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SOME ALGEBRAIC AND MEASURE THEORETIC PROPERTIES OF THE RINGS OF MEASURABLE FUNCTIONS

H. Hejazipour and A. R. Naghipour*

Abstract. Let $M(X, \mathcal{A}, \mu)$ be the ring of real-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$. We show that the maximal ideals of $M(X, \mathcal{A}, \mu)$ are associated with the special measurable sets in $\mathcal{A}$. We also study some other algebraic properties of $M(X, \mathcal{A}, \mu)$.

1. Introduction

A *σ-algebra* on a set X is a collection $\mathcal{A}$ of subsets of X that includes the empty subset which is closed under complement and countable unions. If $\mathcal{A}$ is a σ-algebra on X, then $(X, \mathcal{A})$ is called a *measurable space* and the members of $\mathcal{A}$ are called the *measurable sets* in X. A function μ from a σ-algebra $\mathcal{A}$ to the interval $[0, +\infty]$ is called a *measure* if for all countable collections $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. To avoid trivialities, we shall assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{A}$. A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where X is a set, $\mathcal{A}$ a σ-algebra on X, and μ a measure on $\mathcal{A}$. If Y is a topological space and $f : X \longrightarrow Y$ is a function, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y. The statement "P holds almost everywhere on $(X, \mathcal{A}, \mu)$" (abbreviated to "P holds a.e. on $(X, \mathcal{A}, \mu)$") means that

$$\mu\big(\{x \in X : P \text{ does not hold on } x\}\big) = 0.$$

The sets of real-valued measurable functions with pointwise addition and multiplication are commutative rings with identity. Rings of real-valued measurable functions have been studied in many ways for a long time by many mathematicians (see for example [1, 2, 3, 4, 6, 7, 11, 12, 13, 15, 17, 18, 19]). For notational convenience, we assume that $M(X, \mathcal{A}, \mu)$ is the ring of measurable functions from X to the real line $\mathbb{R}$ with arbitrary σ-algebra $\mathcal{A}$ on X

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*Corresponding author.

and arbitrary measure μ on $\mathcal{A}$. In [9] and [10], Hejazipour and Naghipour presented some properties of $M(X, \mathcal{A}, \mu)$.

We recall that a *complete measure space* is a measure space in which every subset of every set of measure zero is measurable. The *characteristic function* is the function $\chi_A : X \longrightarrow \{0, 1\}$, which for a given measurable set A, has value 1 at elements of A and 0 at elements of $X \backslash A$. For every measurable function f, the *zero set* and the *cozero set* of f are $Z_f := \{x \in X : f(x) = 0\}$ and $\mathrm{co}Z_f := \{x \in X : f(x) \neq 0\}$, respectively. For $a \in X$ and $A \in \mathcal{A}$,

$$I(a) := \{f \in M(X, \mathcal{A}, \mu) : f(a) = 0\}$$

and $I(A) := \{f \in M(X, \mathcal{A}, \mu) : f(x) = 0 \ \forall x \in A\}$ are the ideals of functions that vanish on a and A, respectively. If J is an ideal of $M(X, \mathcal{A}, \mu)$, $V(J) := \{x \in X : f(x) = 0 \ \forall f \in J\}$ is the common vanishing set of the measurable functions in J. The reader is referred to [5, 8, 14, 16] for undefined terms and concepts.

This paper is organized as follows. In Section 2, we investigate the maximal ideals of the rings of real-valued measurable functions with respect to the measures (see Theorems 2.4 and 2.7). In Section 3, we study some algebraic properties of these rings.

2. Maximal ideals of $M(X, \mathcal{A}, \mu)$

To enter into the discussion, we present a basic difference between the rings of real-valued measurable functions with respect to the measures, $M(X, \mathcal{A}, \mu)$, and the rings of real-valued measurable functions without attention to the measures, $M(X, \mathcal{A})$. It is easy to see that if $a \in X$, then $\{f \in M(X, \mathcal{A}) : f(a) = 0\}$ is a maximal ideal of $M(X, \mathcal{A})$. The following theorem shows that this is not true in the case of $M(X, \mathcal{A}, \mu)$.

