\times}, \varphi)$ is a gain graph where $G = (V, \vec{E})$, then $L(\Phi) = \mathrm{H}(\Phi)\mathrm{H}^{\#}(\Phi)$.*

Proof. Let $V = \{v_1, v_2, \dots v_n\}$ be the vertex set and $\vec{E} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m\}$ be the edge set. Let $\mathrm{H}(\Phi) = (\eta_{v_i\vec{e}_j})$ and $\mathrm{H}^{\#}(\Phi) = (\eta^{\theta}_{v_i\vec{e}_j})^T$. Let us represent each row of $\mathrm{H}(\Phi)$ by a vector $R_{v_i} \in F^m$. Now the $(i,j)^{th}$-entry of $\mathrm{H}(\Phi)\mathrm{H}^{\#}(\Phi) = [R_{v_i}, R_{v_j}] = \sum_{\vec{e}\in E} \eta_{v_i\vec{e}}\eta^{\theta}_{v_j\vec{e}}$.

If $i = j$, then this sum becomes $[R_{v_i}, R_{v_i}]$ which is the number of non-zero elements on this row R_{v_i} which is of course $\sum_{\vec{e}\in\vec{E}: v_i\sim\vec{e}} 1 = deg(v_i)$. If $i \neq j$, since we deal with simple graphs, the sum then becomes $\eta_{v_i\vec{e}_k}\eta^{\theta}_{v_j\vec{e}_k}$. This value is $-\varphi(\vec{e}_k)$ or 0, according as $\vec{e}_k$ is the edge joining v_i and v_j or 0, otherwise.

As such, according to the definition of the adjacency matrix, it is the additive inverse of the (i,j)-th entry of $A(\Phi)$. Thus

$$\mathrm{H}(\Phi)\mathrm{H}^{\#}(\Phi) = D(\Phi) - A(\Phi) = L(\Phi).$$

$\square$

Theorem 3.2. *For a gain cycle Φ_C of length n,*

$$\det(\mathrm{H}(\Phi_C)) = (-1)^n[(\varphi(C))^{-1} - 1]$$

and

$$\det(L(\Phi_C)) = 2 - [\varphi(C) + \varphi(C)^{-1}]$$

Proof. Let $C : v_1\vec{e}_2v_2\vec{e}_3v_3\cdots v_{n-1}\vec{e}_nv_n\vec{e}_1v_1$ be the given cycle. Note that $\mathrm{H}(\Phi_C)$ is a square matrix of order n given by

$$\mathrm{H}(\Phi_C) = \begin{bmatrix} -(\varphi(\vec{e}_1))^{-1} & 1 & 0 & \cdots & 0 \\ 0 & -(\varphi(\vec{e}_2))^{-1} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & -(\varphi(\vec{e}_n))^{-1} \end{bmatrix}$$

Define $D = diag(-\varphi(\vec{e}_1), -\varphi(\vec{e}_2), \cdots, -\varphi(\vec{e}_n))$, then

$$\mathrm{H}(\Phi_C)D = \begin{bmatrix} 1 & -\varphi(\vec{e}_2) & 0 & \cdots & 0 \\ 0 & 1 & -\varphi(\vec{e}_3) & \cdots & 0 \\ \cdots & \cdots & \cdots & -\varphi(\vec{e}_4) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\varphi(\vec{e}_n) \\ -\varphi(\vec{e}_1) & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Expanding along the first column

$$\begin{aligned} \det(\mathrm{H}(\Phi_C)D) &= 1 + (-1)^{n-1}(-1)^n\varphi(e_1)\varphi(e_2)\cdots\varphi(e_n) \\ &= 1 - \varphi(C) \end{aligned}$$

Therefore,

$$\begin{aligned} \det(\mathrm{H}(\Phi_C)) &= (1 - \varphi(C))(\det(D))^{-1} \\ &= (1 - \varphi(C))(-1)^n(\varphi(C))^{-1} \\ &= (-1)^n[(\varphi(C))^{-1} - 1] \end{aligned}$$

Similarly $D^{\#}\mathrm{H}^{\#}(\Phi_C)$ can be computed by expanding along the first row to get

$$\det(D^{\#}\mathrm{H}^{\#}(\Phi_C)) = 1 - (\varphi(C))^{-1}$$

which leads to the result $\det(\mathrm{H}^{\#}(\Phi_C)) = (-1)^n[\varphi(C) - 1]$.