Theorem 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $\{a\}$ be a measurable set and $\mu(\{a\}) = 0$. Then $I(a)$ is not a maximal ideal of $M(X, \mathcal{A}, \mu)$.*

Proof. We show that $I(a)$ is not a proper ideal with respect to the measure μ. Suppose that $f \in M(X, \mathcal{A}, \mu)$ and $f(a) \neq 0$. We put

$$g(x) := \begin{cases} f(x) & x \neq a, \\ 0 & x = a. \end{cases}$$

Since $\{a\}$ and $X \backslash \{a\}$ are measurable sets and f is measurable, g is measurable. Now by the definition of g, we have

$$\mu\big(\{x \in X : f(x) \neq g(x)\}\big) = \mu(\{a\}) = 0.$$

This means that $f = g$ a.e. on $(X, \mathcal{A}, \mu)$ and so $f \in I(a)$ a.e. on $(X, \mathcal{A}, \mu)$. Therefore, $I(a) = M(X, \mathcal{A}, \mu)$ a.e. on $(X, \mathcal{A}, \mu)$. $\square$

It is expected that for every $a \in X$ such that $\{a\}$ is measurable and $\mu(\{a\}) \neq 0$, $I(a)$ is a maximal ideal of $M(X, \mathcal{A}, \mu)$. But these ideals are only a small part of the maximal ideals of $M(X, \mathcal{A}, \mu)$. We claim that Theorem 2.4 is the most logical extension of this matter. First, we present an important definition for studying the maximal ideals of $M(X, \mathcal{A}, \mu)$.

Definition 2.2. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, $N \in \mathcal{A}$ and $\mu(N) \neq 0$. The set N is *near-zero* if for every subset $A \subseteq N$ such that $\mu(A) \neq 0$, $A = N$ a.e. on $(X, \mathcal{A}, \mu)$.

If $\mathcal{A} = \{\emptyset, X\}$, then X is a near-zero set. In this paper, to avoid trivialities, we shall also assume that $\mathcal{A} \neq \{\emptyset, X\}$.

Notation 2.3. Let $(X, \mathcal{A}, \mu)$ be a measure space. We set

$$\mathcal{N}_\mu := \big\{N \in \mathcal{A} : N \text{ is a near-zero set in } \mathcal{A}\big\}.$$

In the following theorem, we show that every ideal of a near-zero set is a maximal ideal of $M(X, \mathcal{A}, \mu)$.

Theorem 2.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space and N be a near-zero set. Then $I(N)$ is a maximal ideal of $M(X, \mathcal{A}, \mu)$.*

Proof. Suppose that J is an ideal of $M(X, \mathcal{A}, \mu)$ such that

$$I(N) \subseteq J \subseteq M(X, \mathcal{A}, \mu) \text{ and } f \in J \backslash I(N).$$

We define

$$g(x) := \begin{cases} 0 & x \in N, \\ f(x) - 1 & x \in N^c. \end{cases}$$

Since f is measurable and $N \in \mathcal{A}$, g is a measurable function. By the definition of g, $g \in I(N) \subseteq J$ and so $f - g \in J$. We claim that $f - g$ is a unit in $M(X, \mathcal{A}, \mu)$. For this purpose, it is sufficient to show that $\mu(\{x \in N : f(x) = 0\}) = 0$. We put

$$A := \big\{x \in N : f(x) = 0\big\},$$

and

$$B := \big\{x \in N : f(x) \neq 0\big\}.$$

Since f is measurable, $f^{-1}(\{0\})$ and $f^{-1}(\mathbb{R} \backslash \{0\})$ are measurable sets and so $A = f^{-1}(\{0\}) \cap N$ and $B = f^{-1}(\mathbb{R} \backslash \{0\}) \cap N$ are measurable. On the other hand, N is a near-zero set and so $\mu(A) = 0$ or $\mu(B) = 0$. If $\mu(B) = 0$, then

$f \in I(N)$ a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction. Thus $\mu(A) = 0$ and hence $\mu(\{x \in X : (f-g)(x) = 0\}) = 0$. This means that $f - g$ is a unit element in $M(X, \mathcal{A}, \mu)$ and so $J = M(X, \mathcal{A}, \mu)$ a.e. on $(X, \mathcal{A}, \mu)$. □

As an immediate consequence of Theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $a \in X$ and $\mu(\{a\}) \neq 0$. Then $I(a)$ is a maximal ideal of $M(X, \mathcal{A}, \mu)$.*

Proof. Since $\mu(\{a\}) \neq 0$, $\{a\}$ is a near-zero set. Therefore, $I(a)$ is a maximal ideal of $M(X, \mathcal{A}, \mu)$ by Theorem 2.4. □

The following example shows that the converse of Theorem 2.4 is not true.