Thus, $\det(L(\Phi_C)) = \det(\mathrm{H}(\Phi_C)\mathrm{H}^{\#}(\Phi_C)) = \det(\mathrm{H}(\Phi_C))\det(\mathrm{H}^{\#}(\Phi_C))$ makes $\det(L(\Phi_C)) = 2 - [\varphi(C) + \varphi(C)^{-1}]$. □

Now we define a 1-tree as a connected unicyclic graph and a 1-forest as a disjoint union of 1-trees. To see how the oriented incidence matrix comes into real action, we define an *essential spanning subgraph* Ψ of a gain graph Φ as the spanning subgraph whose union of components form a 1-forest. i.e., a spanning 1-forest is termed as an essential spanning subgraph. We denote by $Œ(\Phi)$, the class of essential spanning subgraphs of Φ. If $Œ(\Phi) = \emptyset$, then we define the sum $\sum_{\Psi \in Œ(\Phi)}$ to be zero. Note that if Ψ is an essential spanning subgraph, then the incidence matrix of Ψ will be a square matrix since $|E(\Psi)| = |V(\Phi)|$.

The direct sum of matrices $A_{m\times n}$ and $B_{p\times q}$, denoted by $A \oplus B$, is a matrix of order $(m+p) \times (n+q)$ which is defined as the block diagonal matrix $diag(A, B)$. i.e., $A \oplus B = diag(A, B) = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$. If there are more than two matrices, say $A_1, A_2, \cdots A_n$, then their direct sum is denoted by $\bigoplus\limits_{i=1}^{n} A_i$. Note that if each constituent matrix is a square matrix, then $\det(\bigoplus\limits_{i=1}^{n} A_i) = \prod\limits_{i=1}^{n} \det(A_i)$. In the same way, we use the notation $A \odot B$ for what we call as the *semidirect sum* of square matrices A and B which is an upper triangular block matrix where the block matrices on the diagonal are still A and B and the other non-zero block matrices in the upper triangular positions may be arbitrary. Here also it can be seen that $\det(\bigodot\limits_{i=1}^{n} A_i) = \prod\limits_{i=1}^{n} \det(A_i)$ where each A_i is a square matrix.

Lemma 3.3. *If $\Phi = (G, F^{\times}, \varphi)$ is a gain graph where $G = (V, E)$ is a 1-tree with the unique cycle C, then* $\det(L(\Phi)) = 2 - [\varphi(C) + \varphi(C)^{-1}]$.

Proof. Define the orientation of edges not on the cycle C as, for $i < j$, the edge $\overrightarrow{e_{i,j}}$ has tail $t(\overrightarrow{e_{i,j}}) = v_i$ and head $h(\overrightarrow{e_{i,j}}) = v_j$. Then,

$$\mathrm{H}(\Phi) = \bigodot_{i=1}^{k} A_i, k = n - l + 1$$

where A_1 is the incidence matrix corresponding to the cycle C of length l and the remaining A_i's are square matrices of the form $[(-\varphi(e))^{-1}]$ of order 1 for the edges e not on the cycle C. Thus,

$$\det(\mathrm{H}(\Phi)) = \prod_{i=1}^{k} \det(A_i) = (-1)^{l(C)}[((\varphi(C))^{-1} - 1] \prod_{e \notin C} (-\varphi(e))^{-1}$$

Similarly

$$\det(\mathrm{H}^{\#}(\Phi)) = (-1)^{l(C)}[\varphi(C) - 1] \prod_{e \notin C} (-\varphi(e))$$

Thus we get, $\det(L(\Phi)) = 2 - [\varphi(C) + \varphi(C)^{-1}]$. ∎

Lemma 3.4. *If $\Phi = (G, F^\times, \varphi)$ is a gain graph where $G = (V, E)$ is a 1-forest, then $\det(L(\Phi)) = \prod_{C \in \Psi}(2 - [\varphi(C) + \varphi(C)^{-1}])$, where the product is taken over all the component 1-trees Ψ of Φ.*