Example 2.6. Let X be the real line $\mathbb{R}$, $\mathcal{A}$ be the Lebesgue σ-algebra on $\mathbb{R}$ and μ be the Lebesgue measure on $\mathcal{A}$. We claim that the Lebesgue σ-algebra has not any near-zero set. Suppose on the contrary that N is a near-zero set. Then there exists $a, b \in \mathbb{R}$ such that $N \subseteq (a, b)$. We put:

$$A_1 := \big\{x \in (a,b) : x \in N, |x-a| \leq |b-a|/2\big\}.$$

Since the absolute value function and its transfers $f_a := |x-a|$ are measurable, $A_1 = f_a^{-1}((-\infty, |b-a|/2))$ is a measurable set. On the other hand, N is a near-zero set and so $A_1 = N$ a.e on $(X, \mathcal{A}, \mu)$ or $N \backslash A_1 = N$ a.e on $(X, \mathcal{A}, \mu)$. Without loss of generality, we assume that $A_1 = N$ a.e on $(X, \mathcal{A}, \mu)$. Again we set:

$$A_2 := \big\{x \in (a,b) : x \in N, |x-a| \leq |b-a|/4\big\}.$$

Similarly, $A_2 = N$ a.e on $(X, \mathcal{A}, \mu)$ or $N \backslash A_2 = N$ a.e on $(X, \mathcal{A}, \mu)$. By repeating this process, for every $n \in \mathbb{N}$ there exists a measurable set

$$A_n := \big\{x \in (a,b) : x \in N, |x-a| \leq |b-a|/2^n\big\}$$

such that $A_n = N$ a.e on $(X, \mathcal{A}, \mu)$ and $\mu(A_n) \leq 2^{-n}$. This implies that for every $n \in \mathbb{N}$, $\mu(N) \leq 2^{-n}$ and hence $\mu(N) = 0$, which is a contradiction. This means that $\mathcal{A}$ has not any near-zero set and so every maximal ideal of $M(X, \mathcal{A}, \mu)$ is not the ideal of a near-zero set.

In the following theorem, we try to present a converse of Theorem 2.4 by the properties of σ-algebras and measures.

Theorem 2.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mathcal{A}$ be a countable set, $\mathcal{N}_\mu \neq \emptyset$ and every measurable cover of X has a finite subcover. Then every maximal ideal of $M(X, \mathcal{A}, \mu)$ is the ideal of a near-zero set.*

Proof. Let J be a maximal ideal of $M(X, \mathcal{A}, \mu)$. In the first step, we show that for every $x \in X$, there exists a near-zero set N_x such that $x \in N_x$. Suppose that $x \in X$ and $N \in \mathcal{N}_\mu$. we consider four cases:

Case 1: $\{x\}$ is measurable and $\mu(\{x\}) \neq 0$. Then $\{x\}$ is a near-zero set.

Case 2: $\{x\}$ is measurable and $\mu(\{x\}) = 0$. Then $N \cup \{x\}$ is a near-zero set.

Case 3: $\{x\}$ is not measurable. we set

$$\mathcal{C} := \big\{B \in \mathcal{A} : N \cup \{x\} \subseteq B\big\}.$$

$\mathcal{C}$ is nonempty since $X \in \mathcal{C}$. Partially order $\mathcal{C}$ by

$$B_1 \leq B_2 \Longleftrightarrow B_1 \supseteq B_2.$$

Assume that $\{B_i\}_{i \in I}$ be a chain of members in $\mathcal{C}$. Let $B := \bigcap_{i \in I} B_i$. Since $\mathcal{A}$ is countable, B is a measurable set. Therefore, B is an upper bound of the chain $\{B_i\}_{i \in I}$. Thus the hypotheses of Zorn's Lemma are satisfied and hence $\mathcal{C}$ contains a maximal element N_x. Since $N \subseteq N_x$, $\mu(N_x) \geq \mu(N) > 0$. If N_x is not a near-zero set, there exist disjoint measurable sets A and B such that $\mu(A)$ and $\mu(B)$ are not zero and $N_x = A \cup B$. Without loss of generality, we consider four cases:

Case 1: $N \subseteq A$ and $x \in A$. Then $N \cup \{x\} \subseteq A$ and $\mu(N_x \backslash A) = \mu(B) \neq 0$, which is a contradiction. Therefore N_x is a near-zero set contains x.

Case 2: $N \subseteq A$, $x \in B$ and B is not near-zero. Then there exist two measurable sets K and L such that $x \in K$, $\mu(K) \neq 0$, $\mu(L) \neq 0$ and $B = K \cup L$. This means that $N \cup \{x\} \subseteq A \cup K$ and

$$\mu(N_x \backslash (A \cup K)) = \mu(L) \neq 0,$$

which is a contradiction. Therefore N_x is a near-zero set contains x.

Case 3: $N \subseteq A$, $x \in B$ and B is a near-zero set. Then B is a near-zero set contains x.

Case 4: $N \nsubseteq A$ and $N \nsubseteq B$ a.e. on $(X, \mathcal{A}, \mu)$. Then

$$\mu(N) = \mu(N \cap A) + \mu(N \cap B) = 0,$$

which is a contradiction. Therefore N_x is a near-zero set contains x.

In the second step, we claim that $V(J)$ is not empty. Suppose to the contrary that $V(J)$ is empty. Then for each $x \in X$, there exists $f_x \in J$ such that $f_x(x) \neq 0$. By the first step, for every $x \in X$, there exists a near-zero set N_x such that $x \in N_x$. We put:

$$K := \big\{x \in X : \mu(N_x \cap \mathrm{co}Z_{f_x}) = 0\big\}.$$

By the hypotheses, we have:

$$\mu(K) \leq \mu\Bigg(\bigcup_{x\in K}(N_x \cap \mathrm{co}Z_{f_x})\Bigg) \leq \sum_{x\in K}\mu(N_x \cap \mathrm{co}Z_{f_x}) = 0.$$

This means that $\{N_x\}_{x\in X\backslash K}$ is a measurable cover for X a.e. on $(X, \mathcal{A}, \mu)$. If $N_x, N_y \in \{N_x\}_{x\in X\backslash K}$, then we consider two cases:

Case 1: $\mu(N_x\cap N_y) = 0$. Then N_x and N_y are disjoint sets a.e. on $(X, \mathcal{A}, \mu)$.

Case 2: $\mu(N_x \cap N_y) \neq 0$. Then $N_x \cap N_y = N_x = N_y$ a.e. on $(X, \mathcal{A}, \mu)$, since N_x and N_y are near-zero set.

This means that we can extract a finite disjoint cover N_{x_1}, N_{x_2},...,N_{x_k} for X, since every measurable cover of X has a finite subcover. By Lemma 2.3 in [9], for every $i = 1, 2, ..., k$, the following function is measurable:

$$g_{x_i}(t) := \begin{cases} 1/f_{x_i}(t) & t \in \mathrm{co}Z_{f_{x_i}}, \\ 0 & t \in Z_{f_{x_i}}. \end{cases}$$

It follows that $f_{x_i}g_{x_i} = \chi_{\mathrm{co}Z_{f_{x_i}}} \in J$ and hence $\chi_{N_{x_i}}\chi_{\mathrm{co}Z_{f_{x_i}}} = \chi_{N_{x_i}} \in J$, for every $i = 1, 2, \ldots, k$. Therefore, $h := \sum_{i=1}^{k} \chi_{N_{x_i}}$ is a unit element in J, which is a contradiction.

In the third step, we claim that $V(J)$ is measurable and $\mu(V(J)) \neq 0$. For every $f \in J$, $f^{-1}(\{0\})$ is measurable. Since $\mathcal{A}$ is countable, $V(J) = \bigcap_{f\in J} f^{-1}(\{0\})$ is a measurable set. If $\mu(V(J)) = 0$, then $X\backslash V(J)$ is measurable and $X = X\backslash V(J)$ a.e. on $(X, \mathcal{A}, \mu)$. Similar to the second step, J has a unit, which is a contradiction.

In the fourth step, we show that there exists a near-zero set N such that $N \subseteq V(J)$. Suppose to the contrary that for every near-zero set N, $N \cap V(J) = \emptyset$ a.e. on $(X, \mathcal{A}, \mu)$. By the hypotheses, the first step and the proof of the second step, there exist disjoint members $N_1, N_2, ..., N_k \in \mathcal{N}_\mu$ such that $X = N_1 \cup N_2 \cup ... \cup N_k$. Now

$$\begin{aligned} \mu\Big(V(J)\Big) &= \mu\Big(V(J) \cap (\bigcup_{i=1}^{k} N_i)\Big) \\ &= \mu\Big(\bigcup_{i=1}^{k}(V(J) \cap N_i)\Big) \\ &= \sum_{i=1}^{k}\mu\Big(V(J) \cap N_i\Big) \\ &= 0, \end{aligned}$$

which is a contradiction.