Proof. By suitably relabelling the vertices and edges, if necessary, the incidence matrix $\mathrm{H}(\Phi)$ can be brought into the form $\bigoplus_{i=1}^{k} H_i$ where each H_i is the incidence matrix of the corresponding 1-tree component of the 1-forest and k is the number of components of the forest. Then $\det(\mathrm{H}(\Phi)) = \prod_{i=1}^{k} \det(H_i)$. To complete the proof it is enough to apply Lemma 3.3 above. $\square$

Lemma 3.5. *Let $\Phi = (G, F^\times, \varphi)$ be a gain graph on n vertices and Ψ be a spanning subgraph of Φ having exactly n edges. Then $\det(L(\Psi)) \neq 0$ implies Ψ is an essential spanning subgraph of Φ.*

Proof. Let Ψ be a spanning subgraph of Φ having exactly $n = V(\Phi)$ edges and let $\det(L(\Psi)) \neq 0$. To prove that Ψ is an essential spanning subgraph of Φ, we have to prove that the components of Ψ are 1-trees. By a suitable reordering of the vertices and edges, the matrix $L(\Psi)$ can be brought to the form $\bigoplus_{i=1}^{k} L_i$ where L_i corresponds to the components of Ψ. Thus, $\det(L(\Psi)) = \prod_{i=1}^{k} \det(L_i))$.
If Ψ contains an isolated vertex, then the matrix $L(\Psi)$ has a zero row which implies $\det(L(\Psi)) = 0$, and if some component is a tree for some i, then also $\det(L_i)) = 0$ which implies $\det(L(\Psi)) = 0$. Hence, both these cases lead to a contradiction.
Thus we have to prove that if A_k is a component of Ψ, then A_k have same number of edges and vertices.
Suppose A_k, for some k, has p vertices and $p + t$ edges where $t \geq 1$. Then the $n - p$ vertices and $n - p - t$ edges not in A_k form either a tree or a disconnected graph having trees as components. Both these cases lead to $\det(L(\Psi)) = 0$, a contradiction. Hence the fact that all the components of Ψ have same number of edges and vertices gives the conclusion that the components of Ψ are 1-trees. That is, Ψ is an essential spanning subgraph of Φ. $\square$

Now the stage is set to have yet another version of the matrix tree theorem for gain graphs.

Theorem 3.6. *If $\Phi = (G, F^\times, \varphi)$ is a connected gain graph, then*
$$\det(L(\Phi)) = \sum_{\Psi \in \mathcal{E}(\Phi)} \prod_{C \in \Psi}(2 - [\varphi(C) + \varphi(C)^{-1}]).$$

Proof. By Theorem 3.1, $\det(L(\Phi)) = \det(\mathrm{H}(\Phi)\mathrm{H}^{\#}(\Phi))$ which on applying Binet-Cauchy theorem and Lemma 3.5 becomes,

$$\begin{aligned}\det(L(\Phi)) &= \sum_{\Psi\in Œ(\Phi)} \det(\mathrm{H}(\Psi))\det(\mathrm{H}^{\#}(\Psi))\\ &= \sum_{\Psi\in Œ(\Phi)} \det(L(\Psi))\\ &= \sum_{\Psi\in Œ(\Phi)} \prod_{C\in\Psi}(2-[\varphi(C)+\varphi(C)^{-1}]).\end{aligned}$$

by using Lemma 3.3 and Lemma 3.4. □

3.1. **Characterization of balance in connected gain graphs.** Before we attempt at general results, the following two lemmas are easy to prove for the trees and the 1-trees. Note that the defintion of the rank of a matrix A, denoted by $\mathrm{rank}(A)$, is the order of largest non-zero minor of A. i.e., the order of the largest square submatrix of A having non-zero determinant.