In the final step, with the help of the fourth step, there exists a near-zero set N_0 such that $N_0 \subseteq V(J)$. Therefore, $J \subseteq I(V(J)) \subseteq I(N_0)$. By Theorem 2.4, $I(N_0)$ is a maximal ideal of $M(X, \mathcal{A}, \mu)$ and so $J = I(N_0)$. □

Remark 2.8. We have not found any example of a measure space to show that the conditions in Theorem 2.7 is necessarily. The lack of such counterexamples, motivates the following question: Can we drop some conditions in Theorem 2.7?

At the end of this section, we give an example to show that measures have played an important role in the structure of the maximal ideals of $M(X, \mathcal{A}, \mu)$.

Example 2.9. Assume that X is the real line $\mathbb{R}$ and $\mathcal{A} = P(\mathbb{R})$, the power set of $\mathbb{R}$, is the σ-algebra on $\mathbb{R}$. We put two measures on the measurable space $(\mathbb{R}, \mathcal{A})$.

(a) For any $B \in \mathcal{A}$, the *counting measure* μ_1 on this measurable space is the measure defined by

$$\mu_1(B) := \begin{cases} n(B) & \text{if } B \text{ is finite,} \\ +\infty & \text{if } B \text{ is infinite.} \end{cases}$$

It is easy to check that μ_1 is a measure on $(\mathbb{R}, \mathcal{A})$. For every $a \in \mathbb{R}$, $\mu(\{a\}) = 1$ and so $\{a\}$ is a near-zero set. This means that

$$\mathcal{N}_{\mu_1} = \big\{\{a\} : a \in \mathbb{R}\big\}.$$

By Theorem 2.4, for every $a \in \mathbb{R}$, $I(a)$ is a maximal ideal of $M(\mathbb{R}, \mathcal{A}, \mu_1)$.

(b) Fix $x_0 \in \mathbb{R}$ and for every $B \in \mathcal{A}$, the *Dirac measure* is

$$\mu_2(B) := \begin{cases} 1 & x_0 \in B, \\ 0 & x_0 \notin B. \end{cases}$$

For every $x \neq x_0$, $\mu_2(\{x\}) = 0$ and so $I(x)$ is not a maximal ideal of $M(\mathbb{R}, \mathcal{A}, \mu_2)$, by Theorem 2.1. If $K, L \in \mathcal{A}$ and $x_0 \in K \cap L$, then $K = L$ a.e. on $(X, \mathcal{A}, \mu_2)$. This implies that

$$\mathcal{N}_{\mu_2} = \big\{B \in \mathcal{A} : x_0 \in B\big\}.$$

By Theorem 2.4, for every $B \in \mathcal{A}$ such that $x_0 \in B$, $I(B)$ is a maximal ideal of $M(\mathbb{R}, \mathcal{A}, \mu_2)$.

3. Some algebraic properties of $M(X, \mathcal{A}, \mu)$

In this section, we study some algebraic properties of the rings of real-valued measurable functions with respect to the measures. The first theorem is about the variety of the prime ideals.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, P be a prime ideal of $M(X, \mathcal{A}, \mu)$ and $V(P) \in \mathcal{A}$. Then $\mu(V(P)) = 0$ or $V(P)$ is a near-zero set.*

Proof. Assume that $\mu(V(P)) \neq 0$. Assume to the contrary that $V(P)$ is not near-zero. Then there exist disjoint measurable sets A and B such that $\mu(A) \neq 0$, $\mu(B) \neq 0$ and $V(P) = A \cup B$. We set $f := \chi_A$ and $g := \chi_B$. We claim that f and g are not in P. If $f \in P$, then $f(V(P)) = \{0\}$ and so $A \cap V(P) = \emptyset$. This implies that $A = \emptyset$ a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction. Similarly, $g \notin P$. This means that $f \notin P$ and $g \notin P$ but $fg = 0 \in P$, which is a contradiction. □

Let $A \subseteq X$. As usual $A \subseteq V(I(A))$. The following theorem shows that if A and $V(I(A))$ are measurable, $A = V(I(A))$ a.e. on $(X, \mathcal{A}, \mu)$.