Theorem 3.7. *Let $\Phi = (G, F^{\times}, \varphi)$ be a connected unbalanced gain graph of order n. Then $\mathrm{rank}(\mathrm{H}(\Phi)) = n$ and $\mathrm{rank}(\mathrm{H}^{\#}(\Phi)) = n$.*

Proof. Let $C = v_1\vec{e_1}v_2\vec{e_2}\dots\vec{e_l}v_1$ be the unbalanced cycle in Φ of length $l, (3 \le l \le n)$ in Φ. Since C is unbalanced, $\varphi(C) \ne 1$. Now choose a spanning 1-tree subgraph, say Ψ, of Φ with C as the unique cycle. Such a selection is possible since Φ is connected. By suitable labeling of vertices and edges, let $\vec{e_{l+1}},\dots,\vec{e_n}$ be the remaining edges in Ψ such that for $i < j$, the edge $e = \overrightarrow{v_iv_j}$ has tail $t(\vec{e}) = v_i$ and head $h(\vec{e}) = v_j$.

Now consider the $n\times n$ submartix of $\mathrm{H}(\Psi)$ obtained by the columns corresponding to the edges in Ψ. Then a simple computation of the determinant gives

$$\det(L(\Psi)) = (-1)^{l(C)}[((\varphi(C))^{-1}-1]\prod_{e\notin C}(-\varphi(e))^{-1}$$

which is not equal to 0, where $l(C)$ denotes the length of the cycle C. This implies, $\mathrm{rank}(\mathrm{H}(\Psi)) = n$ and hence $\mathrm{rank}(\mathrm{H}(\Phi)) = n$. In a similar way,

$$\det(\mathrm{H}^{\#}(\Phi)) = (-1)^{l(C)}[\varphi(C)-1]\prod_{e\notin C}(-\varphi(e)) \ne 0$$

which implies, $\mathrm{rank}(\mathrm{H}^{\#}(\Phi)) = n$. □

Theorem 3.8. *Let $\Phi = (G, F^{\times}, \varphi)$ be a connected balanced gain graph of order n. Then $\mathrm{rank}(\mathrm{H}(\Phi)) = n-1$ and $\mathrm{rank}(\mathrm{H}^{\#}(\Phi)) = n-1$.*

Proof. Since Φ is connected, we can consider a spanning tree of order $n-1$ of Φ and prove in a similar way as that of the above. $\square$

The above results actually prove the following result found in [5] in a different way where $b(\Phi)$ denotes the number of balancing components in the gain graph Φ. In our discussion we deal with connected gain graphs and as such $b(\Phi)$ is either zero or 1 according as Φ is unbalanced or balanced.

Lemma 3.9 ([5],Theorem 2.1)**.** rank$(\mathrm{H}(\Phi)) = n - b(\Phi)$.

Theorem 3.10. *If* $\Phi = (G, F^{\times}, \varphi)$ *is a connected gain graph, then it is balanced if and only if* $\det(L(\Phi)) = 0$.

Proof. If Φ is balanced, then every cycle C in Φ satisfies the condition that $2 - [\varphi(C) + (\varphi(C))^{-1}] = 0$, hence the matrix tree theorem gives $\det(L(\Phi)) = 0$. Conversely, assuming that $\det(L(\Phi)) \neq 0$ implies that $\mathrm{rank}(L(\Phi)) = n$ and hence $\mathrm{rank}(\mathrm{H}(\Phi)) = n$ implying that $b(\Phi) = 0$ using the formula in Lemma 3.9 and it in turn gives Φ is unbalanced in this case. $\square$

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ON TRANSINVERSE OF MATRICES AND ITS APPLICATION

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در باب تراوارون ماتریسها و کاربردهای آن

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به ازای یک ماتریس داده شده A با درایه‌ها از یک میدان با مشخصه‌ی صفر، تراوارون A، نشان داده شده توسط $A^{\#}$، به عنوان ترانهاده‌ی ماتریس حاصل از A با جایگزینی درایه‌های ناصفر با وارون‌شان و عدم تغییر درایه‌های صفر، در صورت وجود، تعریف می‌شود. ویژگی‌های این عملگر ماتریسی را با جزئیاتی بحث و بررسی کرده و و به عنوان کاربردی مهم قضیه‌ی نام آشنای درخت ماتریسی برای گراف‌های (برچسب‌دار یالی) سوددار را بازدید خواهیم کرد. نتیجه‌ای آنی از این بازدید مشخص‌سازی‌ای از تعادل گراف‌های سوددار همبند با استفاده از ماتریس لاپلاسی گراف‌ها است.

کلمات کلیدی: گراف سوددار، گراف علامت‌دار، مقادیر ویژه گراف، گراف لاپلاسی.