Theorem 3.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $A \subseteq X$. If A and $V(I(A))$ are measurable, then*

$$\mu\big(\{x \in X : x \in V(I(A)) \backslash A\}\big) = 0.$$

Proof. Suppose that A and $V(I(A))$ are measurable sets. We define $K := V(I(A)) \backslash A$ and $g := 1 - \chi_A$.

It is easy to check that K is a measurable set and g is a measurable function. Since $g \in I(A)$, $g(V(I(A)) = \{0\}$ and so $g(A \cup K) = 0$ a.e. on $(X, \mathcal{A}, \mu)$. This means that $A \cup K \subseteq A$ a.e. on $(X, \mathcal{A}, \mu)$ and hence $\mu(K) = 0$. □

The ideal of a subset of X is radical. We might hope that every radical ideal is the ideal of some subset of X or equivalently for every ideal J of $M(X, \mathcal{A}, \mu)$, $I(V(J)) = \sqrt{J}$. But every radical ideal is the ideal of some subset of X if and only if every prime ideal is the ideal of some subset of X since every prime ideal is radical and every radical ideal is an intersection of prime ideals. For better understanding of this matter in $M(X, \mathcal{A}, \mu)$, look at the example below.

Example 3.3. Let X be the real line, $\mathcal{A}$ be the power set of $\mathbb{R}$ and μ be the counting measure on $\mathcal{A}$. The set $\{0\}$ is measurable and $\mu(\{0\}) = 1$. Thus $\{0\}$ is near-zero and so $I(0)$ is a maximal ideal of $M(\mathbb{R}, \mathcal{A}, \mu)$, by Theorem 2.4. We set:

$$J := \big\{f \in M(\mathbb{R}, \mathcal{A}, \mu) : f(x) = 0, \text{ for every } x \in [-1, 1]\big\}.$$

The ideal J is not zero since for every $f \in M(X, \mathcal{A}, \mu)$, $\chi_{[-1,1]} f \in J$. If $f := \chi_{[0,+\infty)}$ and $g := \chi_{(-\infty,0]}$ then $f \notin J$ and $g \notin J$, but $fg = 0 \in J$. This means that J is not prime. But J is a radical ideal, for if $h^n \in J$ then for every $x \in [-1, 1]$, $h^n(x) = 0$ and so $h(x) = 0$. Therefore J is a radical ideal properly contained in $I(0)$. Since a radical ideal is the intersection of all prime ideals that contain it, there are prime ideals properly contained in $I(0)$.

Now suppose that P is a prime ideal properly contained in $I(0)$, $J \subseteq P$ and $I(V(P)) = P$. We claim that there exist two disjoint members in $V(P)$. Otherwise, we consider two cases:

Case 1: $V(P)$ is empty. Then $P = M(X, \mathcal{A}, \mu)$ a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction.

Case 2: For some $x_0 \in X$, $V(P) = \{x_0\}$. Then $\mu(\{x_0\}) = 1$ and so $\{x_0\}$ is a near-zero set. By Theorem 2.4, $I(V(P))$ is a maximal ideal, which is a contradiction.

Therefore, there exist two disjoint members $x_1, x_2 \in V(P)$. We define $h := \chi_{(-\infty, \frac{x_1+x_2}{2})}$ and $k := \chi_{(\frac{x_1+x_2}{2}, +\infty)}$. Since $(-\infty, \frac{x_1+x_2}{2})$ and $(\frac{x_1+x_2}{2}, +\infty)$ are measurable sets, h and k are measurable functions. It is easy to check that $h \notin P$ and $k \notin P$ but $hk = 0 \in P$, which is a contradiction. This means that for every prime ideal P such that P contained in $I(0)$ and $J \subseteq P$, $I(V(P)) \neq P$. In the other words, there exist radical ideals that is not the ideals of some subset of X.

For the final theorem of this paper, we present the definition of germs of real-valued measurable functions at the points in X.

Definition 3.4. Suppose that $(X, \mathcal{A}, \mu)$ is measure space. For every $a \in X$,

$$J(a) := \big\{ f \in M(X, \mathcal{A}, \mu) : f|_B = 0, \text{ for some } B \in \mathcal{A} \text{ such that } a \in B \big\}$$

and $M(X, \mathcal{A}, \mu)/J(a)$ is *the ring of germs of measurable functions at* a.

Theorem 3.5. *Let* $(X, \mathcal{A}, \mu)$ *be a measure space and* $a \in X$.

(a) *The prime ideals contained in* $I(a)$ *are in bijection with the prime ideals in* $M(X, \mathcal{A}, \mu)/J(a)$.

(b) *The ring* $M(X, \mathcal{A}, \mu)/J(a)$ *is isomorphic to the localization* $M(X, \mathcal{A}, \mu)_{I(a)}$.

Proof. (a) It suffices to show that $J(a) \subseteq P$ for all $P \subseteq I(a)$. Suppose that $P \subseteq I(a)$ and $f \in J(a)$. Then there exists $B \in \mathcal{A}$ such that $a \in B$ and $f(x) = 0$ for all $x \in B$. We set $g := \chi_{X \setminus B}$. Thus $g \notin I(a)$ and hence $g \notin P$. On the other hand, $fg = 0 \in P$ and so $f \in P$.

(b) Suppose that $f \notin I(a)$. We claim that f becomes a unit in $M(X,\mathcal{A},\mu)/J(a)$. It suffices to show that f^2 is a unit. Let $g := \max\{f^2, f^2(a)/2\}$. It is easy to check that g is a measurable function. Since $f^2(a) > 0$, $g > 0$ and so is a unit. The function $g - f^2$ is measurable and

$$Z_{g-f^2} = \{x \in X : f^2(x) \geq f^2(a)/2\} = (f^2)^{-1}\big([f^2(a)/2, +\infty)\big).$$

Since f^2 is measurable, $(f^2)^{-1}([f^2(a)/2, +\infty))$ is a measurable set and since $a \in (f^2)^{-1}([f^2(a)/2, +\infty))$, $g - f^2 \in J(a)$.

Now we show that the natural map $M(X,\mathcal{A},\mu) \longrightarrow M(X,\mathcal{A},\mu)_{I(a)}$ sends any $f \in J(a)$ to 0. Since $f \in J(a)$, there exists $B \in \mathcal{A}$ such that $a \in B$ and $f(B) = \{0\}$. We set $g := \chi_B$. Then $g \notin I(a)$, $fg = 0$ and so $f/1 = 0/g$. □

4. Conclusion

Let $(X,\mathcal{A},\mu)$ be a measure space and $M(X,\mathcal{A},\mu)$ be the ring of measurable functions from X to the real line $\mathbb{R}$ with arbitrary σ-algebra $\mathcal{A}$ on X and arbitrary measure μ on $\mathcal{A}$. The concept of near-zero set of $(X,\mathcal{A},\mu)$ is very important in this paper. By using this concept, we show that every ideal of a near-zero set of $(X,\mathcal{A},\mu)$ is a maximal ideal of $M(X,\mathcal{A},\mu)$. We also obtain some results about the variety of prime ideals of $M(X,\mathcal{A},\mu)$. Finally, the definition of germs of real-valued measurable functions at the points in X is presented.

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Homayon Hejazipour
Department of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran.
Email: homayhejazipour@gmail.com

Ali Reza Naghipour
Department of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran.
Email: naghipour@sku.ac.ir

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SOME ALGEBRAIC AND MEASURE THEORETIC PROPERTIES

OF THE RINGS OF MEASURABLE FUNCTIONS

H. HEJAZIPOUR AND A. R. NAGHIPOUR

برخی از خواص نظریه جبری و اندازه برای حلقه‌های توابع اندازه‌پذیر

همایون حجازی پور[۱] و علیرضا نقی پور[۲]

[۱,۲]دانشکده علوم ریاضی، دانشگاه شهرکرد، شهرکرد، ایران

فرض کنیم $M(X, \mathcal{A}, \mu)$ حلقه توابع اندازه‌پذیر حقیقی مقدار روی فضای اندازه $(X, \mathcal{A}, \mu)$ باشد. نشان می‌دهیم که ایده‌ال‌های ماکسیمال از $M(X, \mathcal{A}, \mu)$ متناظر با مجموعه‌های به خصوص اندازه‌پذیر در $\mathcal{A}$ هستند. همچنین برخی دیگر از خواص جبری $M(X, \mathcal{A}, \mu)$ را بررسی می‌کنیم.

کلمات کلیدی: فضاهای اندازه، حلقه های‌توابع اندازه پذیر، ایده‌ال‌های ماکسیمال، ایده‌ال‌های اول، چند گونای ایده‌ال‌ها.